ON APPROXIMATING LEBESGUE INTEGRALS BY RIEMANN SUMS

by SZILÁRD GY. RÉVÉSZ AND IMRE Z. RUZSA†

(Received 11 November, 1989)

1. If f is a real function, periodic with period 1, we define

$$(M_n f)(x) = \frac{1}{n}\sum_{i=1}^{n} f\left(x + \frac{i}{n}\right) \quad (n \in \mathbb{N}). \tag{1}$$

In the whole paper we write $\int$ for $\int_0^1$, mE for the Lebesgue measure of $E \cap [0, 1]$, where $E \subset \mathbb{R}$ is any measurable set of period 1, and we also use χ_E for the characteristic function of the set E. Consistent with this, the meaning of $\mathscr{L}^p$ is $\mathscr{L}^p[0, 1]$. For all real x we have

$$\lim_{n\to\infty} (M_n f)(x) = \int f, \tag{2}$$

if f is Riemann-integrable on $[0, 1]$. However, $\int f$ exists for all $f \in \mathscr{L}^1$ and one would wish to extend the validity of (2). As easy examples show, (cf. [**3**], [**7**]), (2) does not hold for $f \in \mathscr{L}^p$ in general if $p < 2$. Moreover, Rudin [**4**] showed that (2) may fail for all x even for the characteristic function of an open set, and so, to get a reasonable extension, it is natural to weaken (2) to

$$\lim_{\substack{n\to\infty \\ n \in S}} (M_n f)(x) = \int f \quad \text{for a.a. } x, \tag{3}$$

where $S \subset \mathbb{N}$ is some "good" increasing subsequence of $\mathbb{N}$. Naturally, for different function classes $\mathscr{F} \subset \mathscr{L}^1$ we get different meanings of being good. That is, we introduce the class of $\mathscr{F}$-good sequences as

$$\mathscr{G}(\mathscr{F}) = \{S \subset \mathbb{N} : (3) \text{ holds for all } f \in \mathscr{F}\}. \tag{4}$$

In 1934 Jessen [**1**], [**2**] proved that if S has the arithmetic property

$$n_k \mid n_{k+1} \text{ for } k \in \mathbb{N}, \quad \text{where} \quad S = \{n_1, n_2, \ldots\}, \tag{5}$$

then S is $\mathscr{L}^1$-good, i.e. $S \in \mathscr{G}(\mathscr{L}^1)$. In 1948 Salem [**5**] proved (3) under certain assumptions on the integral modulus of continuity of f and the lacunarity of the sequences S.

On the other hand Rudin [**4**] introduced the arithmetic condition

$$\exists S_N \subset S, \qquad S_N = \{a_1, \ldots, a_N\} \quad (|S_N| = N),$$
$$a_j \nmid [a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N] \quad (j = 1, \ldots, N), \tag{6}$$

where $[\,.\,,.\,]$ denotes the least common multiple. With this concept Rudin's result runs as follows.

$$S \notin \mathscr{G}(\mathscr{L}^\infty) \text{ if } S \text{ satisfies (6) for every } N \in \mathbb{N}. \tag{7}$$

† Research partially supported by Hungarian National Foundation for Scientific Research, Grant No. 1811 and 27-3-232.

Rudin emphasises that Jessen's results and his imply the importance of the arithmetic properties of S; an immediate corollary is that there exists $S \subset \mathbb{N}$ such that $S \in \mathcal{G}(\mathcal{L}^1)$ and $S+1=\{n+1: n \in S\} \notin \mathcal{G}(\mathcal{L}^\infty)$; cf. [4, Remark A].

2. Clearly if $S' \subset S$ and $S \in \mathcal{G}(\mathcal{F})$ then $S' \in \mathcal{G}(\mathcal{F})$, and the inclusion or omission of finitely many elements can not affect the property $S \in \mathcal{G}(\mathcal{F})$; that is, it is an asymptotic property of S. We are going to construct good sequences in a less trivial manner below. To this end we introduce the least common multiple of two sequences S and T as a new sequence U defined by

$$U=[S,T]=\{[s,t]: s \in S, t \in T\}. \tag{8}$$

Observe that for sequences built up from two disjoint sets of primes we get the usual multiplication of subsets of $\mathbb{N}$. The reason for considering (8) is that for any f and $n, m \in \mathbb{N}$ we have the relation

$$(M_n(M_m f))(x)=\frac{1}{n \,.\, m}\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x+\frac{i}{n}+\frac{j}{m}\right)=(M_{[n,m]}f)(x). \tag{9}$$

THEOREM 1. *If $S, T \in \mathcal{G}(\mathcal{L}^\infty)$ then $U=[S,T]$ is also in $\mathcal{G}(\mathcal{L}^\infty)$.*

