A POTENTIAL THEORETIC MINIMAX PROBLEM ON THE TORUS

BÁLINT FARKAS, BÉLA NAGY AND SZILÁRD RÉVÉSZ

Abstract. We investigate an extension of an equilibrium-type result, conjectured by Ambrus, Ball and Erdélyi, and proved recently by Hardin, Kendall and Saff. These results were formulated on the torus, hence we also work on the torus, but one of the main motivations for our extension comes from an analogous setup on the unit interval, investigated earlier by Fenton.

Basically, the problem is a minimax one, i.e. to minimize the maximum of a function F, defined as the sum of arbitrary translates of certain fixed "kernel functions", minimization understood with respect to the translates. If these kernels are assumed to be concave, having certain singularities or cusps at zero, then translates by y_j will have singularities at y_j (while in between these nodes the sum function still behaves realtively regularly). So one can consider the maxima m_i on each subintervals between the nodes y_j, and look for the minimization of $\max F = \max_i m_i$.

Here also a dual question of maximization of $\min_i m_i$ arises. This type of minimax problems were treated under some additional assumptions on the kernels. Also the problem is normalized so that $y_0 = 0$.

In particular, Hardin, Kendall and Saff assumed that we have one single kernel K on the torus or circle, and $F = \sum_{j=0}^{n} K(\cdot - y_j) = K + \sum_{j=1}^{n} K(\cdot - y_j)$. Fenton considered situations on the interval with *two* fixed kernels J and K, also satisfying additional assumptions, and $F = J + \sum_{j=1}^{n} K(\cdot - y_j)$. Here we consider the situation (on the circle) when *all the kernel functions can be different*, and $F = \sum_{j=0}^{n} K_j(\cdot - y_j) = K_0 + \sum_{j=1}^{n} K(\cdot - y_j)$. Also an emphasis is put on relaxing all other technical assumptions and give alternative, rather minimal variants of the set of conditions on the kernel.

1. Introduction

The present work deals with an ambitious extension of an equilibrium-type result, conjectured by Ambrus, Ball and Erdélyi [2] and recently proved by Hardin, Kendall and Saff [9]. To formulate this equilibrium result, it is convenient to identify the circle (or one dimensional torus) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $[0, 2\pi)$, and call a function $K : \mathbb{T} \to \mathbb{R} \cup \{-\infty\}$ a *kernel*. The setup of [2] and [9] requires that the kernel function is *convex*. However, due to historical reasons we shall suppose that the kernels are *concave*, the transition between the two settings is a trivial multiplication by -1. Accordingly, we take the liberty to reformulate the results of [9] by a multiplication by -1, so in particular for concave kernels, see Theorem 1.1 below.

The setup of our investigation is therefore that some *concave* kernel function K is fixed, meaning that K is concave on $[0, 2\pi)$. Then K is necessarily either finite valued (i.e., $K : \mathbb{T} \to \mathbb{R}$) or it satisfies $K(0) = -\infty$ and $K : (0, 2\pi) \to \mathbb{R}$ (the degenerate situation K is constant $-\infty$ is excluded), and K is upper semi-continuous on $[0, 2\pi)$, and continuous on $(0, 2\pi)$; furthermore, K is necessarily differentiable a.e. and its derivative K' is non-increasing.

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The kernel functions are extended periodically to $\mathbb{R}$ and we consider

$$F(y_0,\dots,y_n,t) := \sum_{j=0}^{n} K(t-y_j).$$

The points $y_0,\dots,y_n$ are called *nodes*. Then we are interested in solutions of the minimax problem

$$\inf_{y_0,\dots,y_n\in[0,2\pi)} \sup_{t\in[0,2\pi)} \sum_{j=0}^{n} K(t-y_j) = \inf_{y_0,\dots,y_n\in[0,2\pi)} \sup_{t\in[0,2\pi)} F(y_0,\dots,y_n,t),$$

and address questions concerning existence and uniqueness of solutions, as well as the distribution of the points $y_0,\dots,y_n$ (mod 2π) in such extremal situations.

In [2] it was shown that for $K(t) := -|\mathrm{e}^{it}-1|^{-2} = -\frac{1}{4}\sin^{-2}(t/2)$, (which comes from the Euclidean distance $|\mathrm{e}^{it}-\mathrm{e}^{is}| = 2\sin((t-s)/2)$ between points of the unit circle on the complex plane), $\max F$ is minimized exactly for the regular configuration of points, i.e., when $y_j = 2\pi j/n$ ($j = 0,\dots,n$) and $\{\mathrm{e}^{iy_j} \ : \ j = 0,\dots,n\}$ forms a regular $n+1$-gon on the circle. (The authors mention that the concrete problem stems from a certain extremal problem called "strong polarization constant problem" by them [1].)

