TILES WITH NO SPECTRA IN DIMENSION 4

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Abstract. We show by a counterexample that the “tiling ⇒ spectral” part of Fuglede’s Spectral Set Conjecture fails already in $\mathbb{Z}^4$ and $\mathbb{R}^4$.

1. Introduction

The *Spectral Set Conjecture* of Fuglede [1] relates the class of *tiling* sets of $\mathbb{R}^d$ to some Fourier analytic property, called *spectrality.* To be able to state the conjecture precisely we recall the appropriate setting. Let G be a locally compact Abelian group (we will only consider $\mathbb{Z}^d$, $\mathbb{R}^d$ and finite commutative groups), the dual group is denoted by $\widehat{G}$. Once for all we fix a Haar-measure on G, and $\widehat{f}$ will stand for the Fourier transform of a function $f : G \to \mathbb{C}$. $Z(f)$ denotes the zero set of the function f. Further we use the notation χ_T for the characteristic function of the set $T \subseteq G$.

Definition. An open set $T \subseteq G$ is called *spectral* with spectrum $L \subseteq \widehat{G}$ if L is a complete orthogonal system in $L_2(T)$.

Definition. An open subset T of G is said to be a *tiling set* (or simply *tile*), if the whole group G can be covered by translated disjoint copies of T up to a set of zero measure. That is there exists a set $T' \subseteq G$, called a *tiling complement of* T such that $T' + T$ is the whole of G except a set of zero measure and for all $t \neq s$, $t, s \in T'$ we have $(t+T) \cap (s+T) = \emptyset$.

Remark 1. It is easy to see – and will be used throughout – that the latter *packing condition* is equivalent to $(T - T) \cap (T' - T') = \{0\}$. In fact, for a finite group G tiling is equivalent to $|G| = |T| \cdot |T'|$ and $(T - T) \cap (T' - T') = \{0\}$.

Now, the Spectral Set Conjecture reads as follows.

A domain $\Omega \subseteq \mathbb{R}^d$ is spectral if and only if it can tile $\mathbb{R}^d$ by translations.

Although there were many results supporting the conjecture (already Fuglede himself proved it in case the tiling complement or the spectrum is assumed to be a lattice), Tao [15] has recently come up with a counterexample, disproving the “spectral ⇒ tiling” part in dimension 5 and higher. Matolcsi [11] has reduced this dimension to 4, and later Kolountzakis and Matolcsi [6] disproved this part in dimension 3. They also clarified a method that could be used to give counterexamples in lower dimensions. Concerning the other, “tiling ⇒ spectral” direction of the conjecture Kolountzakis and Matolcsi [7] have given a counterexample in dimension larger or equal to 5. Our aim is to get down with the dimension to 4.

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The constructions of Tao [15] and Kolountzakis, Matolcsi [7] are based on examples in finite commutative groups. Let us describe the now automatic transition mechanism of transferring a counterexample from a finite Abelian group to $\mathbb{Z}^d$ and $\mathbb{R}^d$ by quoting the following two results of Kolountzakis and Matolcsi from [7].

Theorem 1 (**Kolountzakis–Matolcsi**). *Let* $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ *and consider a set* $A \subseteq G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$. *For the set*

$$T = T(\mathbf{n}, k) = \{0, n_1, 2n_1, \dots, (k-1)n_1\} \times \cdots \times \{0, n_d, 2n_d, \dots, (k-1)n_d\} \tag{1}$$

define $B(k) = A + T$. *Then, for large enough values of* k, *the set* $B(k) \subset \mathbb{Z}^d$ *is spectral in* $\mathbb{Z}^d$ *if and only if* A *is spectral in* G.

Theorem 2 (**Kolountzakis–Matolcsi**). *Suppose* $B \subseteq \mathbb{Z}^d$ *is a finite set and* $Q = [0,1)^d$. *Then* B *is a spectral set in* $\mathbb{Z}^d$ *if and only if* $B + Q$ *is a spectral set in* $\mathbb{R}^d$.