Proof. Let $f \in \mathcal{L}^\infty$, $S=(s_k)$ and $T=(t_j)$ be sequences in $\mathcal{G}(\mathcal{L}^\infty)$ and denote $I=\int f$, $Q=\|f\|_\infty$. Using Egorov's theorem, for any fixed $\varepsilon>0$ we can find a set C, periodic mod 1 and having measure $mC>1-\varepsilon$ such that for any $x \in C$

$$|M_{s_k}f(x)-I|<\varepsilon \quad (k>K) \tag{10}$$

and

$$|M_{t_j}f(x)-I|<\varepsilon \quad (j>J) \tag{11}$$

hold with appropriately chosen K and J depending only on ε, f and C. Consider the following finite subset of $\mathcal{L}^\infty$:

$$\mathcal{E}=\{M_{s_k}f: k \leq K\} \cup \{M_{t_j}f: j \leq J\} \cup \{\chi_{\mathbb{R}\setminus C}\}. \tag{12}$$

Since $S, T \in \mathcal{G}(\mathcal{L}^\infty)$, there exists a set B with $mB=0$ such that if $g \in \mathcal{E}$ and $x \notin B$ then

$$M_{s_k}g(x) \to \int g \quad \text{as} \quad k \to \infty, \qquad M_{t_j}g(x) \to \int g \quad \text{as} \quad j \to \infty.$$

Hence, for $g \in \mathcal{E}$ and $x \notin B$ there exist $K(x) \geq K$ and $J(x) \geq J$ such that

$$\left|M_{s_k}g(x)-\int g\right|<\varepsilon \quad (k>K(x)), \tag{13}$$

$$\left|M_{t_j}g(x)-\int g\right|<\varepsilon \quad (j>J(x)),$$

where of course everything depends on ε. Taking (9) into account, for the remainder we can write

$$R_n(x)=|M_{[s_k,t_j]}f(x)-I|=|M_{s_k}(M_{t_j}f)(x)-I|, \tag{14}$$

where $n=[s_k, t_j]$. From (12)–(14) we get

$$R_n(x)<\varepsilon \text{ if } j \leq J \text{ and } k>K(x) \text{ or } k \leq K \text{ and } j>J(x). \tag{15}$$

Clearly, when we form $U=[S,T]=(u_n)$, there exists an index $L(x)$ with the property that if $u_n=[s_k,t_j]$ for some $n>L(x)$ then either $k>K(x)$ or $j>J(x)$. Hence, for $n>L(x)$ we have either the condition of (15) or $j>J$ and $k>K$ simultaneously. Now for $n>L(x)$ and $j>J(x)$, $k>K$ we write

$$\begin{aligned} R_n(x) &\le \frac{1}{t_j}\sum_{i=1}^{t_j}\left|M_{s_k}f\left(x+\frac{i}{t_j}\right)-I\right| \le \frac{1}{t_j}\sum_{i=1}^{t_j}\left\{\varepsilon\,.\,\chi_C\left(x+\frac{i}{t_j}\right)+2Q\chi_{\mathbb{R}\setminus C}\left(x+\frac{i}{t_j}\right)\right\} \\ &\le \varepsilon+2Q\,.\,M_{t_j}\chi_{\mathbb{R}\setminus C}(x)<(2Q+1)\varepsilon, \quad (x\notin B), \end{aligned} \tag{16}$$

since $\chi_{\mathbb{R}\setminus C}\in\mathscr{E}$, $x\notin B$ and, for $x+(i/t_j)\in C$, (10) applies. Similarly, for $k>K(x)$ and $j>J$ we obtain

$$R_n(x)<(2Q+1)\varepsilon \quad (x\notin B). \tag{17}$$

Now (15), (16) and (17) prove

$$R_n(x)<(2Q+1)\varepsilon \quad (x\notin B, n>L(x)),$$

and consequently

$$\limsup_{n\to\infty} R_n(x)\le(2Q+1)\varepsilon \quad (x\notin B). \tag{18}$$

Take $\varepsilon=1/N$ and denote the resulting set B by B_N. For $x\notin\bigcup_{N=1}^{\infty}B_N=B^*$, (18) holds for every ε; that is, $R_n(x)\to 0$. As $mB^*=0$, the theorem is proved.

Corollary 1. *If there are only finitely many primes that divide the members of the sequence S, then $S\in\mathscr{G}(\mathscr{L}^\infty)$.*

Proof. Let the set of primes dividing elements of S be $\{p_1,\dots,p_d\}$. Then $S_j=\{p_j^k:k\in\mathbb{N}\}$ $(j=1,\dots,d)$ are d sequences in $\mathscr{G}(\mathscr{L}^\infty)$ according to Jessen's Theorem [5]. By Theorem 1, $S_0=\{p_1^{k_1}p_2^{k_2}\dots p_d^{k_d}:k_1,\dots,k_d\in\mathbb{N}\}\in\mathscr{G}(\mathscr{L}^\infty)$; since $S\subset S_0$, this proves the corollary.