Based on this and natural heuristical considerations, Ambrus, Ball and Erdélyi conjectured that the same phenomenon should hold also when $K(t) := -|\mathrm{e}^{it}-1|^{-p}$ ($p > 0$), and, moreover, even when K is any concave kernel (in the above sense). Next, this was proved for $p = 4$ by Erdélyi and Saff [7]. Finally, in [9] the full conjecture of Ambrus, Ball and Erdélyi was indeed settled.

Theorem 1.1 (Hardin, Kendall, Saff)**.** *Let K be any concave kernel function such that $K(t) = K(-t)$ and K is increasing on $(0,\pi)$. For any $0 = y_0 \le y_1 \le \dots \le y_n < 2\pi$ write $\mathbf{y} := (y_1,\dots,y_n)$ and $F(\mathbf{y},t) := K(t) + \sum_{j=1}^{n} K(t-y_j)$. Let $\mathbf{e} := (\frac{2\pi}{n+1},\dots,\frac{2\pi n}{n+1})$ (together with 0 the equidistant node system in $\mathbb{T}$).*

(a) *Then*

$$\inf_{0=y_0\le y_1\le\dots\le y_n<2\pi} \sup_{t\in\mathbb{T}} F(\mathbf{y},t) = \sup_{t\in\mathbb{T}} F(\mathbf{e},t),$$

i.e., the smallest supremum is attained at the equidistant configuration.

(b) *Furthermore, if K is strictly concave, then the smallest supremum is attained at the equidistant configuration* only.

Although this might seem as the end of the story, it is in fact not. The equilibrium phenomenon, captured by this result, is indeed a much more general phenomenon when we interpret it from a proper point of view. However, to generalize further, we should first analyze what more general situations we may address and what phenomena we can expect to hold in the formulated more general situations. Certainly, regularity in the sense of the nodes y_j distributed *equidistantly* is a rather strong property, which is intimately connected to the use of one single and fixed kernel function K. However, this regularity obviously entails *equality of the "local maxima"* (suprema) m_j for all $j = 0,1,\dots,n$, and this is what is usually natural in such equilibrium questions.

We say that the configuration of points $0 = y_0 \le y_1 \le \dots \le y_n \le y_{n+1} = 2\pi$ *equioscillates*, if

$$\sup_{t\in[y_j,y_{j+1}]} F(y_0,\dots,y_n,t) = \sup_{t\in[y_i,y_{i+1}]} F(y_0,\dots,y_n,t)$$

holds for all $i,j\in\{0,\dots,n\}$. Obviously, with one single and fixed kernel K, if the nodes are equidistantly spaced, then the configuration equioscillates. And this—as will be seen from this work—in the more general setup, is a good replacement for the property that a point configuration is equidistant.

To give a perhaps enlightening example of what we have in mind, let us recall here a remarkable, but regrettably almost forgotten result of Fenton (see [8]), in the analogous, yet also somewhat different situation, when the underlying set is not the torus $\mathbb{T}$, but the unit interval $\mathbb{I} := [0, 1]$. In this setting the underlying set is not a group, hence defining translation $K(t - y)$ of a kernel K can only be done if we define the basic kernel function K not only on $\mathbb{I}$ but also on $[-1, 1]$. Then for any $y \in \mathbb{I}$ the translated kernel $K(\cdot - y)$ is well-defined on $\mathbb{I}$, moreover, it will have analogous properties to the above situation, provided we assume $K|_{\mathbb{I}}$ and also $K|_{[-1,0]}$ to be concave. Similarly, for any node systems the analogous sum F will have similar properties to the situation on the torus.

From here one might derive that under the proper and analogous conditions, a similar regularity (i.e., equidistant node distribution) conclusion can be drawn also for $\mathbb{I}$. But this is *not* the result of Fenton, who did indeed dig deeper.

Observe that there is one rather special role, played by the *fixed* endpoint(s) $y_0 = 0$ (and perhaps $y_{n+1} = 1$), since perturbing a system of nodes the respective kernels are translated—but not the one belonging to $K_0 := K(\cdot - y_0)$, since y_0 is fixed. In terms of (linear) potential theory, $K = K(\cdot - y_0) =: K_0$ is a fixed *external field*, while the other translated kernels play the role of a certain "gravitational field", as observed when putting (equal) point masses at the nodes. The potential theoretic interpretation is indeed well observed already in [7], where it is mentioned that the *Riesz potentials* with exponent p on the circle correspond to the special problem of Ambrus, Ball and Erdélyi. From here, it is only a little step further to separate the role of the varying mass points, as generating the corresponding gravitational fields, from the stable one, which may come from a similar mass point and law of gravity—or may come from anywhere else.

