The Periodic Decomposition Problem

Bálint Farkas and Szilárd Gy. Révész

Abstract If a function $f : \mathbb{R} \to \mathbb{R}$ can be represented as the sum of n periodic functions as $f = f_1 + \cdots + f_n$ with $f(x + \alpha_j) = f(x)$ $(j = 1, \ldots, n)$, then it also satisfies a corresponding nth order difference equation $\Delta_{\alpha_1} \ldots \Delta_{\alpha_n} f = 0$. The periodic decomposition problem asks for the converse implication, which may hold or fail depending on the context (on the system of periods, on the function class in which the problem is considered, etc.). The problem has natural extensions and ramifications in various directions, and is related to several other problems in real analysis, Fourier and functional analysis. We give a survey about the available methods and results, and present a number of intriguing open problems. Most results have already appeared elsewhere, while the recent results of [7, 8] are under publication. We give only some selected proofs, including some alternative ones which have not been published, give substantial insight into the subject matter, or reveal connections to other mathematical areas. Of course this selection reflects our personal judgment. All other proofs are omitted or only sketched.

Dedicated to Imre Z. Ruzsa on the occasion of his 60th birthday.

Bálint Farkas was supported in part by the Hungarian National Foundation for Scientific Research, Project #K-100461, Szilárd Gy Révész was supported in part by the Hungarian National Foundation for Scientific Research, Project #'s K-81658, K-100461, NK-104183, K-109789.

B. Farkas
Bergische Universität Wuppertal, Wuppertal, Germany
e-mail: farkas@math.uni-wuppertal.de

Sz. Gy. Révész (✉)
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Kuwait University, Kuwait, Kuwait
e-mail: szilard.revesz@renyi.hu; szilard@sci.kuniv.edu.kw

© Springer-Verlag Berlin Heidelberg 2014
Z. AlSharawi et al. (eds.), *Theory and Applications of Difference Equations and Discrete Dynamical Systems*, Springer Proceedings in Mathematics & Statistics 102,
DOI 10.1007/978-3-662-44140-4_8

Keywords Periodic functions · Periodic decomposition · Difference equation · Almost periodic and mean periodic functions · Transformation invariant functions · Functions with values in a group · Operator semigroups

1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a function with

$$f = f_1 + \cdots + f_n, \qquad f_j(x + \alpha_j) = f_j(x) \quad \forall x \in \mathbb{R}, \quad j = 1, \ldots, n, \tag{1}$$

where $\alpha_j \in \mathbb{R}$ are fixed real numbers. We call this an $(\alpha_1, \ldots, \alpha_n)$-*periodic decomposition* of f. For $\alpha \in \mathbb{R}$ let Δ_α denote the (forward) *difference operator*

$$\Delta_\alpha : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}, \quad \Delta_\alpha g(x) := g(x + \alpha) - g(x).$$

Then the α_i-periodicity of f_i above means $\Delta_{\alpha_i} f_i = 0$. The difference operators commute, so

$$\Delta_{\alpha_1} \Delta_{\alpha_2} \ldots \Delta_{\alpha_n} f = 0. \tag{2}$$

Problem 1.1 *(Ruzsa, 70s)* Does the converse implication "(2) ⇒ (1)" hold true?

Naturally, this question can be posed *in any given function class* $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$.

Definition 1.2 Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ be a set of functions. With $n \in \mathbb{N}$, $n \geq 1$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ given, the function class $\mathcal{F}$ is said to have the *decomposition property with respect to* $\alpha_1, \ldots, \alpha_n$ if for each $f \in \mathcal{F}$ satisfying (2) a periodic decomposition (1) exists with $f_j \in \mathcal{F}$ $(j = 1, \ldots, n)$. Furthermore, the function class $\mathcal{F}$ has the n-*decomposition property* if it has the decomposition property for every choice of $\alpha_1 \ldots, \alpha_n \in \mathbb{R}$, and $\mathcal{F}$ has the *decomposition property* if it has the n-decomposition property for each integer $n \geq 1$.

Notice that we did not speak about uniqueness of decompositions. As we shall see uniqueness is an intriguing problem and in general cannot be expected. Note also that $\mathbb{R}^\mathbb{R}$ or $C(\mathbb{R})$ (space of continuous functions) do *not* have the n-decomposition property for $n \geq 2$. Indeed, let $n = 2$ and $\alpha_1 = \alpha_2 = \alpha$. The identity function $\mathrm{id}(x) := x$ satisfies $\Delta_\alpha \Delta_\alpha \mathrm{id} = 0$, but it fails to be α-periodic. So the implication "(2) ⇒ (1)" fails. As a matter of fact, a function class containing the identity does not have the decomposition property.

The above choice for α_1, α_2 hides the nature of the problem a bit: The existence of periodic decompositions may depend on the system $\alpha_1, \ldots, \alpha_n$ of prescribed periods. If we take $\alpha_1 = 1$ and $\alpha_2 = \sqrt{2}$ the arguments above do not work. And in fact, *if* α_1 *and* α_2 *are incommensurable* (i.e., $\alpha_1\mathbb{Z} \cap \alpha_2\mathbb{Z} = \{0\}$) then $f = \mathrm{id} : \mathbb{R} \to \mathbb{R}$ has a decomposition as $f = f_1 + f_2$, $\Delta_{\alpha_j} f_j = 0$.

Proposition 1.3 *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be incommensurable. Then each function $f : \mathbb{R} \to \mathbb{R}$ satisfying* (2) *can be written as $f = f_1 + f_2$, with f_1, f_2 being α_1 and α_2 periodic, respectively. That is, $\mathbb{R}^{\mathbb{R}}$ has the decomposition property with respect to any system of two incommensurable reals.*

Proof Using the *axiom of choice*, we can select one representative from each of the classes of the equivalence relation $x \sim y \Leftrightarrow x - y \in \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z}$.

On each class we construct our f_j as follows. For the fixed class representative $y \in \mathbb{R}$ take $f_1(y + k\alpha_1 + m\alpha_2) := f(y + m\alpha_2)$ and $f_2(y + k\alpha_1 + m\alpha_2) := f(y + k\alpha_1) - f(y)$. Then f_j are α_j-periodic and by (2)

$$\begin{aligned} f(y + k\alpha_1 + m\alpha_2) &= f(y + m\alpha_2) + f(y + k\alpha_1) - f(y) \\ &= f_1(y + k\alpha_1 + m\alpha_2) + f_2(y + k\alpha_1 + m\alpha_2). \end{aligned}$$

This ends the construction of a periodic decomposition. □

Of course, the decomposition given in the preceding proof depends on the particular choice of the representatives for the equivalence classes, hence uniqueness cannot be expected. In fact, by adding and subtracting a function constant on $\alpha_1\mathbb{Z} + \alpha_2\mathbb{Z}$ to f_1 and f_2 respectively, we immediately obtain different decompositions. In Sect. 8 below we shall return to this matter. The above decomposition can be far worse than the function itself. E.g., $f = \mathrm{id}$ is continuous, while f_1 and f_2 are certainly not, for continuous periodic functions, hence also their sums, are necessarily bounded. That $f = \mathrm{id}$ does not even have a measurable decomposition, is proved in [34] by a somewhat involved argumentation.

In fact, no function with $\lim_{x\to\infty} f(x) = \infty$ can have a measurable periodic decomposition. To see this, let $\varepsilon, \eta > 0$ be arbitrarily fixed, and assume that f has a measurable decomposition (1). Then for each $j = 1, \ldots, n$, f_j must be bounded on $[0, \alpha_j]$ by some constant $K_j < \infty$ apart from an exceptional set $A_j \subseteq [0, \alpha_j]$ of Lebesgue measure $|A_j| < \eta$. Therefore, on any interval I of length ℓ (large), f is bounded by $K := K_1 + \cdots + K_n < \infty$ apart from an exceptional set $A \subseteq I$ of measure $|A| < (\lceil \ell/\alpha_1 \rceil + \cdots + \lceil \ell/\alpha_n \rceil)\eta < \varepsilon\ell$, if η is chosen small enough. So f is "locally almost bounded": for any $\varepsilon > 0$ there is $K < \infty$ such that on any sufficiently large interval I, $|\{x \in I : |f(x)| > K\}| < \varepsilon|I|$.

One would think that the bug here is with the axiom of choice, the huge number of "ugly", non-measurable functions, so that once a continuous function has a relatively nice—say, measurable—decomposition, then it must also have a continuous one. However, the contrary is true:

Proposition 1.4 (Keleti [24]) *There exists $f \in \mathrm{C}(\mathbb{R})$ having measurable decomposition* (1) *but without a continuous periodic decomposition.*

For the proof see [23, Theorem 4.8].

We can also look for further immediate solutions of (2): For example polynomials of degree $m < n$ satisfy this difference equation. So, we can ask for *quasi-decompositions* with *periodic functions and polynomials*

$$f = p + f_1 + \cdots + f_n, \quad \text{with } \Delta_{\alpha_j} f_j = 0 \quad \text{and} \quad \deg p < n \quad \text{a polynomial.} \tag{3}$$

Theorem 1.5 (Ruzsa and Szegedy (unpublished)) *There exist continuous, unbounded solutions of* (2) *with* $\lim_{x\to\infty} f(x)/x = 0$.

As a consequence $C(\mathbb{R})$ does not have the quasi-decomposition property either. For a discussion see [29, pp. 338–339]. It can be precisely described which functions in $C(\mathbb{R})$ have continuous periodic quasi-decompositions (3).