3. We say that a sequence S has finite Rudin dimension d if (6) is valid for $N\le d$ but not for $N>d$. If S does not have a finite dimension, then it has dimension ∞. The smallest possible Rudin dimension, 1, occurs for the sequences of Jessen in (5) which are $\mathscr{L}^1$-good sequences. The other extremity is dimension ∞, occurring for the sequences of Rudin used in (7). According to this theorem of Rudin any $\mathscr{G}(\mathscr{L}^\infty)$ sequence must have a finite Rudin dimension, and the least common multiple of two $\mathscr{G}(\mathscr{L}^\infty)$ sequences cannot be of dimension ∞ in view of Theorem 1. This also follows from the following.

Proposition 1. *If A and B are sequences having Rudin dimension α and β respectively then $C=[A,B]$ has dimension $\gamma\le\alpha+\beta$.*

Proof. If $c_j=[a_j,b_j]$ for $j=1,\dots,\alpha+\beta+1$ are $\alpha+\beta+1$ elements of C, then we have at least $\beta+1$ indices $j_1,\dots,j_{\beta+1}$ such that the corresponding a_{j_m} divides the least common multiple of the other $\alpha+\beta$ a_j's for each $m=1,\dots,\beta+1$. Among the corresponding $b_{j_1},\dots,b_{j_{\beta+1}}$ we again find at least one b_k with the property that the least common multiple of the other β b_{j_m}'s is a multiple of b_k. Now consider $c_k=[a_k,b_k]$. As both a_k and b_k divide the least common multiples of the other a_j and b_j respectively, we obtain $c_k\mid[c_1,\dots,c_{k-1},c_{k+1},\dots,c_{\alpha+\beta+1}]$. This completes the proof.

Easy examples show that equality can occur in this proposition, but γ can also be any number not exceeding $\alpha+\beta$. As a particular example, the sequence of all integers built up from a given d-element set of primes has Rudin dimension d. This example is similar to Corollary 1 and suggests that all sequences of larger dimension can be built up from sequences of smaller dimension. However, this is not the case.

THEOREM 2. *There exists a sequence S of dimension* 3 *which is not a subsequence of the least common multiple of a finite number of sequences of dimension* 1.

Proof. We say that a set A has the property Z_l if from any $l+1$ of its elements one can select three, say a, b, c, such that $a \mid [b, c]$.

First we show that if A is contained in the least common multiple of the sets $B_1, \ldots, B_k$ of dimension 1, then A has property Z_l for some $l=l(k)$. Indeed, take l elements $a_1, \ldots, a_l$ of A. Each a_i has a representation in the form

$$a_i=[b_i^{(1)}, \ldots, b_i^{(k)}], \qquad b_i^{(t)} \in B_t.$$

Consider the complete graph on the vertices $a_1, \ldots, a_l$. Take an edge $(a_i, a_{i'})$, $i<i'$. For certain values of $t=1, \ldots, k$ the divisibility $b_i^{(t)} \mid b_{i'}^{(t)}$ holds and for other values it may not hold (but then the reverse $b_{i'}^{(t)} \mid b_i^{(t)}$ must hold); there are altogether 2^k possibilities. We color the graph with 2^k colors accordingly. We recall Ramsey's theorem: for every pair of integers u, v there is a number $R(u, v)$ such that for every coloring of any graph of more than $R(u, v)$ points with u colors there must be a complete monochromatic subgraph of v points. In particular, for a suitable $l=l(k)$ there must be a monochromatic triangle in our graph, say with vertices a_i, $a_{i'}$, $a_{i''}$, $i<i'<i''$. For every t either $b_i^{(t)} \mid b_{i'}^{(t)} \mid b_{i''}^{(t)}$ or $b_{i''}^{(t)} \mid b_{i'}^{(t)} \mid b_i^{(t)}$ must hold. In either case we conclude that

$$b_{i'}^{(t)} \mid [b_i^{(t)}, b_{i''}^{(t)}] \mid [a_i, a_{i''}],$$

which yields $a_{i'} \mid [a_i, a_{i''}]$ as wanted.

Next, for a fixed l, we find a set A_l of $l+1$ elements that has dimension 3 but does not have property Z_l.