Note that this potential theoretic external field consideration is far from being really new. To the contrary, it is the fundamental point of view of studying weighted polynomials (in particular, orthogonal polynomial systems with respect to a weight), which has been introduced by the breakthrough paper of Mhaskar and Saff [12] and developed into a far-reaching theory in [14] and several further treatises. So in retrospect we may interpret the factual result of Fenton as an early (in this regard, not spelled out and very probably not thought of) external field generalization of the equilibrium setup considered above.

Theorem 1.2 (Fenton)**.** *Let $K : [-1, 1] \to \mathbb{R} \cup \{-\infty\}$ be a C^2 kernel function concave and monotonic both on $(-1, 0)$ and $(0, 1)$ with $K'' < 0$ and $D_\pm K(0) = \pm\infty$. Let $J : [0, 1] \to \mathbb{R}$ be a strictly concave functions. For $\mathbf{y} \in [0, 1]^n$ consider*

$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} K(t - y_j).$$

We set $y_0 = 0$, $y_{n+1} = 1$. Then the following are true:

(a) *There are $0 \le w_1 \le \cdots \le w_n \le 1$ such that*

$$\inf_{0 \le y_1 \le \cdots \le y_n \le 1} \max_{j=0,\ldots,n-1} \sup_{t \in [y_j, y_{j+1}]} F(\mathbf{y}, t) = \sup_{t \in [0,1]} F(\mathbf{w}, t).$$

(b) *The configuration from* (a) $\mathbf{w}$ *equioscillates, i.e.,*

$$\sup_{t \in [w_j, w_{j+1}]} F(\mathbf{w}, t) = \sup_{t \in [w_i, w_{i+1}]} F(\mathbf{w}, t)$$

for all $i, j \in \{0, \ldots, n\}$.

(c) *We have*

$$\inf_{0\le y_1\le\cdots\le y_n\le 1}\ \max_{j=0,\dots,n-1}\ \sup_{t\in[y_j,y_{j+1}]} F(\mathbf{y},t) = \sup_{0\le y_1\le\cdots\le y_n\le 1}\ \min_{j=0,\dots,n-1}\ \sup_{t\in[y_j,y_{j+1}]} F(\mathbf{y},t).$$

(d) *If* $0 \le z_1 \le \cdots \le z_n \le 1$ *is a configuration which equioscillates, then* $\mathbf{w} = \mathbf{z}$.

This gave us the first clue and impetus to the further, more general investigations, which, however, were executed for the torus setup. As regards Fenton's setup, i.e., similar questions on the interval, we plan to return them in a subsequent paper. The two setups are rather different in technical details, and we found it difficult to explain them simultaneously—while in principle they should indeed be the same. Such an equivalency is at least exemplified also in this paper, when we apply our results to the problem of Bojanov on so-called "restricted Chebyshev polynomials": In fact, the original result of Bojanov (and our generalization of it) is formulated on an interval. So in order to use our results, valid on the torus, we must work out both some corresponding (new) results on the torus itself, and also a method of transference (working well at least in the concrete Bojanov situation). The transference seems to work well in symmetric cases, but becomes untractable for non-symmetric ones. Therefore, it seems that to capture full generality, not the transference, but direct, analogous arguments should be used. This explains our decision to restrict current considerations to the case of the torus only.

Nevertheless, as for generality of the results, the reader will see that we indeed make a further step, too. Namely, we will allow not only an external field (which, for the torus case, would already be an extension of Theorem 1.1, analogous to Theorem 1.2), but we will study situations when *all the kernels*, fixed or translated, may as well be different. (Definitely, this makes it worthwhile to work out subsequently the analogous questions also for the interval case.) It is not really easy to interpret this situation in potential theoretical terms anymore. However, one may argue that in physics we do encounter some situations, e.g., in sub-atomic scales, when simultaneously different forces and laws can be observed: strong kernel forces, electrostatic and gravitational forces etc. In any case, the reader will see that the generality here is clearly a powerful one: e.g., the above mentioned new proof (and generalization and extension to the torus) of Bojanov's problem of restricted Chebyshev polynomials requires (although only slightly) this generality. Hopefully, in other equilibrium type questions the generality of the current investigation will prove to be of use, too.

In this introduction it is not yet possible to precisely formulate our results, for we need to discuss a couple of technical issues first, to be settled in Section 2. One such, but not only technical, matter is the loss of symmetry with respect to the *ordering* of the nodes. Indeed, while in case of a fixed kernel to be translated (even if the external field is different), all permutations of the nodes $y_1, \dots, y_n$ are equivalent, for different kernels $K_1, \dots, K_n$ we of course must distinguish between situations when the ordering of the nodes differ. Also, the original extremal problem can have *different interpretations* according to consideration of *one fixed order* of the kernels (nodes), or *simultaneously all possible orderings* of them. We will treat *both* type of questions, but the *answers will be different.* This is not only a technical matter: We will see that, e.g., it can well happen that in some prescribed ordering of the nodes (i.e., the kernels) the extremal configuration has equioscillation, while in some other ordering that fails.