Theorem 1.6 (Laczkovich and Révész [29]) *For a function* $f \in C(\mathbb{R})$ *the existence of a quasi-decomposition* (3) *is equivalent to* (2) *together with the Whitney condition*

$$\delta_n(f) := \sup\Big\{\sum_{j=0}^{n}(-1)^j\binom{n}{j}f(x+jh) \;:\; x,h\in\mathbb{R}\Big\} < \infty.$$

Proof Notice first $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(x+jh) = \Delta_h^n f(x)$. Hence, if $f = p + f_1 + \cdots + f_n$ as in (3), then $\Delta_h^n p = 0$. Since f_j is α_j-periodic and continuous $\delta_n(f_j) \le 2^n \sup_{t\in[0,\alpha_j]}|f_j(t)|$. So that (3) implies both (2) and $\delta_n(f) < \infty$. Conversely, a result of Whitney [38] says that $\delta_n(f) < \infty$ entails that f can be approximated by a polynomial p of degree $\deg p < n$ within a bounded distance: $\|f - p\|_\infty < \infty$. Thus, for $g := f - p \in BC(\mathbb{R})$ we have $\Delta_{\alpha_1}\ldots\Delta_{\alpha_n} g = 0$ and it remains to establish the decomposition property of $BC(\mathbb{R})$, postponed to Sect. 4.1. □

2 Continuous Periodic Decompositions

In view of the foregoing discussion it is natural to pose the boundedness condition on the occurring functions and look at subclasses $\mathcal{F}$ of the space $BC(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. Note that if f has a continuous periodic decomposition it is *uniformly almost periodic* (alternatively, Bohr or Bochner almost periodic), i.e., the set

$$\big\{f(\cdot + t) : t \in \mathbb{R}\big\} \subseteq BC(\mathbb{R})$$

of its translates is relatively compact with respect to the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$. Denote by $UAP(\mathbb{R})$ the set of all such functions, which becomes a Banach space, actually a C^*-algebra, if endowed with the supremum norm and pointwise operations, see [2, Chap. 1]. Evidently, a solution of (2) in $\mathcal{F} \subseteq BC(\mathbb{R})$ must be contained by $UAP(\mathbb{R})$ if $\mathcal{F}$ has the decomposition property.

Proposition 2.1 *The space* UAP($\mathbb{R}$) *has the decomposition property.*

At this point, we give a proof only for the case of incommensurable periods to illustrate the use of Fourier analytic techniques. The complete proof will be given in Sect. 3 as a special case of a more general result, see Example 3.7.

Proof Suppose $\alpha_1, \ldots, \alpha_n$ are incommensurable and let $f \in \mathrm{UAP}(\mathbb{R})$. Any $f \in \mathrm{UAP}(\mathbb{R})$ has a mean value

$$Mf := \lim_{t\to\infty} \frac{1}{t} \int_0^t f(s)\mathrm{d}s \in \mathbb{C}$$

by [2, Sect. 1.3], and M is a continuous linear functional on UAP($\mathbb{R}$). Moreover, for $\lambda \in \mathbb{R}$ the Fourier coefficients of f are defined as $a(\lambda) := M(f(s)\mathrm{e}^{-is\lambda})$ among which only countably many are nonzero, denote these by c_k and the corresponding "frequences" by λ_k. We say that f has the Fourier series $f \sim \sum_k c_k \mathrm{e}^{ix\lambda_k}$.

Let $\alpha \in \mathbb{R}$. In what follows M is understood with respect to the variable s and Δ_α with respect to the variable x. We have

$$\begin{aligned} c_k \Delta_\alpha\Big(\mathrm{e}^{ix\lambda_k}\Big) &= \Delta_\alpha\Big(M(f(s)\mathrm{e}^{-is\lambda_k})\mathrm{e}^{ix\lambda_k}\Big) \\ &= M\Big(f(s)\mathrm{e}^{-i(s-(x+\alpha))\lambda_k}\Big) - M\Big(f(s)\mathrm{e}^{-i(s-x)\lambda_k}\Big) \\ &= M\Big((f(s+\alpha) - f(s))\mathrm{e}^{-i(s-x)\lambda_k}\Big) = M\Big(\Delta_\alpha f(s)\mathrm{e}^{-is\lambda_k}\Big)\mathrm{e}^{ix\lambda_k}. \end{aligned}$$

So that the difference equation (2) implies $\Delta_{\alpha_1} \ldots \Delta_{\alpha_n} c_k \mathrm{e}^{ix\lambda_k} = 0$. Since $c_k \neq 0$, this is only possible if $\lambda_k = 2\pi\ell/\alpha_j$ for some $\ell \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. Since $\alpha_1, \ldots, \alpha_n$ are incommensurable there can be at most one such j.

On the other hand, by Sect. 1.8.6° in [2] $\frac{1}{N}\sum_{k=1}^N f(s+k\alpha_j)$ converges uniformly (in s) as $N \to \infty$ to an α_j-periodic continuous function f_j, whose (non-zero) Fourier coefficients are precisely those Fourier coefficients $a(\lambda)$ of f for which $\lambda \in (2\pi/\alpha_j)\mathbb{Z}$. We see therefore that $f_1 + \cdots + f_n$ and f have the same Fourier coefficients, hence they coincide by Theorem I.4.7° in [2]. □

That is to say if we *a priori* know that f is uniformly almost periodic, then the difference equation (2) implies the periodic decomposition (1).

The next step is to deduce this almost periodicity. Let $\mu \in \mathrm{M}_c(\mathbb{R})$, i.e., a compactly supported finite (signed) Borel measure on $\mathbb{R}$, and let $f \in \mathrm{C}(\mathbb{R})$. Then

$$f * \mu(x) := \int_{\mathbb{R}} f(x-t)\mathrm{d}\mu(t)$$

defines a continuous function, the *convolution* of f and μ. The convolution of two measures $\mu, \nu \in \mathrm{M}_c(\mathbb{R})$ is defined by $f * (\mu * \nu) := (f * \mu) * \nu$ (for $f \in \mathrm{C}(\mathbb{R})$): As a continuous linear functional on the locally convex space $\mathrm{C}(\mathbb{R})$, $\mu * \nu$ is a compactly supported measure, i.e., $\mu * \nu \in \mathrm{M}_c(\mathbb{R})$. It is also easy to see that convolution is commutative and associative in $\mathrm{M}_c(\mathbb{R})$.

Now denote $\mu_\alpha := \delta_{-\alpha} - \delta_0$, where δ_β is the Dirac measure at $\beta \in \mathbb{R}$. Then $f * \mu_\alpha = f * (\delta_{-\alpha} - \delta_0) = \Delta_\alpha f$. With this Eq. (2) takes the form

$$f * (\mu_{\alpha_1} * \cdots * \mu_{\alpha_n}) = f * \big((\delta_{-\alpha_1} - \delta_0) * \cdots * (\delta_{-\alpha_n} - \delta_0)\big) = 0.$$

Definition 2.2 *(Schwartz [35])* A function $f \in \mathrm{C}(\mathbb{R})$ is *mean periodic* if there exists a compactly supported Borel measure μ on $\mathbb{R}$ with $f * \mu = 0$. i.e. $\int_{-\infty}^{\infty} f(x - t)\mathrm{d}\mu(t) = 0$.

Let us recall from [21, p. 44] the following.

Proposition 2.3 (Kahane) *A bounded uniformly continuous mean periodic function is uniformly almost periodic.*

An immediate consequence of this and of Proposition 2.1 is the following.

Proposition 2.4 (Gajda [13]) *The Banach space* $\mathrm{BUC}(\mathbb{R})$ *has the decomposition property.*

Gajda proved this results with a different argument (using Banach limits) that can be easily extended to the case of translations on locally compact Abelian groups (see Corollary 7.2).

However, the result of Gajda for $\mathrm{BUC}(\mathbb{R})$ falls short of the complete truth, in the extent that it does not tell that a continuous function satisfying (2) is necessarily uniformly continuous, a fact that would imply even the decomposition property of the whole $\mathrm{BC}(\mathbb{R})$ itself.

No direct proof of the implication "$f \in \mathrm{BC}(\mathbb{R})$ & (2) $\Rightarrow f \in \mathrm{BUC}(\mathbb{R})$" is known, so the decomposition property of $\mathrm{BC}(\mathbb{R})$ lies deeper. In fact, to prove that a bounded continuous solution of (2) is uniformly continuous, we have no other known ways than this periodic decomposition result on $\mathrm{BC}(\mathbb{R})$ itself.

Before proceeding let us formulate the following more general question than Problem 1.1.

Problem 2.5 Let μ, ν (or $\mu_1, \ldots, \mu_n$) be given Borel measures of compact support on $\mathbb{R}$. Clearly, if

$$f = g + h \quad \text{with} \quad g, h \in \mathrm{C}(\mathbb{R}) \quad \text{such that} \quad g * \mu = 0,\ h * \nu = 0, \tag{4}$$

then $f * (\mu * \nu) = 0$. Find conditions, under which we have the converse implication: If $f \in \mathrm{C}(\mathbb{R})$, and $f * (\mu * \nu) = 0$, then (4) holds. Or find conditions on μ ensuring that a solution $f \in \mathrm{BC}(\mathbb{R})$ of $f * \mu = 0$ is almost periodic.

In this formulation we use no assumption on boundedness or uniform continuity. Clearly, then additional assumptions are needed. E.g. additional functional equations must also be satisfied? Spectra must be simple? Spectra of μ and ν should be distinct? Several variations may be considered.

Remark 2.6 In the problem above f is by default mean periodic. However, convergence of mean periodic Fourier expansions was shown only in a complicated, complex sense. Perhaps, recent developments in the Fourier synthesis and representation of mean periodic functions can be used, see Székelyhidi [37]. Then again, boundedness and uniform continuity could be of use by means of Proposition 2.3 of Kahane.

Wierdl [39] showed that the space $\mathrm{BC}(\mathbb{R})$ of bounded continuous functions has the 2-decomposition property. Subsequently, Laczkovich and Révész proved this for general n as the main result of [29], which was the first internationally published paper in this topic (but see also the preceding paper [28]).

Theorem 2.7 (Laczkovich and Révész [29]) *The Banach space* $\mathrm{BC}(\mathbb{R})$ *has the decomposition property.*

Although many generalizations and interpretations have since been described and various tools could be invoked depending on the setup, oddly enough this first non-trivial result could be covered by neither extensions. To date, we have no other proof than the essentially elementary yet tricky original argument. In Sect. 4.1 we present a proof of this result utilizing the operator theoretic approach to be developed next.