Note that obviously in the above constructions for a tile $A \subset G$ we must also have that $B = B(k) \subset \mathbb{Z}^d$ tiles $\mathbb{Z}^d$ (for any $k \in \mathbb{N}$) and for $B \subset \mathbb{Z}^d$ tiling $\mathbb{Z}^d$ also $B + Q$ tiles $\mathbb{R}^d$. Whence it is now straightforward that our task is reduced to exhibit a counterexample in a finite group G.

2. Counterexample in a finite group

To construct tiling complements of a set T a natural tool is provided by the following lemma.

Lemma 3 (**Szegedy** [14]). *Let* G *be a finite Abelian group,* $T \subseteq G$ *and suppose that there exists a homomorphism* $\varphi : G \to H$ *such that* φ *is injective on* T *and* $\varphi(T)$ *is a tile in* H. *Then* T *tiles also* G, *and a tiling complement is given by* $\varphi^{-1}(\tilde{T})$ *where* $\tilde{T}$ *is a complement of* $\varphi(T)$.

To exhibit a counterexample in $\mathbb{R}^4$, we follow the idea of Kolountzakis and Matolcsi [7], which is based on arguments in $\mathbb{Z}_6^5$ and the extension of the finite counterexamples to $\mathbb{Z}^5$ and $\mathbb{R}^5$ (hereafter $\mathbb{Z}_n$ denotes the cyclic group of n elements, for convenience regarded as $\mathbb{Z}/n\mathbb{Z}$). However, to go down with the dimension to 4, we have to modify the starting point, and to construct an example of "tiling $\not\Rightarrow$ spectral" first in the group $\mathbb{Z}^4$, based on considerations in $\mathbb{Z}_6^4$.

When working with $d \times r$ matrices over a finite commutative group G, the column and row vectors are regarded as elements of G^d and $\widehat{G^r}$, respectively. Particularly for cyclic groups $G = \mathbb{Z}_n$ the duality pairing between G^d and $\widehat{G^d}$ in this identification takes the following form

$$\gamma(g) = e^{\frac{2\pi \imath}{n} \gamma \cdot g} \qquad \text{for } g \in G^d = \mathbb{Z}_n^d,\ \gamma \in \widehat{G^d} = (\mathbb{Z}_n^d)^\top.$$

We will also "identify" any matrix with the set of its columns or rows; the meaning should be obvious from the context. For example, consider the mod 6 matrices

$$(*) \qquad T := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad L := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 2 \\ 4 & 4 & 2 & 0 \\ 4 & 2 & 4 & 2 \end{pmatrix}.$$

Then $T \subseteq \mathbb{Z}_6^4$ is a spectral set with spectrum $L \subseteq \widehat{\mathbb{Z}}_6^4$. This is so because $L \cdot T = K$ holds mod 6, with

$$K := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 4 & 4 \\ 0 & 2 & 0 & 4 & 4 & 2 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 4 & 4 & 2 & 0 & 2 \\ 0 & 4 & 2 & 4 & 2 & 0 \end{pmatrix},$$

and $\frac{1}{6}K$ (considered now as a matrix of real numbers) is a log-Hadamard matrix. (Recall that Matolcsi [11] calls a square matrix $H = [h_{jk}]_{j,k=1,\dots,n}$ a *log-Hadamard matrix* if the entrywise exponential of $2\pi iH$, that is, $\left[e^{2\pi i h_{jk}}\right]_{j,k=1,\dots,n}$, is a complex Hadamard matrix, i.e. a complex matrix with orthogonal rows and all entries having absolute value 1). The matrix K first appeared in the context of the Spectral Set Conjecture in Tao [15]. Later, Kolountzakis and Matolcsi [7] used it to construct a counterexample to the "tiling $\Rightarrow$ spectral" part of Fuglede's conjecture, and also the above decomposition (originally mod 3) was utilized in [11] to bring down the dimension in the disproof of the other, "spectral $\Rightarrow$ tiling" direction of the conjecture.

In finite groups G there is a very straightforward way of justifying that a subset $T \subseteq G$ is not a tile. Namely if the number of elements $|T|$ does not divide the order $|G|$ of the group, then T cannot be a tile. Unfortunately we have no such immediate evidence for being not spectral. However, a convenient reformulation of being a spectrum is the following.