We shall progress methodologically, defining notation, properties and discussing details step by step. Our main result will only be formulated later in Section 11. In the next section (Section 2) we will first introduce the setup precisely, hoping that the reader will be satisfied with the motivation provided by this introduction. In subsequent sections we will discuss various aspects—such as continuity properties in Section 3, limits and approximations in Section 4, concavity, distributions of local extrema—without providing more motivation or explanation, hoping that the final results will justify also the otherwise seemingly unmotivated technical terms in this course of investigation. Finally, in Section 13 we shall describe, how Bojanov's results (and extensions of it) can be derived via our equilibrium results.

2. The setting of the problem

For given 2π-periodic kernel functions $K_0, \dots, K_n : \mathbb{R} \to [-\infty, \infty)$ we are interested in solutions of minimax problems like

$$\inf_{y_0,\dots,y_n \in [0,2\pi)} \sup_{t\in[0,2\pi)} \sum_{j=0}^{n} K_j(t-y_j),$$

and address questions concerning existence and uniqueness of solutions, as well as the distribution of the points $y_0, \dots, y_n$ (mod 2π) in such extremal situations. In the case when $K_0 = \cdots = K_n$ similar problems were studied by Fenton [8] (on intervals), Saff et al. [9] (on the unit circle). For C^2 kernels an abstract framework for handling of such minimax problems was developed by Shi [15], which is based on the fundamental works of Kilgore [10], [11], de Boor, Pinkus [6] concerning an interpolation theoretic conjecture of Bernstein and Erdős. Apart from the fact that we do not pose any smoothness conditions on the kernels, it will turn out that Shi's framework is not applicable in this general setting (cf. Example 5.12 and Section 9 below). The exact references will be given at the relevant places below, but let us stress already here that we do not assume the functions K_j to be smooth (in contrast to [15]), and that they may be different (in contrast to [8] and [9]). For a precise formulation of the problems we wish to study, some preparations are needed.

For convenience we shall identify the unit circle (torus) $\mathbb{T}$ with the interval $[0, 2\pi)$ (with addition mod 2π), and consider 2π-periodic functions also as functions on $\mathbb{T}$; we shall use the terminology of both frameworks, whichever comes more handy. So that we may speak about concave functions on $\mathbb{T}$ (i.e., on $[0, 2\pi)$), just as about arcs in $[0, 2\pi)$ (i.e., on $\mathbb{T}$); this shall cause no ambiguity. We also use the notation

$$d_{\mathbb{T}}(x,y) = \min\big\{|x-y|, 2\pi - |x-y|\big\} \quad (x, y \in [0, 2\pi]),$$

and

$$d_{\mathbb{T}^m}(\mathbf{x},\mathbf{y}) = \max_{j=1,\dots,m} d_{\mathbb{T}}(x_j, y_j) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{T}^m).$$

Note that the metric $d_{\mathbb{T}}(x,y)$ is equivalent to the Euclidean metric $|x-y|$ on the unit circle $\mathbb{T}$ (identified with $[0, 2\pi)$).

For $n \in \mathbb{N}$ and $j = 0, \dots, n$ let K_j be a strictly concave kernel function on $(0, 2\pi)$ that has an infinite cusp at $0 \in \mathbb{T}$, i.e., it is such that $\lim_{\substack{t\to 0\\ t\in\mathbb{T}}} K_j(t) = -\infty$ meaning that

$$(\infty) \qquad \lim_{t\downarrow 0} K_j(t) = -\infty = \lim_{t\uparrow 2\pi} K_j(t).$$

Denote by $D_- f$ and $D_+ f$ the left and right derivatives of a function f defined on an interval, respectively. A *concave* function f, defined on an open interval possesses at each points left and right derivatives, and $D_- f$, $D_+ f$ are decreasing functions. Then, under condition (∞) it is obvious that we must also have that

$$\lim_{t\downarrow 0} K_j(t) = \lim_{t\uparrow 2\pi} K_j(t) =: K_j(0),$$

$$(\infty'_-) \qquad \text{and} \qquad \lim_{t\uparrow 2\pi} D_+K_j(t) = \lim_{t\uparrow 2\pi} D_-K_j(t) = -\infty,$$

$$(\infty'_+) \qquad \text{and} \qquad \lim_{t\downarrow 0} D_-K_j(t) = \lim_{t\downarrow 0} D_+K_j(t) = \infty$$

(equivalently written in the form $D_\pm K_j(0) = \pm\infty$ or $K'_j(\pm 0) = \pm\infty$). The two conditions (∞'_-) and (∞'_+) together constitute

$$(\infty') \qquad D_-K_j(0) = -\infty \quad \text{and} \quad D_+K_j(0) = \infty.$$

To distinguish from the case of an actual singularity, this, for concave kernels less restrictive condition than (∞), will be spelled out as the concave kernel functions K_j having a (finite) *cusp* (at $0 \in \mathbb{T}$).