3 Generalizations to Linear Operators

For $\alpha \in \mathbb{R}$ the translation by α acts as a homeomorphism on $\mathbb{R}$. Consider the so-called *Koopman (or composition) operator*, in this case called the *shift operator*,

$$T_\alpha : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}, \quad T_\alpha f(x) := f(x+\alpha).$$

Observe that the solutions of the difference equation (2) form the subspace

$$\ker(T_{\alpha_1} - \mathrm{I}) \cdots (T_{\alpha_n} - \mathrm{I})$$

(where I denotes the identity operator), while the functions having a periodic decomposition (1) are the elements of

$$\ker(T_{\alpha_1} - \mathrm{I}) + \cdots + \ker(T_{\alpha_n} - \mathrm{I}).$$

Then Problem 1.1 can be rephrased so as whether the equality

$$\ker(T_{\alpha_1} - \mathrm{I}) \cdots (T_{\alpha_n} - \mathrm{I}) = \ker(T_{\alpha_1} - \mathrm{I}) + \cdots + \ker(T_{\alpha_n} - \mathrm{I}) \tag{5}$$

holds? Of course, one can restrict the question by considering linear subspaces of $\mathbb{R}^{\mathbb{R}}$ that are invariant under the occurring operators. The equality then means the decomposition property of $\mathcal{F}$. Or more generally one can ask the following:

Problem 3.1 Let E be a linear space and let $T_1, \ldots, T_n : E \to E$ be commuting linear operators. Find conditions such that

$$\ker(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I}) = \ker(T_1 - \mathrm{I}) + \cdots + \ker(T_n - \mathrm{I}). \tag{6}$$

Remark 3.2 For a system of pairwise commuting operators $T_1, \ldots, T_n$ the inclusion "$\ker(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I}) \supseteq \ker(T_1 - \mathrm{I}) + \cdots + \ker(T_n - \mathrm{I})$" trivially holds. This corresponds to the trivial implication "(1) $\Rightarrow$ (2)".

The first result in this direction is the following:

Theorem 3.3 (Laczkovich and Sz. Révész [30]) *Let X be a topological vector space and $T_1, \ldots, T_n$ be commuting continuous linear operators on X. Suppose that for every $x \in X$ and $j \in \{1, \ldots, n\}$ the closed convex hull of $\{T_j^m x : m \in \mathbb{N}\}$ contains a fixed point of T_j, that is*

$$\overline{\mathrm{conv}} \left\{ T_j^m x : m \in \mathbb{N} \right\} \cap \ker(T_j - \mathrm{I}) \neq \emptyset.$$

Then (6) holds.

We shall give the proof of this theorem in a special case only, see Proposition 3.6, because that proof yields some extra information about the obtained decompositions. For the proof of the general statement we refer to [30]. For a Banach space E we denote by $\mathscr{L}(E)$ the space of bounded linear operators on E. Here are some consequences of the previous theorem:

Corollary 3.4 *Let E be a Banach space and let $T_1, \ldots, T_n \in \mathscr{L}(E)$ be commuting power bounded operators. Suppose an additional vector topology τ is given on E such that the unit ball $\mathrm{B} := \{x \in E : \|x\| \leq 1\}$ is τ-compact, and the operators T_j are τ-continuous. Then (6) holds.*

The proof is the application of the foregoing result and the Markov–Kakutani fixed point theorem (see, e.g., [5, Sect. 10.1]) to the closed convex hull $\overline{\mathrm{conv}}\{T_j^m x : m \in \mathbb{N}\}$, which was assumed to be τ-compact.

The above together with the Banach–Alaoglu theorem yields the following:

Proposition 3.5 *Let X be a normed space, $E := X^*$ and let $\tau := \sigma(X^*, X)$ be the weak* topology on X^*. If $T_1, \ldots, T_n \in \mathscr{L}(E)$ are commuting, power bounded weakly* continuous operators, then (6) holds.*

Let E be a Banach space. Suppose $T_1, \ldots, T_n \in \mathscr{L}(E)$ are power bounded, then the fixed point condition in Theorem 3.3 means precisely the mean ergodicity of $T_1, \ldots, T_n$, see [5, Theorem 8.20]. Recall that $T \in \mathscr{L}(E)$ is *mean ergodic* if

$$Px := \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} T^j x$$

exists for every $x \in E$. In this case the limit P is a bounded projection onto $\ker(T - \mathrm{I})$, the so-called *mean ergodic projection*, and one has $E = \mathrm{rg}\, P \oplus \ker P$ and $\ker P = \overline{\mathrm{rg}}(T - \mathrm{I})$, where rg and $\overline{\mathrm{rg}}$ stand for the range and the closure of the range of an operator, respectively, see [5, Sect. 8.4].

If T, S are commuting mean ergodic operators with mean ergodic projections P, Q, then $PS = SP$ (and $TQ = QT$), so that $PQ = QP$.

Proposition 3.6 *Let E be a Banach space and $T_1, \ldots, T_n \in \mathscr{L}(E)$ be commuting mean ergodic operators. Then the equality* (6) *holds.*

Proof Since the operators $T_1, \ldots, T_n$ commute, so do the mean ergodic projections $P_1, \ldots, P_n$, and actually all operators occurring in this proof commute with each other. A moment's thought explains that the *direct* decomposition

$$\begin{aligned} E &= \mathrm{rg}\, P_1 \oplus \mathrm{rg}\, P_2(\mathrm{I} - P_1) \oplus \cdots \oplus \mathrm{rg}\big(P_n(\mathrm{I} - P_{n-1}) \cdots (\mathrm{I} - P_1)\big) \\ &\quad \oplus \mathrm{rg}\big((\mathrm{I} - P_n)(\mathrm{I} - P_{n-1}) \cdots (\mathrm{I} - P_1)\big) \end{aligned}$$

is valid, i.e. for any $x \in E$ we can uniquely write $x = x_1 + \cdots + x_n + y$ with $x_i \in \mathrm{rg}\, P_i = \ker(T_i - \mathrm{I})$ and $y \in \mathrm{rg}(\mathrm{I} - P_1) \cdots (\mathrm{I} - P_n)$. Let now $x \in \ker(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I})$: then $(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I})y = 0$. It follows that $y \in \ker(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I}) \subseteq \ker(\mathrm{I} - P_1) \cdots (\mathrm{I} - P_n)$, thus $y \in \mathrm{rg}(\mathrm{I} - P_1) \cdots (\mathrm{I} - P_n) \cap \ker(\mathrm{I} - P_1) \cdots (\mathrm{I} - P_n)$. However, $(\mathrm{I} - P_1) \cdots (\mathrm{I} - P_n)$ is a projection, so from this $y = 0$ follows. □

Actually, the proof above and the result itself appears in [19] in a slightly more general form, and as a matter of fact even much earlier in [30]. None of the papers however formulated it by using the notion of mean ergodicity.

Example 3.7 Since shift operators T_α are all mean ergodic on $E = \mathrm{UAP}(\mathbb{R})$ we obtain a (complete) proof of Proposition 2.1. To see that T_α is mean ergodic it suffices to note that $\{T_\alpha^n : n \in \mathbb{N}\}$ is compact in the strong operator topology and to invoke [5, Theorem 8.20]; or alternatively one can use [2, Sect. 1.8.6°] as in the proof of Proposition 2.1 given for incommensurable periods.

Remark 3.8

(a) Notice, that $Q_j = P_j(\mathrm{I} - P_{j-1}) \cdots (\mathrm{I} - P_2)(\mathrm{I} - P_1)$ is a bounded projection on E. One trivially has $\|P_j\| \le \|T_j\|$ and $\|\mathrm{I} - P_j\| \le 1 + \|T_j\|$. If we suppose $\|T_j\| \le 1$ for $j = 1, \ldots, n$, then $\|Q_j\| \le 2^{j-1}$. The proof above yields that the decomposition obtained is actually

$$\begin{aligned} x &= P_1 x + P_2(\mathrm{I} - P_1)x + \cdots + P_n(\mathrm{I} - P_{n-1}) \cdots (\mathrm{I} - P_2)(\mathrm{I} - P_1)x \\ &= Q_1 x + Q_2 x + \cdots + Q_n x. \end{aligned}$$

Hence x has a decomposition $x = x_1 + \cdots + x_n$ with $x_j \in \ker(T_j - \mathrm{I})$ and

$$\max_{j=1,\dots,n} \|x_j\| \le 2^{n-1}\|x\|.$$

(b) If E is a Hilbert space, then the mean ergodic projections P_j are orthogonal, see [5, Theorem 8.6]. So that $\mathrm{I} - P_j$ is also an orthogonal, hence contractive, projection. This implies that $x \in \ker(T_1 - \mathrm{I})\cdots(T_n - \mathrm{I})$ has a decomposition $x = x_1 + \cdots + x_n$ with $x_j \in \ker(T_j - \mathrm{I})$ and

$$\max_{j=1,\dots,n} \|x_j\| \le \|x\|.$$

(c) In the original setting of the decomposition problem Laczkovich and Révész have shown that on $E = \mathrm{BC}(\mathbb{R})$ with T_j being translations by a_j a function f satisfying (2) has a decomposition $f = f_1 + \cdots + f_n$ with

$$\max_{j=1,\dots,n} \|f_j\|_\infty \le 2^{n-2}\|f\|.$$

The estimate is sharp for $n = 2$, see [29].

Problem 3.9 Find the best constant C_n such that any $x \in \ker(T_1 - \mathrm{I})\cdots(T_n - \mathrm{I})$ has some decomposition $x = x_1 + \cdots + x_n$ with $x_j \in \ker(T_j - \mathrm{I})$ and

$$\max_{j=1,\dots,n} \|x_j\| \le C_n\|x\|.$$

We saw $C_n \le 2^{n-1}$ in general, $C_n \le 2^{n-2}$ for translations on $\mathrm{BC}(\mathbb{R})$. Are these estimates sharp? Is it true that $C_n = 1$ for translations on $\mathrm{BC}(\mathbb{R})$ for every $n? \in \mathbb{N}$, $n \ge 1$? Under which conditions on E and/or $T_1, \cdots, T_n$ does $C_n = 1$ hold?

Example 3.10 It is a classical result that a power bounded operator on a reflexive Banach space E is mean ergodic. As a consequence, commuting power bounded operators on a reflexive Banach space E fulfill the conditions of Proposition 3.6, hence (6) holds true. See also [30, Corollary 2.6]

Definition 3.11 Let E be a Banach space, or, more generally, a topological vector space. We say that E has the *decomposition property with respect to the pairwise commuting operators* $T_1, \dots, T_n \in \mathscr{L}(E)$ if (6) holds. Moreover, if $\mathscr{A} \subseteq \mathscr{L}(E)$ and E has the decomposition property for each system of n pairwise commuting operators $T_1, \dots, T_n \in \mathscr{A}$, then E is said to have the *n-decomposition property with respect to $\mathscr{A}$*. Finally, if this holds for all $n \in \mathbb{N}$, then E is said to have the *decomposition property with respect to $\mathscr{A}$*.

So that e.g. Example 3.10 means that a reflexive Banach space has the decomposition property with respect to (commuting) power bounded operators. This new terminology shall not cause any ambiguity in connection with the decomposition property of function classes $\mathscr{F} \subseteq \mathbb{R}^{\mathbb{R}}$ (in Definition 1.2).