Let p_{ij}, $i \neq j$, $1 \leqslant i, j \leqslant l+1$ be a collection of primes such that $p_{ij}=p_{ji}$ but the p_{ij} are otherwise all distinct. Define

$$n=\prod_{i,j} p_{ij}, \qquad m_i=\prod_{j \neq i} p_{ij}, \qquad n_i=n/m_i.$$

For different subscripts i, j, k we clearly have

$$n_k \nmid [n_i, n_j]=\frac{n}{p_{ij}};$$

consequently the set $A_l=\{n_1, \ldots, n_{l+1}\}$ does not have the Z_l property. We show that its dimension is at most 3. Take three elements n_i, n_j, n_k. Since a prime p_{uv} is missing only from two of the numbers n_t, namely from n_u and n_v, we have $p_{uv} \mid [n_i, n_j, n_k]$; consequently $[n_i, n_j, n_k]=n$ is divisible by any fourth number n_z, a property actually somewhat stronger than necessary.

Finally, we combine these sets into one by putting $A=\bigcup q_l A_l$, where the integers q_l are taken so that q_l is a multiple of all the numbers in $q_1 A_1 \cup \ldots \cup q_{l-1} A_{l-1}$. This union clearly will not have property Z_l for any l. We must show that it still has dimension 3.

Take any four elements of A. If they are from the same q_jA_j, then any one divides the least common multiple of the other three by the corresponding property of A_j. If they come from different sets, then the one which comes from q_jA_j with the smallest j divides the least common multiple of the others (in fact it divides any of the others) by the choice of the numbers q_j.

4. Our results do not determine whether it is possible to characterize $\mathscr{G}(\mathscr{L}^\infty)$ in terms of the Rudin dimension. For a concrete sequence it may be quite difficult to decide whether it belongs to $\mathscr{G}(\mathscr{L}^\infty)$ or to determine its Rudin dimension. The following result asserts that any sequence having sufficiently many elements has an infinite Rudin dimension, and hence is not in $\mathscr{G}(\mathscr{L}^\infty)$.

THEOREM 3. *Every sequence S of Rudin dimension d satisfies*

$$S(x) < c_d(\log x)^d,$$

where c_d is a constant depending on d and $S(x)$ denotes the number of elements of S in the interval $[1, x]$.

Proof. Let $f_d(n)$ denote the maximal number of sets that can be selected from the subsets of a set of cardinality n with the property that if $X_1, \dots, X_{d+1}$ are selected then we always have

$$X_i \subset \bigcup_{\substack{j=1 \\ j\neq i}}^{d+1} X_j \tag{19}$$

for some i. We have $f_d(n) \leq C_d n^d$; see **[6]**.

For an integer N, let $F_d(N)$ be the maximal number of integers that can be selected from the divisors of N with the property that from any $d+1$ selected numbers some one divides the least common multiple of the rest (Rudin dimension $\leq d$). We claim

$$F_d(N) \leq f_d(\Omega(N)) \leq C_d(\Omega(N))^d, \tag{20}$$

where $\Omega(N)$ denotes the number of prime divisors of N, counted with multiplicity. Indeed, to every $M \mid N$ let us assign the set of *prime-powers* that divide M. This maps the divisors of N onto the subsets of a set of cardinality $\Omega(N)$ and the divisor property corresponds to condition (19). Substituting the estimate $\Omega(N) \leq (\log N)/(\log 2)$ into (20) we obtain

$$F_d(N) \leq C'_d(\log N)^d, \qquad C'_d = (\log 2)^{-d} C_d.$$

Now consider our set S of Rudin dimension d. Fix x, and let N denote the least common multiple of all the numbers $s \in S$, $s \leq N$. We have obviously $S(x) \leq F_d(N)$; we have to estimate N.

N was defined as the least common multiple of some elements of S. Observe that not all elements are necessary to form this least common multiple; among any $d+1$ elements there is one that divides the least common multiple of the rest, and can hence be omitted. Repeating this argument, we find that N is the least common multiple of a collection of at most d elements of S; thus $N \leq x^d$. Substituting this estimate into our previous equations we find

$$S(x) \leq c_d(\log x)^d, \qquad c_d = d^d C'_d.$$

REFERENCES

1. B. Jessen, On the approximation of Lebesgue integrals by Riemann sums, *Ann. of Math.* **35** (1934), 248–251.

2. B. Jessen, The theory of integration in a space of an infinite number of dimensions, *Acta Math.* **63** (1934), 249–323.

3. J. Marczinkiewicz and A. Zygmund, Mean values of trigonometrical polynomials, *Fund. Math.* **28** (1937), 131–166.

4. W. Rudin, An arithmetic property of Riemann sums, *Proc. Amer. Math. Soc.* **15** (1964), 321–324.

5. R. Salem, Sur les sommes Riemanniennes des fonctions sommables, *Mat. Tidsskr. B* **1948** (1948), 60–62.

6. N. Sauer, On the density of families of sets, *J. Combin. Theory Ser. A.* **13** (1972), 145–147.

7. H. D. Ursell, On the behaviour of a certain sequence of functions derived from a given one, *J. London Math. Soc.* **12** (1937), 229–232.

MATHEMATICAL RESEARCH INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13–15, POB 127
H-1364 HUNGARY