These will usually be our assumptions, but we refer to them explicitly whenever they are needed.

For a fixed $n \in \mathbb{N}$ we take $n+1$ points $y_0, y_1, y_2, \ldots, y_n \in [0, 2\pi)$, called *nodes*. As a matter of fact, for definiteness, we shall always take $y_0 = 0 \equiv 2\pi \mod 2\pi$. Then $\mathbf{y} = (y_1, \ldots, y_n)$ is called a *node system*. For convenience we also set $y_{n+1} = y_0$. For a given node system $\mathbf{y}$ we consider the function

$$F(\mathbf{y}, t) := \sum_{j=0}^{n} K_j(t - y_j) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j).$$

For a permutation σ of $\{1, \ldots, n\}$ we introduce the notation $\sigma(0) = 0$ and $\sigma(n+1) = n + 1$, and define the simplex

$$S_\sigma := \big\{\mathbf{y} \in \mathbb{T}^n : 0 = y_{\sigma(0)} < y_{\sigma(1)} < \cdots < y_{\sigma(n)} < y_{\sigma(n+1)} = 2\pi\big\}.$$

In this paper the term *simplex* is reserved exclusively for domains of this form. Then S_σ is open subset of $\mathbb{T}^n$ with

$$\bigcup_\sigma \overline{S}_\sigma =: \mathbb{T}^n \quad \text{and for} \quad X := \bigcup_\sigma S_\sigma$$

the set $\mathbb{T}^n \setminus X$ is the union of less than n-dimensional simplexes. Given a permutation σ and $\mathbf{y} \in \overline{S}_\sigma$, for $k = 0, \ldots, n$ we define the arc $I_{\sigma(k)}$ (in the counterclockwise direction)

$$I_{\sigma(k)}(\mathbf{y}) := [y_{\sigma(k)}, y_{\sigma(k+1)}].$$

For $j = 0, \ldots, n$ we have $I_j = [y_j, y_{\sigma(\sigma^{-1}(j)+1)}]$. Of course, a priori, nothing prevents that some of these arcs I_j reduce to a singleton, but their lengths sum up to 2π

$$\sum_{j=0}^{n} |I_j| = 2\pi.$$

Given $\mathbf{y} \in \mathbb{T}^n$ the arcs $I_j(\mathbf{y})$ are defined uniquely as soon as we specify σ with $\mathbf{y} \in \overline{S}_\sigma$. This is, in particular, the case if $\mathbf{y} \in S_\sigma$, because different (open) simplexes are disjoint. However, for $\sigma \neq \pi$ and for $\mathbf{y} \in \overline{S}_\sigma \cap \overline{S}_\pi$ on the (common) boundary, the system of arcs is still well defined but their numbering does depend on the permutations π and σ.

We set

$$m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

We also introduce the functions

$$\begin{aligned} \overline{m} &: \mathbb{T}^n \to [-\infty, \infty), \quad \overline{m}(\mathbf{y}) := \max_{j=0,\dots,n} m_j(\mathbf{y}) = \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \\ \underline{m} &: \mathbb{T}^n \to [-\infty, \infty), \quad \underline{m}(\mathbf{y}) := \min_{j=0,\dots,n} m_j(\mathbf{y}). \end{aligned}$$

Of interest are then the following two minimax type expressions:

$$M := \inf_{\mathbf{y} \in \mathbb{T}^n} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y} \in \mathbb{T}^n} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y} \in \mathbb{T}^n} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \tag{1}$$

$$m := \sup_{\mathbf{y} \in \mathbb{T}^n} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y} \in \mathbb{T}^n} \min_{j=0,\dots,n} m_j(\mathbf{y}) \tag{2}$$

Or more specifically for any given simplex $S = S_\sigma$ we may consider the problems:

$$M(S) := \inf_{\mathbf{y} \in S} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y} \in S} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y} \in S} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \tag{3}$$

$$m(S) := \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y} \in S} \min_{j=0,\dots,n} m_j(\mathbf{y}). \tag{4}$$

For notational convenience for any given set $A \subseteq \mathbb{T}^n$ we also define

$$\begin{aligned} M(A) : &= \inf_{\mathbf{y} \in A} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y} \in A} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y} \in A} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \\ m(A) : &= \sup_{\mathbf{y} \in A} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y} \in A} \min_{j=0,\dots,n} m_j(\mathbf{y}). \end{aligned}$$