Remark 3.12 If 1 is not an eigenvalue of say T_1, then the questioned equality (6) reduces to $\ker(T_2 - \mathrm{I}) \ldots (T_n - \mathrm{I}) = \ker(T_2 - \mathrm{I}) + \ldots + \ker(T_n - \mathrm{I})$. That is to say the order n reduces to order $n-1$. In particular, if 1 is not a spectral value for every $T_1, \ldots, T_n$, then (6) is satisfied trivially, both sides being $\{0\}$.

Note the following border-line feature of our subject matter. It is only interesting to look at cases when $\|T_1\| \geq 1, \ldots, \|T_n\| \geq 1$ (since $\mathrm{I} - T$ is invertible for $\|T\| < 1$). On the other hand, if $T_1, \ldots, T_n$ are power bounded and commute, we can equivalently renorm E by $|||x||| := \sup_{k_1,\ldots,k_n \in \mathbb{N}} \left\| T_1^{k_1} \ldots T_n^{k_n} x \right\|$, such that for the new norm each operator becomes a contraction. Hence in the end with the assumption $\|T_1\| = \cdots = \|T_n\| = 1$ one loses no generality (for the particularly fixed power bounded operators $T_1, \ldots, T_n$).

Recall that a Banach space E is called *m-quasi-reflexive* if E has codimension m in its bidual E^{**}.

Theorem 3.13 (Kadets and Shumyatskiy [20])

(a) A 1-quasi reflexive Banach space E has the 2-decomposition property with respect to any pair of commuting linear transformations S, T of norm 1.

(b) If E is m-quasi reflexive with $m > 1$, then there exist commuting linear transformations $S, T \in \mathscr{L}(E)$ of norm 1 such that E fails to have the 2-decomposition property with respect to S, T.

Also Kadets and Shumyatskiy proved the following:

Theorem 3.14 (Kadets and Shumyatskiy [19]) *Neither the space c_0 of null sequences, nor ℓ^1 has the 2-decomposition property with respect to operators of norm 1.*

See [19] for the proofs and for further information on averaging techniques which can be used in connection with the periodic decomposition problem. Several natural questions arise, see [20]:

Problem 3.15

1. Is it true that in a 1-quasi reflexive space E has the decomposition property with respect to any finite system of commuting operators of norm 1?
2. Does the 2-decomposition property with respect to contractions imply the n-decomposition property with respect to contractions?
3. Does the 2-decomposition property with respect to power bounded operators characterizes m-quasi reflexive Banach spaces with $m \leq 1$?

Let us finally remark that a recent result of Fonf et al. [12] states that a separable 1-quasi reflexive space can be equivalently renormed such that every contraction with respect to the new norm becomes mean ergodic. Also a classical result of theirs, see [11], is that a Banach space E is reflexive if (and only if) every power bounded operator is mean ergodic. These indicate the possible difficulty of Problem 3.15.

The original setting of the decomposition problem has a special feature, namely that the translation operators T_t on translation invariant subspaces E of $\mathbb{R}^{\mathbb{R}}$ form a one-parameter (semi)group of linear operators. In the rest of this section we shall study this aspect from a more general point of view. Given a Banach space E, a *one-parameter semigroup* T is a unital semigroup homomorphism $T : [0, \infty) \to \mathscr{L}(E)$, i.e., $T(t+s) = T(t)T(s)$ and $T(0) = \mathrm{I}$ are fulfilled for every $t, s \geq 0$. Whereas a *one-parameter group* defined analogously as a group homomorphism (into the group of invertible operators). On $\mathbb{R}^{\mathbb{R}}$ one can define the translation group by $T(t)f(x) = f(t+x)$, which is then, as said above, a one-parameter group.

Problem 3.16 Under which conditions does a Banach space E have the decomposition property with respect to operators $T_1, \ldots, T_n$ coming from a one-parameter (semi)group T as $T_j = T(t_j)$ for some $t_j > 0$, $j = 1, \ldots, n$?

A one-parameter (semi)group is called a C_0*-(semi)group* if it is strongly continuous, i.e., continuous into $\mathscr{L}(E)$ endowed with the strong (i.e., pointwise) operator topology. The translation group is not strongly continuous on the Banach space $\mathrm{B}(\mathbb{R})$ of bounded functions or on $\mathrm{BC}(\mathbb{R})$, but it is strongly continuous on the Banach space $\mathrm{BUC}(\mathbb{R})$ of bounded unformly continuous functions. A one-parameter (semi)group is called bounded if $\|T(t)\| \leq M$ for all $t \in [0, \infty)$ (or $t \in \mathbb{R}$). See [6] for the general theory.

Theorem 3.17 (Kadets and Shumyatskiy [20]) *Let T be a bounded C_0-group, and let $t_1, t_2 > 0$. Then*

$$\ker(T(t_1) - \mathrm{I})(T(t_2) - \mathrm{I}) = \ker(T(t_1) - \mathrm{I}) + \ker(T(t_2) - \mathrm{I}). \tag{7}$$

Translations on $\mathrm{BUC}(\mathbb{R})$ is a C_0-group of isometries, providing another proof of the 2-decomposition property of $\mathrm{BUC}(\mathbb{R})$, formulated in Proposition 2.4.

In general the idea is to find a closed subspace $F \subseteq E$ invariant under the semigroup operators $T(t)$, such that one can apply Proposition 3.6 to the restricted operators. Concerning the nature of the problem there is one immediate candidate for this subspace. In what follows T will be a fixed bounded C_0-semigroup. A vector $x \in E$ is called *asymptotically almost periodic* (with respect to the semigroup T) if the orbit $\{T(t)x : t \geq 0\}$ is relatively compact in E. Denote by $E_{\mathrm{aap}}$the collection of asymptotically almost periodic vectors, which is easily seen to be a closed subspace of E invariant under the semigroup operators. It can be proved that if T is a bounded C_0-group then for $x \in E_{\mathrm{aap}}$ one actually has also the relative compactness of the entire orbit $\{T(t)x : t \in \mathbb{R}\}$. The proof of Theorem 3.17 by Kadets and Shumyatskiy establishes the fact that $\ker(T(t_1) - \mathrm{I})(T(t_2) - \mathrm{I}) \subseteq E_{\mathrm{aap}}$.

The only known extensions/variations of the Kadets–Shumyatskiy result follow the same strategy (or some modifications of it) and are the following:

Theorem 3.18 (Farkas [8]) *Let E be a Banach space and let T be a bounded C_0-group. Suppose that E does not constain an isomorphic copy of the Banach space c_0 of null sequences. Then for every $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$ we have*

$$\ker(T(t_1)-I)\cdots(T(t_n)-I)=\ker(T(t_1)-I)+\cdots+\ker(T(t_n)-I). \quad (8)$$

It is not surprising that Bohl–Bohr–Kadets type theorems (see [1, 18]) play an important role here. In this regard let us mention just the following:

Theorem 3.19 (Basit [1], Farkas [7]) *A separable Banach space E does not contain an isomorphic copy of c_0 if and only if for every $x \in E$, $T \in \mathscr{L}(E)$ invertible with T and T^{-1} both power bounded the following statements are equivalent:*

(i) $\{T^{n+1}x - T^n x : n \in \mathbb{N}\}$ *is relatively compact.*
(ii) $\{T^{n+m}x - T^n x : n \in \mathbb{N}\}$ *is relatively compact for some $m \in \mathbb{N}$, $m \geq 1$.*
(iii) $\{T^{n+m}x - T^n x : n \in \mathbb{N}\}$ *is relatively compact for all $m \in \mathbb{N}$.*
(iv) $\{T^n x : n \in \mathbb{N}\}$ *is relatively compact.*

The next class of C_0-semigroups for which the decomposition problem has positive solution is of those that are *norm-continuous at infinity*, including also *eventually norm-continuous* semigroups, see [31] or [6, Sect. 2.1] for these notions.

Theorem 3.20 (Farkas [8]) *Let T be a bounded C_0-semigroup that is norm-continuous at infinity. Then for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \geq 0$ (8) holds.*

Problem 3.21 1. Is the Kadets–Shumyatskiy theorem true for every n? I.e., can one drop the geometric assumptions on E from Theorem 3.18?
2. What about the case of C_0-semigroups? Can one get rid of the eventual norm-continuity in Theorem 3.20?
3. None of the above covers the decomposition property of $\mathrm{BC}(\mathbb{R})$. What can be said about one-parameter semigroups that are only strongly continuous with respect to some weaker topology on the Banach space E? Can one cover the decomposition property of $\mathrm{BC}(\mathbb{R})$ by some extension of the results for one-parameter semigroups?

4 Application of the Operator Theoretic Results

In this section we present some applications of the results in the foregoing section.

4.1 The decomposition property of BC($\mathbb{R}$)

We devote this subsection to the proof of Theorem 2.7. We slightly differ from the original proof of [29], in exploiting the previous results and in particular Proposition 2.1.

For $n = 1$ the statement is trivial, so we argue by induction. Suppose $f \in \mathrm{BC}(\mathbb{R})$ satisfies (2). We group the periods according to commensurability:

$$\{\alpha_1, \ldots, \alpha_n\} = \{\alpha_1, \ldots, \alpha_a\} \cup \{\beta_1, \ldots, \beta_b\} \cup \cdots \cup \{\rho_1, \ldots, \rho_r\}.$$

Denote the *least common multiple* of these by $\alpha, \beta, \ldots, \rho$, i.e., α is the non-negative generator of the cyclic group $\bigcap_{j=1}^{a} \alpha_j \mathbb{Z}$ etc. Then from (2) we obtain

$$\Delta_\alpha^a \ldots \Delta_\rho^r f = 0. \tag{9}$$

Lemma 4.1 *Let $f \in \mathrm{B}(\mathbb{R})$ (a bounded function) and $\alpha \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$. If $\Delta_\alpha^n f = 0$, then $\Delta_\alpha f = 0$.*

Proof Obviously, it suffices to work out the proof for $n = 2$. Let $g := \Delta_\alpha f$. By condition, $\Delta_\alpha g = 0$, so g is α-periodic. Therefore,

$$f(x + N\alpha) = f(x) + \sum_{i=0}^{N-1} \Delta_\alpha f(x + i\alpha) = f(x) + Ng(x),$$

thus f cannot be bounded if $g(x) \neq 0$. □

As a consequence, from (9) we obtain

$$\Delta_\alpha \ldots \Delta_\rho f = 0. \tag{10}$$

Hence in case $\alpha_1, \ldots, \alpha_n$ are not all pairwise incommensurable then f is also a solution of a difference equation of order less than n. We can therefore apply the induction hypothesis providing that f has an $(\alpha, \ldots, \rho)$-decomposition. So in particular $f \in \mathrm{UAP}(\mathbb{R})$, which space has the decomposition property in view of Proposition 2.1, and so we are done.
It remains to handle the case when $\alpha_1, \ldots, \alpha_n$ are pairwise incommensurable. The crux of the proof is thus the following:

Lemma 4.2 *Let $\alpha_1, \ldots, \alpha_n$ be pairwise incommensurable, and let $f \in \mathrm{BC}(\mathbb{R})$ satisfy (2). Then f has an $(\alpha_1, \ldots, \alpha_n)$-decomposition in $\mathrm{BC}(\mathbb{R})$.*

To prove this lemma it is natural to get rid of one period and reduce the situation to a difference equation of order $n - 1$ by considering $g := \Delta_{\alpha_n} f$, which then satisfies $\Delta_{\alpha_1} \ldots \Delta_{\alpha_{n-1}} g = 0$, and thus by the induction hypothesis, by Remark 3.8(a) and by Example 3.7

$$g = g_1 + \cdots + g_{n-1} \qquad (\Delta_{\alpha_j} g_j = 0,\ j = 1, \ldots, n-1),$$

where $g_j = Q_j g$ for some bounded projection Q_j on $\mathrm{UAP}(\mathbb{R})$. If f were subject to the representation (1), then we could guess $\Delta_{\alpha_n} f_j = g_j$. So we try to "lift up" the g_j to some functions f_j with $\Delta_{\alpha_j} f_j = \Delta_{\alpha_j} g_j = 0$ and $\Delta_{\alpha_n} f_j = g_j$. Suppose this works, we find such $f_j \in \mathrm{BC}(\mathbb{R})$. Then

$$f_n := f - (f_1 + \cdots + f_{n-1}) \in \mathrm{BC}(\mathbb{R}),$$

and $\Delta_{\alpha_n} f_n = g - (g_1 + \cdots + g_{n-1}) = 0$, so f has a decomposition (1). So it remains to show the possibility of a lift-up for any incommensurable periods.

Lemma 4.3 *Let* $g \in \mathrm{C}(\mathbb{R})$, *let* $\beta, \gamma \in \mathbb{R}$ *be incommensurable, and suppose* $\Delta_\beta g = 0$. *Then the following are equivalent:*

(i) There exists $K > 0$ *such that*

$$\left|\sum_{i=0}^{k-1} g(x + i\gamma)\right| < K \quad (\textit{for } x \in \mathbb{R}, k \in \mathbb{N}).$$

(ii) There is $h \in \mathrm{C}(\mathbb{R})$ *such that* $\Delta_\beta h = 0$ *and* $\Delta_\gamma h = g$.

Proof This is a special case of a well-known ergodic theory result, see [14, Theorem 14.11, p.135], as putting $Y := \mathbb{R}/\gamma\mathbb{Z}$, the homeomorphism $\Theta(x) := x + \beta \mod \gamma$ has minimal orbit-closure Y for every x. □

To complete the proof of Theorem 2.7 we need to check that condition (i) in the preceding lemma is fulfilled. For $j \in \{1, \ldots, n-1\}$ the projection Q_j commutes with translations so that

$$\left|\sum_{i=0}^{k-1} g_j(x + i\alpha_n)\right| = \left|\sum_{i=0}^{k-1} (Q_j g)(x + i\alpha_n)\right|$$
$$= \left|Q_j \sum_{i=0}^{k-1} g(x + i\alpha_n)\right| = \left|Q_j\big(f(x + k\alpha_n) - f(x)\big)\right| \le 2\|Q_j\| \cdot \|f\|_\infty$$

for every $x \in \mathbb{R}$, $k \in \mathbb{N}$. The proof is hence complete.

4.2 Applications to L^p spaces

Let (X, Σ, μ) be a measure space. In this subsection our standing assumption is as follows:

Condition 4.4 *For* $j = 1, \ldots, n$ *let* $T_j : X \to X$ *be pairwise commuting measurable mappings such that* $\mu(T_j^{-1}(A)) \le \mu(A)$ *for every* $A \in \Sigma$.

Then the Koopman operators, denoted by the same letter and defined by

$$T_j f := f \circ T_j$$

are contractions on all of the spaces $\mathrm{L}^p(X, \Sigma, \mu)$. In particular the condition above is fulfilled if the T_js are measure-preserving, in which case the Koopman operators T_j become isometries on each of the L^p spaces.

For the reflexive range the next corollary of Proposition 3.6 is immediate:

Corollary 4.5 *Let* $1 < p < \infty$. *Under Condition 4.4 consider the Koopman operators* T_j *on* $\mathrm{L}^p(X, \Sigma, \mu)$. *Then* (6) *holds true.*

The same result is true for the case $p = 1$, but the proof is different since infinite dimensional L^1 spaces are non-reflexive. We remark however that if (X, Σ, μ) is finite, then the Koopman operators T_j are simultaneous L^1 and L^∞ contractions, so-called Dunford–Schwartz operators, that are known to be mean ergodic on L^1, see, e.g., [5, Sect. 8.4].

Proposition 4.6 (Laczkovich and Révész [30]) *Under Condition 4.4 consider the Koopman operators* T_j *on* $\mathrm{L}^1(X, \Sigma, \mu)$. *Then* (6) *holds true.*

We do not give the proof here, but note that the mean ergodicity of the operators can be replaced by an application of Birkhoff's pointwise ergodic theorem, see, e.g. [5, Chap. 11]. See [30] for the detailed proof.

The case of $p = \infty$ is more subtle. Let us recall the following notion.

Definition 4.7 A measure space (X, Σ, μ) is called *localizable* if the dual of the Banach space $\mathrm{L}^1(X, \Sigma, \mu)$ is $\mathrm{L}^\infty(X, \Sigma, \mu)$ (with the usual identification).

As a matter of fact, the original definition of Segal (see [36, Sect. 5]) was different, but is equivalent to the one above. Known examples of localizable measure spaces include:

Example 4.8

1. σ-finite measure spaces,
2. (X, Σ, μ) with X a set $\Sigma = \mathscr{P}(X)$ the power set, μ the counting measure,
3. (X, Σ, μ) purely atomic,
4. (X, Σ, μ), X a locally compact group, Σ the Baire algebra, μ a (left/right) Haar measure.

Hence, in all of these cases the results below apply. In particular if one considers commuting left- (or right) translations on some locally compact group G, then the respective Koopman operators will satisfy (6). Note that the left and the right Haar measures are absolutely continuous with respect to each other, so we can fix each and any of them for our considerations below.

Theorem 4.9 (Laczkovich and Révész [30]) *Let* (X, Σ, μ) *be a localizable measure space, and suppose that for the pairwise commuting measurable mappings* $T_j : X \to X$ $(j = 1, \ldots, n)$ *the push-forward measures* $\mu \circ T_j^{-1}$ *are all absolutely continuous with respect to* μ. *Then for the Koopman operators* T_j *on* $\mathrm{L}^\infty(X, \Sigma, \mu)$ (6) *holds true.*

The proof relies on the fact that under the conditions of localizability of (X, Σ, μ) and absolute continuity of the push-forward measures, the operators T_j will be weak* continuous on $\mathrm{L}^\infty(X, \Sigma, \mu)$ hence one can apply Proposition 3.5. For the details see [30].

Problem 4.10 Can one drop the localizability assumption?

Corollary 4.11 (Gajda [13], Laczkovich and Révész [30]) *The space* $\mathrm{B}(X)$ *of bounded functions on a set* X *has the decomposition property with respect to any system of commuting Koopman operators.*

This follows from Theorem 4.9 and from Example 4.8(2) above. The proof of Gajda uses Banach limits, see also Sect. 7. To sum up we have:

Corollary 4.12 *The Banach spaces* $\mathrm{L}^p(\mathbb{R})$ *(*$1 \le p \le \infty$*, Lebesgue measure) have the decomposition property.*

Of course, 0 is the only periodic function in $\mathrm{L}^p(\mathbb{R})$ if $p < \infty$, hence the message of the previous result is that (2) has 0 as the only L^p-solution if $p < \infty$. This follows also from a more general result of Edgar and Rosenblatt [4, Corollary 2.7] stating that the translates of a function $0 \ne f \in \mathrm{L}^p(\mathbb{R}^d)$, $p < 2d/(d-1)$ are linearly independent.

4.3 More Spaces with the Decomposition Property

Proposition 4.13 (Laczkovich and Révész [30]) *The following spaces of real-valued functions on* $\mathbb{R}$ *have the decomposition property:*

(a) $\mathrm{BV}^1(\mathbb{R}) := \{f : f \in \mathrm{B}(\mathbb{R})$ *with unif. bdd. variation on* $[x, x+1], x \in \mathbb{R}\}$
(b) $\mathrm{Lip}_b(\mathbb{R}) := \{f : f$ *is bounded and Lipschitz continuous*$\}$
(c) $\mathrm{Lip}_b^k(\mathbb{R}) := \{f : f \in \mathrm{BC}(\mathbb{R})$ *k times differentiable with* $f^{(k)}$ *Lipschitz* $\}$

The cases (a) and (b) can be handled by introducing an appropriate norm turning the spaces under consideration into Banach spaces, then by noting that the unit ball is compact for the pointwise topology. Hence Theorem 3.3 is applicable. Details are in [30]. Part (c) relies on the following result interesting in its own right:

Proposition 4.14 (Laczkovich and Révész [30]) *Let* $\mathcal{F} \subseteq \mathrm{C}(\mathbb{R})$ *be a function class with the property that whenever* $f \in \mathcal{F}$ *and* $c \in \mathbb{R}$ *then* $f + c \in \mathcal{F}$*. Let* $k \in \mathbb{N}$ *and define*

$$\mathcal{G} := \{f : f \in \mathrm{BC}(\mathbb{R}) \text{ is } k \text{ times differentiable with } f^{(k)} \in \mathcal{F}\}.$$

If the function class $\mathcal{F}$ *has the decomposition property so does* $\mathcal{G}$*.*

Problem 4.15 There are several interesting Banach function spaces. Which of them do have the decomposition property? Just take your favorite non-reflexive translation invariant Banach function space on $\mathbb{R}$. Does it have the decomposition property? Denote by $\mathrm{L}^1_p(\mathbb{R})$ the set of functions with

$$\|f\|_{1,p} := \sup_{x \in \mathbb{R}} \left(\int_x^{1+x} |f(t)|^p \,\mathrm{d}t \right)^{1/p} < \infty,$$

and by $S^p(\mathbb{R})$ the closure of trigonometric polynomials in this norm. The elements of $S^p(\mathbb{R})$ are called *Stepanov almost periodic functions*, see [2]. Does the Banach space $L^1_p(\mathbb{R})$ have the decomposition property? If the answer were affirmative it would follow that $f \in L^1_p(\mathbb{R})$ and (2) imply that $f \in S^p(\mathbb{R})$. (This is because periodic functions belong to $S^p(\mathbb{R})$.) So, is an $L^1_p(\mathbb{R})$ solution of (2) Stepanov almost periodic?