Proposition 4 (**Kolountzakis** [5] **p. 37, Kolountzakis–Matolcsi** [7]**).** *The set $S \subseteq \widehat{G}$ is a spectrum of the set $R \subseteq G$ if and only if $|S| = |R|$ and $S - S \subseteq Z(\widehat{\chi}_R) \cup \{0\}$.*

Since we want to find a tile which is not spectral, we will use the above proposition together with a duality argument (see [7]).

Imagine the following situation: $R \subseteq G$ is a tile in G and there is a subset $L \subseteq \widehat{G}$ with $|R| \cdot |L| = |G|$ and $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$. Now if S was a spectrum of R, then $|S| = |R|$ and $S - S \subseteq Z(\widehat{\chi}_R) \cup \{0\}$ in view of Proposition 4, and hence the packing condition $(S - S) \cap (L - L) = \{0\}$ would hold. Since by condition we also have $|\widehat{G}| = |G| = |R| \cdot |L| = |S| \cdot |L|$, this packing condition and Remark 1 ensures that $S + L$ is in fact a *tiling* of $\widehat{G}$. So if we can construct an $R \subset G$ together with a corresponding $L \subset \widehat{G}$ satisfying the above properties, but such that L is *not* tiling $\widehat{G}$, then no spectra S of R can exist, and hence R is not spectral (although being a tile by assumption). Therefore, ultimately, our goal is to establish this situation.

Note that in this case for any tiling complement T of R we have $|R| \cdot |T| = |G|$ and also $\chi_{R+T} = \chi_G$, $\widehat{\chi}_G = \widehat{\chi}_R \cdot \widehat{\chi}_T$, and thus $Z(\widehat{\chi}_R) \cup Z(\widehat{\chi}_T) \cup \{0\} = \widehat{G}$. Hence the assumption $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$ leads to $Z(\widehat{\chi}_T) \cup \{0\} \supseteq (L - L)$ and so by $|L| = |G|/|R| = |T|$ we find that L is a spectrum of T according to Proposition 4. That is, T is tiling with complement R, and is spectral with spectrum L, but also $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$.

As a first step, we construct a set $T \subseteq \mathbb{Z}_6^4$ which is tiling and spectral with some spectrum L and further for each element $\mathbf{z} \in L^\top - L^\top$ ($\mathbf{z}$ is 4-dimensional column vector) there exists a tiling complement $R_\mathbf{z}$ of T such that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_\mathbf{z}})$. Then at the end of the proof we will construct a larger finite group, $\mathcal{G}$, where using these $R_\mathbf{z}$s and L, the above described situation will finally be achieved for some $\mathcal{R}$ and $\mathcal{L}$. Also the non-tiling of $\mathcal{G}$

by $\mathcal{L}$ will then be seen, using the observation that whenever a subset Q tiles a group G so that we also have $Q \subset G_0 \leq G$ for some subgroup G_0, then Q must also tile even G_0, (which, in turn, will be excluded by an obvious divisibility criterion).

Given a set T, the easiest way to produce a tiling complement of T is to apply the pull-back procedure described in Lemma 3. One defines a group homomorphism $\varphi : \mathbb{Z}_6^4 \to H$ with some group H such that $\varphi(T)$ tiles H and φ is injective on T. Then one can apply Lemma 3 to pull back the tiling complement of $\varphi(T)$ into $\mathbb{Z}_6^4$ showing T to be a tile. Kolountzakis and Matolcsi [7] have applied this method with one-dimensional group homomorphisms $\varphi : \mathbb{Z}_6^5 \to \mathbb{Z}_6$, in connection with a 5-dimensional decomposition of the matrix K. Their construction led to a counterexample in dimension 5.

To reduce the dimension to 4 we need to give a suitable 4-dimensional decomposition of K. The above, most straightforward, choice $K = L \cdot T$ could be a good candidate, since as remarked T is spectral and also tiling. However, executing some calculations it turns out that in this case there exist some vectors $\mathbf{z} \in L^\top - L^\top$ for which there is no one-dimensional homomorphism producing a tiling complement R' of T such that it satisfies the above non-vanishing requirement $\widehat{\chi}_{R'}(\mathbf{z}^\top) \neq 0$.