It will be proved in Proposition 3.11 below that $m(S) = m(\overline{S})$ and $M(S) = M(\overline{S})$. Observe that then we can also write

$$M = \min_{\sigma} \inf_{\mathbf{y} \in \overline{S}_\sigma} \overline{m}(\mathbf{y}) = \min_{\sigma} M(\overline{S}_\sigma), \tag{5}$$

$$m = \max_{\sigma} \sup_{\mathbf{y} \in \overline{S}_\sigma} \underline{m}(\mathbf{y}) = \max_{\sigma} m(\overline{S}_\sigma). \tag{6}$$

We are interested in whether the infimum or supremum are always attained, and if so, what can be said about the extremal configurations.

Example 2.1. If the kernels are only concave and not strictly concave, then the minimax problem (3) may have many solutions, even on the boundary ∂S of $S = S_\sigma$. Let n be fixed, $K_0 = K_1 = \cdots = K_n = K$ and let K be a symmetric ($K(t) = K(2\pi - t)$) kernel which is constant c_0 on the interval $[\delta, 2\pi - \delta]$, where $\delta < \frac{\pi}{n+1}$. Then for any node system $\mathbf{y}$ we have $\max_{t \in \mathbb{T}^n} F(\mathbf{y}, t) = (n+1)c_0$, because the 2δ long intervals around the nodes cannot cover $[0, 2\pi]$.

Proposition 2.2. *For every $\delta > 0$ there is $L = L(K_0, \dots, K_n, \delta) \geq 0$ such that for every $\mathbf{y} \in \mathbb{T}^n$ and for every $j \in \{0, \dots, n\}$ with $|I_j(\mathbf{y})| > \delta$ one has $m_j(\mathbf{y}) \geq -L$.*

Proof. Let $\delta \in (0, 2\pi)$. Each K_j, $j = 0, \dots, n$ is bounded from below by $-L_j(\delta) \leq 0$ on $\mathbb{T} \setminus (-\delta/2, \delta/2)$. So that for $\mathbf{y} \in \mathbb{T}^n$ the function $F(\mathbf{y}, t)$ is bounded from below by $-L := -(L_0 + \cdots + L_n)$ on $B := \mathbb{T} \setminus \bigcup_{j=0}^n (y_j - \delta/2, y_j + \delta/2))$. Let $\mathbf{y} \in \mathbb{T}^n$ and $j \in \{0, \dots, n\}$ be such that $|I_j(\mathbf{y})| > \delta$, then there is $t \in B \cap I_j(\mathbf{y})$, hence $m_j(\mathbf{y}) \geq -L$. $\square$

Corollary 2.3. *For each simplex $S := S_\sigma$ we have that $m(S), M(S)$ are finite, in particular $m, M \in \mathbb{R}$.*

Proof. Since $K_0, \dots, K_n$ are bounded from above, say by $C \geq 0$, $F(\mathbf{y}, t) \leq (n+1)C$ for every $t \in \mathbb{T}$ and $\mathbf{y} \in \mathbb{T}^n$. This yields $m(S), M(S) \leq (n+1)C$.

Take any $\mathbf{y} \in S$ consisting of distinct nodes, so that $m_j(\mathbf{y}) > -\infty$ for each $j = 0, \dots, n$. Hence $m(S) \geq \min_{j=0,\dots,n} m_j(\mathbf{y}) > -\infty$.

For $\delta := \frac{2\pi}{n+2}$ take $L \geq 0$ as in Proposition 2.2. Then for every $\mathbf{y} \in \mathbb{T}^n$ there is $j \in \{0, \dots, n\}$ with $|I_j(\mathbf{y})| > \delta$, so that for this j we have $m_j(\mathbf{y}) \geq -L$. This implies $M(S) \geq -L > -\infty$. $\square$

3. Continuity properties

In this section we study the continuity properties of the various functions defined in Section 2. And as a consequence, we prove that for each of the problems (3), (4) extremal configurations exist. For this the assumption about the strict concavity of the kernels is not needed, it perfectly suffices that they are continuous (in the extended sense as described below).

First, a node system $\mathbf{y}$ determines $n+1$ arcs on $\mathbb{T}$, and we would like to look at the continuity (in some sense) of the arcs as function of the nodes. The technical difficulties are that the nodes may coincide and they may jump over $0 \equiv 2\pi$. Note that passing from one simplex to another one may indeed cause jumps in the definitions of the arcs $I_j(\mathbf{y})$, entailing jumps[1] also in the definition of the corresponding m_j.