5 Results for Arbitrary Transformations

Treating the periodic decomposition problem for various classes $\mathcal{F}$ of real functions a natural approach would be to split the question into two. That is first looking for a periodic decomposition into arbitrary periodic functions with the given periods, and then investigating whether the existence of such arbitrary decomposition entails the existence of a decomposition within the function class $\mathcal{F}$. In this section we address the first question, which can be actually done in a far more general setup. We consider this problem interesting in its own right, even if we already know that for some important function classes (e.g., for $C(\mathbb{R})$, see the paragraph after Proposition 1.3) the answer to the second question is in the negative.

Let X be a non-empty set. The decomposition problem can be formulated in *the whole space of functions* $\mathbb{R}^X$ with respect to *arbitrary commuting transformations* in X^X. To do that to a self map $T : X \to X$, called *transformation*, we associate the Koopman operator (denoted by the same letter) $Tf := f \circ T$, and the *T-difference operator* $\Delta_T f := Tf - f$. A function f satisfying $\Delta_T f = 0$ is then called *T-invariant*. A $(T_1, \dots, T_n)$*-invariant decomposition* of some function f is a representation

$$f = f_1 + \cdots + f_n\,, \qquad \text{where} \qquad \Delta_{T_j} f_j = 0 \quad (j = 1, \dots, n). \tag{11}$$

For pairwise commuting transformations T_i the functional equation

$$\Delta_{T_1} \dots \Delta_{T_n} f = 0 \tag{12}$$

is evidently necessary for the existence of invariant decompositions. On the example of translations on $\mathbb{R}$ we saw that it is not sufficient. Now in this general setting our basic question sounds:

Problem 5.1 Give necessary and sufficient conditions, containing (12), in order to have some $(T_1, \dots, T_n)$-invariant decomposition (11). Or give restrictions either on the transformations or on X (but not on the function class $\mathbb{R}^X$) such that (12) becomes also sufficient.

More precisely, we focus on complementary conditions, functional equations, on the functions, which they must satisfy in case of existence of an invariant decomposition (11) and which equations will also imply existence of such a decomposition.

Difference equations (of higher order) and/or inequalities occur here naturally, as is also suggested by the appearance of the Whitney condition in Theorem 1.6.

Further necessary conditions can be easily obtained. Indeed, as the transformations commute, (12) implies

$$\Delta_{T_1^{k_1}} \dots \Delta_{T_n^{k_n}} f = 0 \qquad (\forall k_1, \dots, k_n \in \mathbb{N}). \tag{13}$$

Now the major difficulties come from the following features:

1. The transformations T_j may not be invertible.
2. The "mix-up" of transformations can be completely irregular: $T^5 S^3 x = T^7 S^2 x$ for some $x \in X$ and nothing similar for other points $y \in X$.
3. Functions on X lack any structure beyond the obvious linear one—no boundedness, continuity, measurability, compatibility with underlying structure of X, nothing—so not much theoretical mathematics but pure combinatorics can be invoked.

For two transformations, i.e., $n = 2$, the answer is completely known:

Theorem 5.2 (Farkas and Révész [10]) *Let X be a non-empty set, let $S, T : X \to X$ be commuting transformations, and let $f \in \mathbb{R}^X$. The following are equivalent:*

(i) *There exists a decomposition $f = g + h$, with g and h being S- and T-invariant, respectively.*

(ii) $\Delta_S \Delta_T f = 0$, *and if for some $x \in X$ and $k, n, k', n' \in \mathbb{N}$ the equality*

$$T^k S^n x = T^{k'} S^{n'} x \tag{14}$$

holds, then

$$f(T^k x) = f(T^{k'} x).$$

(iii) $\Delta_S \Delta_T f = 0$, *and if for some $x \in X$ and $k, n, k', n' \in \mathbb{N}$ (14) holds, then*

$$f(S^n x) = f(S^{n'} x).$$

Of course, the equivalence of (ii) and (iii) is due to symmetry, if one knows that any one of them is equivalent to (i). We do not give the proof (see [10]), but mention an idea that will be useful also below. First we partition the set X with respect to an *equivalence relation*: $x, y \in X$ are equivalent if there exist $k, n, k', n' \in \mathbb{N}$ such that $T^k S^n x = T^{k'} S^{n'} y$. X splits into equivalence classes $X/\sim$, from which *by the axiom of choice* we choose a representation system. Obviously, it is enough to define g and h on each of these equivalence classes. Indeed, for $x \in X$ the elements x, Tx and Sx are all equivalent, so the invariance of the desired functions g, h is decided already in the common equivalence class. So the task is now reduced to defining the functions g and h on a fixed, but arbitrary equivalence class.

For general $n \in \mathbb{N}, n \geq 2$ the following difference equation type necessary conditions can be found:

Condition (*) *For every* $N \leq n$, *disjoint* N *-term partition* $B_1 \cup B_2 \cup \cdots \cup B_N = \{1, 2, \ldots, n\}$, *distinguished elements* $h_j \in B_j$ $(j = 1, \ldots, N)$, *indices* $0 < k_j, l_j, l'_j \in \mathbb{N}$, $(j = 1, \ldots, N)$ *and* $z \in X$ *once the conditions*

$$T_{h_j}^{k_j} T_i^{l_i} z = T_i^{l'_i} z \qquad \text{for all } i \in B_j \setminus \{h_j\}, \text{for all } j = 1, \ldots, N \tag{15}$$

are satisfied, then

$$\Delta_{T_{h_1}^{k_1}} \ldots \Delta_{T_{h_N}^{k_N}} f(z) = 0. \tag{16}$$

Theorem 5.3 (Farkas and Révész [10]) *Let* $T_1, \ldots, T_n$ *be commuting transformations of* X *and let* f *be a real function on* X. *In order to have a* $(T_1, \ldots, T_n)$*-invariant decomposition* (11) *of* f *Condition* $(*)$ *is necessary.*

If the blocks B_j are all singletons the condition (15) is empty, so (16) expresses exactly (13). In particular, Condition $(*)$ contains (12).
For $n = 3$ transformations Condition $(*)$ is not only necessary but also sufficient for the existence of invariant decompositions.

Theorem 5.4 (Farkas and Révész [10]) *Suppose that* T_1, T_2 *and* T_3 *commute and that the function* f *satisfies Condition* $(*)$. *Then* f *has a* (T_1, T_2, T_3)*-invariant decomposition.*

Again the proof is combinatorially involved, so let us just state one main ingredient, the "lift-up lemma" corresponding to Lemma 4.3 above. It is proved itself in a series of lemmas, which we do not detail here.

Lemma 5.5 *Let* T, S *be commuting transformations of* X *and let* $g : X \to \mathbb{R}$ *be a function satisfying* $\Delta_S g = 0$. *Then there exists a function* $h : X \to \mathbb{R}$ *satisfying both* $\Delta_S h = 0$ *and* $\Delta_T h = g$ *if and only if for every* $x \in X$ *it holds*

$$\sum_{i=0}^{k-1} g(T^i x) = 0 \quad \textit{whenever} \quad T^k S^l x = S^{l'} x \textit{ with some } k, l, l' \in \mathbb{N}. \tag{17}$$

Problem 5.6 Is Condition $(*)$ equivalent to (11) for all $n \in \mathbb{N}$ $(n \geq 4)$?

5.1 Unrelated Transformations

If the orbits of the transformations show no recurrence then a satisfactory answer can be given. The relevant notion is the following.

Definition 5.7 We call two commuting transformations S, T on X *unrelated* if $T^n S^k x = T^m S^l x$ can occur only if $n = m$ and $k = l$. In particular, then neither of the two transformations can have any cycles in their orbits, nor do their joint orbits have any recurrence.

If all pairs T_i and T_j $(1 \le i \ne j \le n)$ are unrelated, then Condition $(*)$ degenerates, as in (15) we necessarily have that all blocks B_j are singletons. Hence Condition $(*)$ reduces merely to (13) or, equivalently, to (12).

Theorem 5.8 (Farkas and Révész [10]) *Suppose the transformations $T_1, \ldots, T_n$ are pairwise commuting and unrelated. Then the difference equation* (12) *is equivalent to the existence of some invariant decomposition* (11).

Proof Only sufficiency is to be proved. We argue by induction. The cases of small n are obvious. Let $F := \Delta_{T_{n+1}} f$. Then F satisfies a difference equation of order n, hence by the inductive hypothesis we can find an invariant decomposition of F in the form $F = F_1 + \cdots + F_n$, where $\Delta_{T_j} F_j = 0$ for $j = 1, \ldots, n$. Since the transformation are unrelated, condition (17) in Lemma 5.5 is void, and therefore the "lift-ups" f_j with $\Delta_{T_j} f_j = 0$, $\Delta_{T_{n+1}} f_j = F_j$ exist for all $j = 1, \ldots, n$. Therefore, $f_{n+1} := f - (f_1 + \cdots + f_n)$ provides a function satisfying $\Delta_{T_{n+1}} f_{n+1} = F - (F_1 + \cdots + F_n) = 0$. Thus a required decomposition of f is established. □

5.2 Invertible Transformations

When the transformations T_j are invertible, the situation simplifies somewhat. Denote by $G \subseteq X^X$ the group generated by $T_1, \ldots, T_n$. As before, we work on equivalence classes, now *orbits* $O := \{Tx : T \in G\}$ for some $x \in X$, under the action of the transformation group G. Given a group G denote by $\langle a \rangle$ the cyclic group generated by a i.e., $\langle a \rangle := \{a^n : n \in \mathbb{Z}\}$, and for $H \subseteq G$ let $[H] := \bigcap_{h \in H} \langle h \rangle$.