Now, there are two possibilities, if we are sticking to Lemma 3. Either we look for non one-dimensional homomorphisms or we choose a different T. Let us observe the instructive number theoretic reason of lacking such good one-dimensional φ-s: the last column of T is $0 \mod 2$. Thus we modify the above T so that this obstacle vanishes. To do this, we will keep the above L and K and alter only T. Since there are only even entries in L, we can freely add 3 to any of the elements of T, while $K = L \cdot T \mod 6$ will still hold, showing T to be spectral in $\mathbb{Z}_6^4$ with the same spectrum L. If we modify T carefully, also the existence of a universal spectrum (to all tiling components of T) can be avoided (which we, of course, aim at on our way to construct a tiling but non spectral set). So in the following, we fix T mod 3 as in $(*)$, but we will only specify it mod 2 later.

Let us fix $\mathbf{z}^\top \in L - L$. We have to find a tiling complement $R_{\mathbf{z}}$ of T (whatever T is), such that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_{\mathbf{z}}})$. Since our idea is to produce this tiling complement $R_{\mathbf{z}}$ as $\ker \varphi$ for some one-dimensional homomorphism, we look for the homomorphism in the form $\varphi(\mathbf{x}) = \mathbf{v}^\top \cdot \mathbf{x}$ with some $\mathbf{v} \in \mathbb{Z}_6^4$ to be chosen appropriately (column vector, hence $\mathbf{v}^\top$ is a row vector). Then for the Fourier transform

$$\widehat{\chi}_{R_{\mathbf{z}}}(\mathbf{w}^\top) = \sum_{\mathbf{x} \in R_{\mathbf{z}}} e^{\frac{2\pi \imath}{6} \mathbf{w}^\top \cdot \mathbf{x}}, \tag{2}$$

a choice of $\mathbf{v}$ satisfying $\alpha \mathbf{v} = \mathbf{z}$ with some $\alpha \in \mathbb{Z}_6$ will ensure

$$\widehat{\chi}_{R_{\mathbf{z}}}(\mathbf{z}^\top) = \sum_{\mathbf{x} \in R_{\mathbf{z}}} e^{\frac{2\pi \imath}{6} \alpha \mathbf{v}^\top \cdot \mathbf{x}} = \sum_{\mathbf{x} \in R_{\mathbf{z}}} e^{\frac{2\pi \imath}{6} \alpha \varphi(\mathbf{x})} = \sum_{\mathbf{x} \in R_{\mathbf{z}}} e^0 = |R_{\mathbf{z}}| > 0. \tag{3}$$

So the homomorphism φ should be given in such a way that $\mathbf{z}^\top$ becomes a scalar multiple of $\mathbf{v}^\top$.

To find a suitable $\mathbf{v}$ we let $\mathbf{k} := \mathbf{z}^\top \cdot T \in K - K$. Notice that although T has been defined only mod 3 so far, this nevertheless makes sense, since $\mathbf{z}$ has even coordinates, and $\mathbf{k}$ is a permutation of $(0, 0, 2, 2, 4, 4)$ mod 6. Now "divide" $\mathbf{k}$ by 2 (mod 6) (this is because of the above consideration with α); then for each entry we have two possibilities, as $0 = 2 \cdot 0 = 2 \cdot 3$, $2 = 2 \cdot 1 = 2 \cdot 4$, $4 = 2 \cdot 2 = 2 \cdot 5$.