These problems can be overcome by the next considerations:

Remark 3.1. Let us fix any node system $\mathbf{y}_0$, together with a small $0 < \delta < \pi/(2n+2)$—then there exists an arc $I_j(\mathbf{y}_0)$, together with its center point $c = c_j$ such that $|I_j(\mathbf{y}_0)| > 2\delta$, so in a (uniform-) δ-neighborhood $U := U(\mathbf{y}_0, \delta) := \{\mathbf{x} \in \mathbb{T}^n \ : \ \|\mathbf{x} - \mathbf{y}_0\|_\infty < \delta\}$ of $\mathbf{y}_0 \in \mathbb{T}^n$, no node can reach c. We cut the torus at c and represent the points of the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ by the interval $[c, c+2\pi) \sim [0, 2\pi)$ and use the ordering of this interval. For $i = 1, \dots, n$ we define

$$\ell_i(\mathbf{y}) := \min\{t \in [c, c+2\pi) \ : \ \#\{k : y_k \leq t\} \geq i\}$$
$$r_i(\mathbf{y}) := \max\{t \in [c, c+2\pi) \ : \ \#\{k : y_k \leq t\} \leq i\}$$
$$\hat{I}_i(\mathbf{y}) := [\ell_i(\mathbf{y}), r_i(\mathbf{y})]$$

and we set

$$\hat{I}_0(\mathbf{y}) := [c, \ell_1(\mathbf{y})] \cup [r_n(\mathbf{y}), c+2\pi] =: [\ell_0(\mathbf{y}), r_0(\mathbf{y})] \subseteq \mathbb{T} \quad \text{(as an arc)}.$$

Then $\hat{I}_i(\mathbf{y})$ is the i^{th} arc in this **cut** of torus along c corresponding to the node system $\mathbf{y}$. We immediately see the continuity of the mappings

$$\mathbb{T}^n \ni \mathbf{y} \mapsto \ell_i(\mathbf{y}) \in \mathbb{T} \quad \text{and} \quad \mathbb{T}^n \ni \mathbf{y} \mapsto r_i(\mathbf{y}) \in \mathbb{T}$$

at $\mathbf{y}_0$ for each $i = 0, \dots, n$. Obviously, the system of arcs $\{I_j \ : \ j = 0, \dots, n\}$ is the same as $\{\hat{I}_i \ : \ i = 0, \dots, n\}$.

Next we turn to the continuity of the function F and m_j. To facilitate the argumentation we shall consider $\bar{\mathbb{R}} = [-\infty, \infty]$ endowed with the metric $d : [-\infty, \infty] \to \mathbb{R}$, $d(x, y) := |\arctan(x) - \arctan(y)|$ which makes it a compact metric space, with convergence meaning the usual convergence of real sequences to some finite or infinite limit. In this way, we may speak about uniformly continuous functions with values in $[-\infty, \infty]$. Moreover, the mapping $\arctan : [-\infty, \infty] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is an order

[1] Indeed, at points $\mathbf{y} \in \mathbb{T}^n \setminus X$, on the (common) boundary of some simplexes, the change of the arcs I_j may be discontinuous. E.g., when y_j and y_k changes place (ordering changes between them, e.g., from $y_\ell < y_j \leq y_k < y_r$ to $y_\ell < y_k < y_j < y_r$), then the three arcs between these points will change from the system $I_\ell = [y_\ell, y_j], I_j = [y_j, y_k], I_k = [y_k, y_r]$ to the system $I_\ell = [y_\ell, y_k], I_k = [y_k, y_j], I_j = [y_j, y_r]$. This also means that the functions m_j may be defined *differently* on a boundary point $\mathbf{y} \in \mathbb{T}^n \setminus X$ depending on the simplex we use: the interpretation of the equality $y_j = y_k$ as part of the simplex with $y_j \leq y_k$ in general furnishes a different value of m_j (which is then $F(\mathbf{y}, z_j) = F(\mathbf{y}, y_j)$) than the interpretation as (boundary) part of the simplex with $y_k \leq y_j$ (when it becomes $\max_{t \in [y_j, y_r]} F(\mathbf{y}, t)$).

preserving homeomorphism, and hence $[-\infty, \infty]$ is order complete, and therefore a continuous function defined on a compact set attains maximum and minimum (possibly ∞ and $-\infty$).

Now the condition (∞) expresses the fact that the kernels $K_j : (0, 2\pi) \to \mathbb{R}$ can be continuously extended to $[0, 2\pi]$ having then values in $\bar{\mathbb{R}}$.

Proposition 3.2. *Suppose the kernels are continuous bounded from above, meaning that they do not take the value ∞. Then the function*

$$F : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$$

is uniformly continuous (in the above defined extended sense).