Condition(**) For all orbits O of G, for all partitions

$$B_1 \cup B_2 \cup \cdots \cup B_N = \left\{T_1 \mid_O, T_2 \mid_O, \ldots, T_n \mid_O\right\}$$

and any element $S_j \in [B_j]$, $j = 1, \ldots, N$, we have that

$$\Delta_{S_1} \ldots \Delta_{S_N} f \mid_O = 0 \quad \text{holds.} \tag{18}$$

The next is the main result in this setting:

Theorem 5.9 (Farkas et al. [9]) *Let $T_1, \ldots, T_n$ be pairwise commuting invertible transformations on a set X. Let $f : X \to \mathbb{R}$ be any function. Then f has a $(T_1, T_2, \ldots, T_n)$-invariant decomposition* (11) *if and only if it satisfies Condition (**).*

The proof relies on a variant of Lemma 5.5.

6 Decompositions on Groups

Let us see some consequences of the result in the foregoing section. Let G be a group, and let $a_1, \dots, a_n \in G$. Consider the actions of $a_1, \dots, a_n$ on G as left multiplications. For a function $f : G \to \mathbb{R}$ we introduce the *left a-difference operator* $\Delta_a f(x) := f(ax) - f(x)$. The function f is called *left a-invariant* (or left a-periodic) if $\Delta_a f = 0$. Since the actions are transitive we get:

Corollary 6.1 *Let G be a group and $a_1, \dots, a_n \in G$ pairwise commuting. Then a function $f : G \to \mathbb{R}$ decomposes into a sum of left a_j-invariant functions, $f = f_1 + \dots + f_n$, if and only if for all partitions $B_1 \cup B_2 \cup \dots \cup B_N = \{a_1, \dots, a_n\}$ and for each element $b_j \in [B_j]$*

$$\Delta_{b_1} \dots \Delta_{b_N} f = 0.$$

In a torsion free Abelian group A for $B \subseteq A$ the generator of the cyclic group $[B]$ is uniquely determined (up to taking inverse). In [10] we called this (maybe two) element(s) the *least common multiple* of the elements in B. For instance, with this terminology we have that the least common multiple of 1 and $\sqrt{2}$ in the group $(\mathbb{R}, +)$ is 0. Then we have the next result:

Corollary 6.2 *Let A be a torsion free Abelian group and $a_1, \dots, a_n \in A$. A function $f : A \to \mathbb{R}$ decomposes into a sum of a_j-periodic functions, $f = f_1 + \dots + f_n$, if and only if for all partitions $B_1 \cup B_2 \cup \dots \cup B_N = \{a_1, \dots, a_n\}$ and b_j being the least common multiple of the elements in B_j one has*

$$\Delta_{b_1} \dots \Delta_{b_N} f = 0. \tag{19}$$

If we take $A = \mathbb{R}$ and $\alpha_1, \dots, \alpha_n$ incommensurable we obtain the following.

Corollary 6.3 (Mortola and Peirone [32], Farkas and Révész [10]) *Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be incommensurable. Then a function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) if and only if it has periodic decomposition (1).*

The above results remain true if one considers functions with values in torsion free groups Γ. The proof of the following is the same as for Theorem 5.9 with the new aspect that taking averages in Γ requires some additional care.

Theorem 6.4 (Farkas et al. [9]) *Let A, Γ be torsion free Abelian groups and $a_1, \dots, a_n \in A$. A function $f : A \to \Gamma$ decomposes into a sum of a_j-periodic functions $f_j : A \to \Gamma$, $f = f_1 + \dots + f_n$ if and only if for all partitions $B_1 \cup B_2 \cup \dots \cup B_N = \{a_1, \dots, a_n\}$ and b_j being the least common multiple of the elements in B_j one has* (19).

Let A be a torsion free Abelian group. By the previous theorem for $\Gamma = \mathbb{R}$ and for $\Gamma = \mathbb{Z}$, we obtain that for a function $f : A \to \mathbb{Z}$ the existence of a real-valued periodic decomposition and the existence of an integer-valued periodic decomposition are both equivalent to the same condition.

Corollary 6.5 *If an integer-valued function f on a torsion free Abelian group A decomposes into the sum of a_j-periodic real-valued functions for some $a_1, \dots, a_n$, then f decomposes into the sum of a_j-periodic integer-valued ones.*

There are examples showing that one cannot get rid of the torsion freeness of A in Corollary 6.5 or Theorem 6.4, see [9].

Note that in crystallography and other applications, reconstruction or at least unique identification of integer-valued functions or characteristic functions of sets from various (partial) information concerning their Fourier transform are rather important. This also motivates the interest in integer-valued periodic decompositions or decompositions with values within a subgroup. In turn, support of a Fourier transform can reveal the existence of a periodic decomposition, see e.g. [27, 2.7 and 2.8], or the analogous idea of the proof for Proposition 2.1. For more about this see [27] and the references therein.

7 Actions of Semigroups

Let X be a non-empty set and let $T : X \to X$ be an arbitrary mapping. If a function $f : X \to \mathbb{R}$ is invariant under T, i.e., $\Delta_T f = 0$, then it is evidently invariant under each iterate T^n of T for $n \in \mathbb{N}$. Given commuting mappings $T_1, \dots, T_n : X \to X$ consider the generated semigroups

$$S_j := \left\{T_j^n : n \in \mathbb{N}\right\}. \tag{20}$$

The corresponding semigroup of the Koopman operators on $\mathbb{R}^X$ is denoted by $\mathscr{S}_j$. (Recall that we use the same symbol T for the Koopman operator of $T \in X^X$.) For a subset $\mathscr{A}$ of linear operators on $\mathbb{R}^X$ we introduce the notations $\ker \mathscr{A} := \bigcap_{A\in\mathscr{A}} \ker A$. Then the equality

$$\ker(T_1 - \mathrm{I}) \cdots (T_n - \mathrm{I}) = \ker(T_1 - \mathrm{I}) + \cdots + \ker(T_n - \mathrm{I}) \tag{21}$$

is easily seen to be equivalent to

$$\ker(\mathscr{S}_1 - \mathrm{I}) \cdots (\mathscr{S}_n - \mathrm{I}) = \ker(\mathscr{S}_1 - \mathrm{I}) + \cdots + \ker(\mathscr{S}_n - \mathrm{I}). \tag{22}$$

In what follows we study this equality when $\mathscr{S}_j$ are general, not necessarily cyclic, semigroups.

Let S be a discrete semigroup with unit element, and let S_j, $j = 1, \dots, n$ unital subsemigroups of S that all act on the non-empty set X (from the left), the unit acting as the identity. Suppose furthermore $st = ts$ for all $s \in S_j$ and $t \in S_i$ with $i \neq j$ (the actions of different S_js are commuting).

Theorem 7.1 (Farkas [8]) *Suppose that for* $j = 1, \dots, n$ *the unital semigroups* S_j *on the set* X *are (right-)amenable and that the actions of the different* S_j *are commuting. Denote by* $\mathscr{S}_j$ *the semigroups of the Koopman operators. Then* (22) *holds in the space* $\mathrm{B}(X)$. *Furthermore, if* X *is uniform (topological) space and the action of* S_j *on* X *is uniformly equicontinuous, then* (22) *holds in the space* $\mathrm{BUC}(X)$.

This result and its proof generalizes those of Gajda [13], who used Banach limits (i.e., amenability of $\mathbb{Z}$ or $\mathbb{N}$) to establish the above for $\mathbb{Z}$ and $\mathbb{N}$ actions, i.e., for semigroups as in (20). The next consequence immediately follows.

Corollary 7.2 (Gajda [13]) *Let* A *be a locally compact Abelian group, and let* $a_1, \dots, a_n \in A$. *Then* (21) *holds in* $\mathrm{BUC}(\mathbb{R})$ *for* T_j *being the shift operator by* a_j. *In particular* $\mathrm{BUC}(\mathbb{R})$ *has the decomposition property.*

Let us finally return to the purely linear operator setting on an arbitrary Banach space E. A subsemigroup $\mathscr{S} \subseteq \mathscr{L}(E)$ of bounded linear operators is called *mean ergodic* if the closed convex hull $\overline{\mathrm{conv}}(\mathscr{S}) \subseteq \mathscr{L}(E)$ contains a zero element P, i.e., $PT = P = TP$ for every $T \in \mathscr{S}$. In this case P is a projection, called the *mean ergodic projection* of $\mathscr{S}$, and it holds (see [33])

$$E = \mathrm{rg}\, P \oplus \mathrm{rg}(\mathrm{I} - P) \quad \text{with} \quad \mathrm{rg}\, P = \ker(\mathscr{S} - \mathrm{I})\ .$$

Theorem 7.3 (Farkas [8]) *Let* $\mathscr{S}_1, \mathscr{S}_2, \dots, \mathscr{S}_n \subseteq \mathscr{L}(E)$ *be mean ergodic operator semigroups and suppose that* $ST = TS$ *whenever* $T \in \mathscr{S}_i$, $S \in \mathscr{S}_j$ *with* $i \neq j$. *Then* (22) *holds.*

Since an operator T is mean ergodic if and only if the semigroup $\{T^n : n \in \mathbb{N}\}$ is mean ergodic, the previous result contains Proposition 3.6. Moreover, the obvious modification of Theorem 3.3 (using fixed points in the closed convex hull) for this semigroup setting is easily proved, but this we will not pursue here. Furthermore, the analogue of Corollary 3.4 can be formulated for amenable semigroups instead of cyclic ones, where of course one applies Day's fixed point theorem, see [3], instead of the one of Markov and Kakutani.

Problem 7.4 Does the space $\mathrm{BC}(A)$ of bounded and continuous functions, where A is a locally compact Abelian group, has the decomposition property with respect to translations? If A is compact or discrete or $A = \mathbb{R}$, this is so by the previous results. What about $A = \mathbb{R}^2$?

8 Further Results

We briefly touch upon topics that, regrettably, could not be covered in detail.

First we take a second glimpse at the original problem.

Theorem 8.1 (Natkaniec and Wilczyński [34]) *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ be incommensurable. A function $f : \mathbb{R} \to \mathbb{R}$ has a decomposition* (1) *with $f_1, \ldots, f_n$ Darboux functions if and only if* (2) *holds.*

See [34] for the proof where also the decomposition property of Marczewski measurable functions is studied for incommensurable periods. It is also shown that the identity is not the sum of periodic functions having the Baire property. For classes of measurable functions we have, e.g., the following.