So we fix $\mathbf{e}$ among the possible "halves" of $\mathbf{k}$ such that it will be a permutation of $(0, 1, 2, 3, 4, 5)$ and, moreover that the matrix equation $\mathbf{v}^\top \cdot T = \mathbf{e}$ has a solution mod 6 in $\mathbf{v}$ with an appropriate choice of T (which, up to here, is given only mod 3 but is still to be specified mod 2). These assumptions will ensure that the homomorphism φ defined by $\mathbf{v}$ is surjective (hence tiling) and injective on T. Observe that $K-K$ consists of all the vectors with first coordinate 0 and the rest 5 coordinates being any permutation of $(0,2,2,4,4)$. Thus for any choice of $\mathbf{e}$, by $2\cdot 2 = 1$ mod 3 we will have $\mathbf{e} = 2\mathbf{k}$ mod 3. That is, a solution $\mathbf{v}_3$ of $\mathbf{v}^\top \cdot T = \mathbf{e}$ mod 3 undoubtedly exists, because $L\cdot T = K$ mod 3, hence $2\cdot(L-L)\cdot T$ covers $2(K-K)$ containing $\mathbf{k}$ mod 3. We show that with an appropriate choice of $\mathbf{e}$ and T one also finds a mod 2 solution. Clearly the first coordinate e_1 of $\mathbf{e} = (e_1,e_2,e_3,e_4,e_5,e_6)$ can be fixed as 0. Among the coordinates of $\mathbf{k}$ there are exactly two falling into each of the mod 3 classes. These we call *pairs.* Now we have to distinguish between these pairs mod 2. Notice that we can assume that among e_2, e_3 and e_4 exactly two are odd. Indeed, among these three elements there is either a pair from the same mod 3 class, or all three elements differ mod 3. In either case we can prescribe e_2, e_3 and e_4 such that $e_2 = 1$ mod 2, and e_3 and e_4 have different parity, while for the rest two coordinates e_5 and e_6 of $\mathbf{e}$ the only restriction is that the mod 3 pairs have to be mod 2 different. Choosing $\mathbf{e}$ in such a way and, for instance,

$$T := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \text{mod } 2,$$

an easy calculation shows that $\mathbf{v}^\top \cdot T = (0, v_1, v_2, v_3, v_4, v_2+v_3+v_4) = \mathbf{e}$ mod 2 has a solution $\mathbf{v}_2$ mod 2.

Now the desired $\mathbf{v}$ can be computed from $\mathbf{v}_2$ and $\mathbf{v}_3$ because the moduli are relative primes. So finally our T will be

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \qquad \text{mod } 6.$$

Let us make the whole more comprehensible by means of a particular example. E.g., let $\mathbf{z}^\top := (0,2,2,4) \in L-L$. Then $\mathbf{k} = \mathbf{z}^\top \cdot T = (0,0,2,2,4,4)$ mod 6. So $\mathbf{e} = \mathbf{k}/2 = (0,0,1,1,2,2)$ mod 3, then as described above we can choose $\mathbf{e} = (0,1,1,0,1,0)$ mod 2, resulting in $\mathbf{e} = (0,3,1,4,5,2)$ mod 6. The solution vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are the following $\mathbf{v}_2^\top = (1,1,0,1)$ mod 2 and $\mathbf{v}_3^\top = (0,1,1,2)$ mod 3, hence $\mathbf{v}^\top = (3,1,4,5)$.

The above T, though chosen together with $\mathbf{e}$, is independent of $\mathbf{z}$, and has all necessary properties. Indeed, it is a tile, as the tiling complements $R_{\mathbf{z}} := \ker\varphi$ show (with the φ above *depending* on $\mathbf{z}$), and also spectral by definition with spectrum L. It remains only to show that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_{\mathbf{z}}})$, but this is obvious by construction. In fact, let $\mathbf{x} \in R_{\mathbf{z}}$: this means $\mathbf{v}^\top \cdot \mathbf{x} = 0$. On the other hand, $2\mathbf{v}^\top \cdot T = 2\mathbf{e}$ and $\mathbf{z}^\top \cdot T = \mathbf{k} = 2\mathbf{e}$. From this $2\mathbf{v} = \mathbf{z}$ follows, whence $0 = 2\mathbf{v}^\top \cdot \mathbf{x} = \mathbf{z}^\top \cdot \mathbf{x}$, so keeping (3) in mind gives $\widehat{\chi}_{R_{\mathbf{z}}}(\mathbf{z}^\top) > 0$.

Let now $L^\top - L^\top = \{\mathbf{z}_j : j = 1,\dots,k\}$, say ($\mathbf{z}_j$ is a column vector). Take $\mathcal{L} \subseteq \widehat{\mathcal{G}} := \mathbb{Z}_6^4 \times \mathbb{Z}_p$ to be the set of the elements of L extended by a 0 in the fifth coordinate (i.e. considering $L \subseteq \widehat{G} \cong \widehat{G} \times \{0\} =: \widehat{\mathcal{G}}_0$ as imbedded into $\mathcal{G}$, which trivial identification – as well as the similar, dual imbedding of G into $\mathcal{G}$ – we do not mention further on). We put together the desired tiling but not spectral set from the above constructed tiling complements $R_j := R_{\mathbf{z}_j}$ of T. So let $p \geq k$ be relatively prime to 6, and let us augment the sequence $R_1,\dots,R_p$ by listing the R_js and then repeating R_k additionally $p-k$ times.

Consider the group $\mathcal{G} = \mathbb{Z}_6^4 \times \mathbb{Z}_p$ (which is, on the other hand, isomorphic to $\mathbb{Z}_6^3 \times \mathbb{Z}_{6p}$) and the set

$$\mathcal{R} = \bigcup_{j=1}^{p} \left(R_j + (0,0,0,0,j)^\top\right).$$

Consider now the sets $\mathcal{R}$ and $\mathcal{L}$. First, $\mathcal{R} + T$ is a tiling, as for all $j = 1, \dots, p$ $\left(R_j + (0,0,0,0,j)^\top\right) + T$ is a tiling of the translated subgroups $\mathcal{G}_0 + (0,0,0,0,j)^\top$ of $\mathcal{G}_0 := \mathbb{Z}_6^4 \times \{0\}$. Hence $\mathcal{R}$ is a tile of $\mathcal{G}$ with the tiling complement T.

Moreover, L is a spectrum of T, hence we get $|\mathcal{L}| = |L| = |T| = |\mathcal{G}|/|\mathcal{R}|$. (It can also bee seen easily that $\mathcal{L}$ is a spectrum of T, but we do not need this here.) We need to show that also $Z(\widehat{\chi}_{\mathcal{R}}) \cap (\mathcal{L} - \mathcal{L}) = \emptyset$. So let $0 \neq \mathbf{z} \in \mathcal{L}^\top - \mathcal{L}^\top$ be any element; it corresponds to $\mathbf{z}_j$ for some index $j \leq p$. Then the Fourier transform of $\chi_{\mathcal{R}}$ evaluated at $\mathbf{z}^\top$ is

$$\widehat{\chi}_{\mathcal{R}}(\mathbf{z}^\top) = \widehat{\chi}_{R_1}(\mathbf{z}^\top) + \dots + \widehat{\chi}_{R_{k-1}}(\mathbf{z}^\top) + (p-k+1)\widehat{\chi}_{R_k}(\mathbf{z}^\top) > 0,$$

because all the terms are non-negative (all R_ms being subgroups), and by construction the j^{th} term is strictly positive in view of (3). So $\mathcal{R}$ and $\mathcal{L}$ fulfill the initial requirements for a pair of sets for a counterexample.

Furthermore, $\mathcal{L}$ is not a tile. To see this note that $\mathcal{L} \subset \widehat{\mathcal{G}}_0$, hence $\mathcal{L}$ can be a tile if only it tiles also the subgroup $\widehat{\mathcal{G}}_0$, that is, if L tiles $\widehat{G}$. But since L consists of vectors with all coordinates even, it is in fact a subset of the subgroup $E \leq \widehat{G}$ with even coordinates, hence in order to tile $\widehat{G}$, it has to tile even E. However, this is not possible since $|L| = 6$, which does not divide $|E| = 3^4$. Thus we see that the sets $\mathcal{R}$ and $\mathcal{L}$ provide all the properties of the construction we were aiming at, whence $\mathcal{R}$ is tiling $\mathcal{G}$ while being non spectral.

Having a counterexample in $\mathcal{G} \cong \mathbb{Z}_6^3 \times \mathbb{Z}_{6p}$, the counterexample in $\mathbb{Z}^4$ and $\mathbb{R}^4$ is obtained by an application of Theorems 1 and 2.

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