Theorem 8.2 (Keleti [26]) *None of the following classes $\mathcal{F}$ have the decomposition property:*

(a) $\mathcal{F} = \{f : f : \mathbb{R} \to \mathbb{Z},\ f \in \mathrm{L}^\infty(\mathbb{R})\}$,
(b) $\mathcal{F} = \{f : f : \mathbb{R} \to \mathbb{Z}$ *is bounded measurable*$\}$,
(c) $\mathcal{F} = \{f : f : \mathbb{R} \to \mathbb{R}$ *is a.e. integer-valued and* $f \in \mathrm{L}^\infty(\mathbb{R})\}$,
(d) $\mathcal{F} = \{f : f : \mathbb{R} \to \mathbb{Z}$ *is a.e. integer-valued, bounded and measurable*$\}$.

For more information on measurable decompositions see also [23–25]. Next we turn to integer-valued decompositions on Abelian groups. We mention only three exemplary results from [22]:

Theorem 8.3 (Károlyi et al. [22])

(a) Suppose $f : \mathbb{Z} \to \mathbb{Z}$ has an $(\alpha_1, \ldots, \alpha_n)$-periodic decomposition into real-valued functions with $a_j \in \mathbb{Z}$. Then it has an $(\alpha_1, \ldots, \alpha_n)$-periodic integer-valued decomposition.
(b) For $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ the class of $\mathbb{Z} \to \mathbb{Z}$ functions has the decomposition property.
(c) Let A be a torsion-free Abelian group. Then the class of bounded $A \to \mathbb{Z}$ functions has the decomposition property if and only if A is isomorphic to an additive subgroup of $\mathbb{Q}$.

For a proof and for an abundance of further information we refer to [22], and remark that part c) above implies that the class of bounded and integer-valued functions does not have the decomposition property known also from Theorem 8.2, see also [22, Corollary 3.4].
Finally, we discuss some aspects of uniqueness of decompositions. Of course, one cannot expect uniqueness in the original setting, since appropriate constant functions can be added to the summands in (1) not affecting the validity of (2). If one restricts to certain function classes then only this trivial procedure can produce different decompositions (for incommensurable periods).

Theorem 8.4 (Laczkovich and Révész [30]) *For incommensurable periods a periodic decomposition in $\mathrm{L}^\infty(\mathbb{R})$ of a function $f \in \mathrm{L}^\infty(\mathbb{R})$ is unique up to additive constants.*

In the original setting of the decomposition problem, i.e., in $\mathbb{R}^\mathbb{R}$ the situation is somewhat more complicated. E.g. consider $n = 2$, $f = f_1 + f_2$ with $\Delta_{a_j} f_j = 0$,

$j = 1, 2$. Let h be a not identically 0 function that is both α_1- and α_2-periodic. Then $f = (f_1 + h) + (f_2 - h)$ is a different decomposition.

In general two decompositions $f = g_1 + \cdots + g_n$ and $f = f_1 + \cdots + f_n$ with $\Delta_{\alpha_j} g_j = \Delta_{\alpha_j} f_j = 0$ $j = 1, \ldots, n$ are called *essentially the same* if there are functions $h_{ij} \in \mathbb{R}^{\mathbb{R}}$ for $i, j = 1, \ldots, n$ with $h_{ii} = 0$, $h_{ij} = -h_{ji}$, $\Delta_{\alpha_i} h_{ij} = 0$, $\Delta_{\alpha_j} h_{ij} = 0$ such that for all $j = 1, \ldots, n$ one has $f_j - g_j = \sum_{i=1}^{n} h_{ij}$. Note that for incommensurable periods $\alpha_i/\alpha_j \notin \mathbb{Q}$ we necessarily have h_{ij} =constant on each equivalence class of $\mathbb{R}$ (for the equivalence relation as in the paragraph after Theorem 5.2), whence in presence of continuity on the whole real line.

Essential uniqueness of decomposition depends very much on the periods:

Theorem 8.5 (Harangi [17]) *For $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ the following assertions are equivalent:*

(i) If any three numbers $\alpha_i, \alpha_j, \alpha_k$ from $\alpha_1, \ldots, \alpha_n$ are pairwise linearly independent over $\mathbb{Q}$, then they are linearly independent over $\mathbb{Q}$.
(ii) Any two $(\alpha_1, \ldots, \alpha_n)$-periodic decomposition of a function f are essentially the same, i.e., the decomposition is essentially unique.
(iii) If a function $f : \mathbb{R} \to \mathbb{Z}$ has an $(\alpha_1, \ldots, \alpha_n)$-periodic decomposition into bounded real-valued functions, then it has also one into bounded integer-valued functions.

See also [15], [17] or [16] for details and further directions.

We end this survey by posing the following problem:

Problem 8.6 Study the periodic decomposition problem for functions f on $\mathbb{R}$, or on an Abelian group, with values in $\mathbb{R}$ mod 1 (or in an Abelian group).

References

1. Bazit, R.B.: Generalization of two theorems of M.I. Kadets concerning the indefinite integral of abstract almost periodic functions. Matematicheskie Zametki **9**(3), 311–321 (1971)
2. Besicovitch, A.S.: Almost Periodic Functions. Dover Publications Inc., New York (1955)
3. Day, M.M.: Fixed-point theorems for compact convex sets. Ill. J. Math. **5**, 585–590 (1961)
4. Edgar, G.A., Rosenblatt, J.M.: Difference equations over locally compact abelian groups. Trans. Amer. Math. Soc. **253**, 273–289 (1979)
5. Eisner, T., Farkas, B., Haase, M., Nagel, R.: Operator Theoretic Aspects of Ergodic Theory. Graduate Text in Mathematics. Springer, Berlin (2013, In Press)
6. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, vol. 194. Springer, New York (2000)
7. Farkas, B.: A Bohl-Bohr-Kadets type theorem characterizing Banach spaces not containing c_0 (2013). (Submitted); arxiv:1301.6250
8. Farkas, B.: A note on the periodic decomposition problem for semigroups (2014). (Submitted); arxiv:1401.1226
9. Farkas, B., Harangi, V., Keleti, T.: Révész, Sz.: Invariant decomposition of functions with respect to commuting invertible transformations. Proc. Amer. Math. Soc. **136**(4), 1325–1336 (2008)

10. Farkas, B.: Révész, Sz.: Decomposition as the sum of invariant functions with respect to commuting transformations. Aequationes Math. **73**(3), 233–248 (2007)
11. Fonf, V.P., Lin, M., Wojtaszczyk, P.: Ergodic characterizations of reflexivity of banach spaces. J. Funct. Anal. **187**(1), 146–162 (2001)
12. Fonf, V.P., Lin, M., Wojtaszczyk, P.: A non-reflexive banach space with all contractions mean ergodic. Isr. J. Math. **179**, 479–491 (2010)
13. Gajda, Z.: Note on decomposition of bounded functions into the sum of periodic terms. Acta Math. Hungar. **59**(1–2), 103–106 (1992)
14. Gottschalk, W.H., Hedlund, G.A.: Topological Dynamics. Amer. Math. Soc. Colloquium Publications. American Mathematical Society, Providence, R.I. (1955)
15. Harangi, V.: Periodic decompositions of functions. Real Anal. Exchange (31st Summer Symposium Conference), 185–188 (2007)
16. Harangi, V.: Sums of periodic functions. Master's thesis, Eötvös Loránd Univeristy, Budapest (2007). (In Hungarian)
17. Harangi, V.: On the uniqueness of periodic decomposition. Fund. Math. **211**(3), 225–244 (2011)
18. Kadec, M.I.: The integration of almost periodic functions with values in a Banach space. Funkcional. Anal. i Priložen. **3**(3), 71–74 (1969)
19. Kadets, V.M., Shumyatskiy, B.M.: Averaging technique in the periodic decomposition problem. Mat. Fiz. Anal. Geom. **7**(2), 184–195 (2000)
20. Kadets, V.M., Shumyatskiy, B.M.: Additions to the periodic decomposition theorem. Acta Math. Hungar. **90**(4), 293–305 (2001)
21. Kahane, J.P.: Lectures on Mean Periodic Functions. Tata Institute, Bombay (1959)
22. Károlyi, G., Keleti, T., Kós, G., Ruzsa, I.Z.: Periodic decomposition of integer valued functions. Acta Math. Hungar. **119**(3), 227–242 (2008)
23. Keleti, T.: Difference functions of periodic measurable functions. Ph.D. thesis
24. Keleti, T.: On the differences and sums of periodic measurable functions. Acta Math. Hungar. **75**(4), 279–286 (1997)
25. Keleti, T.: Periodic decomposition of measurable integer valued functions. Real Anal. Exchange (31st Summer Symposium Conference), 189–193 (2007)
26. Keleti, T.: Periodic decomposition of measurable integer valued functions. J. Math. Anal. Appl. **337**(2), 1394–1403 (2008)
27. Keleti, T. and Kolountzakis, M.: On the determination of sets by their triple correlation in finite cyclic groups. Online J. Anal. Comb. 1, 4, 18 pp. (electronic) (2006)
28. Laczkovich, M., Révész, Sz: Periodic decompositions of functions. Real Anal. Exch. **13**(1), 126–128 (1987)
29. Laczkovich, M.: Révész, Sz.: Periodic decompositions of continuous functions. Acta Math. Hungar. **54**(3–4), 329–341 (1989)
30. Laczkovich, M.: Révész, Sz.: Decompositions into the sum of periodic functions belonging to a given Banach space. Acta Math. Hungar. **55**(3–4), 353–363 (1990)
31. Martinez, J., Mazon, J.M.: C_0-semigroups norm continuous at infinity. Semigroup Forum **52**(2), 213–224 (1996)
32. Mortola, S., Peirone, R.: The sum of periodic functions. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **2**(2), 393–396 (1999)
33. Nagel, R.: Mittelergodische Halbgruppen linearer Operatoren. Ann. Inst. Fourier (Grenoble) **23**(4), 75–87 (1973)
34. Natkaniec, T., Wilczyński, W.: Sums of periodic Darboux functions and measurability. Atti Sem. Mat. Fis. Univ. Modena **51**, 369–376 (2003)
35. Schwartz, L.: Théorie générale des fonctions moyenne-périodiques. Ann. Math. **2**(48), 857–929 (1947)
36. Segal, I.E.: Equivalences of measure spaces. Amer. J. Math. **73**, 275–313 (1951)
37. Székelyhidi, L.: On mean periodic functions. Aequationes Math. **58**(3), 203–213 (1999)
38. Whitney, H.: On functions with bounded nth differences. J. Math. Pures Appl. **9**(36), 67–95 (1957)
39. Wierdl, M.: Continuous functions that can be represented as the sum of finitely many periodic functions. Mat. Lapok **32**(1–3), 107–113 (1981/1984)