Proof. Continuity of F is trivial since the K_j are continuous in the sense described in the preceding paragraph, and they are bounded from above. □

Proposition 3.3. *Suppose the kernels are continuous bounded from above, meaning that they do not take the value ∞. Let $\mathbf{y}_0 \in \mathbb{T}^n$ be a node system and let c be as in Remark 3.1, where we cut the torus. Then for $i = 0, \ldots, n$ the functions*

$$\mathbf{y} \mapsto \hat{m}_i(\mathbf{y}) := \sup_{t \in \hat{I}_i(\mathbf{y})} F(\mathbf{y}, t) \in [-\infty, \infty]$$

are continuous at $\mathbf{y}_0$.

Proof. By Proposition 3.2 the function $\arctan \circ F : \mathbb{T}^n \times \mathbb{T} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is continuous at $\{\mathbf{y}_0\} \times \mathbb{T}$. Hence $f_i(\mathbf{y}) := \max_{t \in \hat{I}_i(\mathbf{y})} \arctan \circ F(\mathbf{y}, t)$ (and thus also $\hat{m}_i = \tan \circ f_i$) is continuous, since ℓ_i and r_i are continuous (cf. Remark 3.1). □

The continuity of $\hat{m}_i$ for fixed i involves the cut of the torus at c. However, if we consider the system $\{m_0, \ldots, m_n\} = \{\hat{m}_0, \ldots, \hat{m}_n\}$ the dependence on the cut of the torus can be cured. For $\mathbf{x} \in \mathbb{T}^{n+1}$ define

$$T_i(\mathbf{x}) := \min\{t \in [c, c + 2\pi) \ : \ \exists k_0, \ldots, k_i \text{ s.t. } x_{k_0}, \ldots, x_{k_i} \le t\} \quad (i = 0, \ldots, n)$$

and

$$T(\mathbf{x}) := (T_0(\mathbf{x}), \ldots, T_n(\mathbf{x})).$$

The mapping T arranges the coordinates of $\mathbf{x}$ increasingly.

Corollary 3.4. *Suppose the kernels are continuous and do not take the value ∞. The mapping*

$$\mathbb{T}^n \ni \mathbf{y} \mapsto T(m_0(\mathbf{y}), \ldots, m_n(\mathbf{y}))$$

is (uniformly) continuous.

Proof. We have $T(m_0(\mathbf{y}), \ldots, m_n(\mathbf{y})) = T(\hat{m}_0(\mathbf{y}), \ldots, \hat{m}_n(\mathbf{y}))$ for any $\mathbf{y} \in \mathbb{T}$, while $\mathbf{y} \mapsto (\hat{m}_0(\mathbf{y}), \ldots, \hat{m}_n(\mathbf{y}))$ is continuous at any given point $\mathbf{y}_0 \in \mathbb{T}^n$ and for any given cut. But the left-hand term here does not depend on the cut, so the assertion is proved. □

Corollary 3.5. *Suppose the kernels are continuous and do not take the value ∞. The functions*

$$\overline{m} : \mathbb{T}^n \to [-\infty, \infty] \quad \text{and} \quad \underline{m} : \mathbb{T}^n \to [-\infty, \infty]$$

are (uniformly) continuous.

Corollary 3.6. *Suppose the kernels are continuous and do not take the value ∞. Let $S := S_\sigma$ be a simplex. For $j = 0, \ldots, n$ the functions*

$$m_j : \overline{S} \to [-\infty, \infty]$$

are (uniformly) continuous.

Proof. Let $\mathbf{y}_0 \in \overline{S}$, then there is a cut at some c (cf. Remark 3.1) and there is some i, such that we have $m_j(\mathbf{y}) = \hat{m}_i(\mathbf{y})$ for all $\mathbf{y}$ in a small neighborhood U of $\mathbf{y}_0$ in S. So the continuity follows from Proposition 3.3. □

Remark 3.7. Suppose that the kernel functions are concave and at least one of them is strictly concave. For fixed $\mathbf{y}$ also $F(\mathbf{y}, \cdot)$ is strictly concave on the interior of each arc $I_j(\mathbf{y})$ and continuous on $I_j(\mathbf{y})$, so there is a *unique* $z_j(\mathbf{y}) \in I_j(\mathbf{y})$ with

$$m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$$

(this being trivially true if $I_j(\mathbf{y})$ is degenerate). The unique maximum point in $\hat{I}_i(\mathbf{y})$ is denoted by $\hat{z}_i(\mathbf{y})$ (cf. Remark 3.1).

Remark 3.8. Note that in view of the jumps in the (numbering of the) arcs, neither z_j nor m_j can be continuous on the whole $\mathbb{T}^n$. However, as we are to discuss now, the *system of arcs* is continuous, whence even the *system of points* z_j, as well as the *system of values* m_j will still depend continuously on $\mathbf{y} \in \mathbb{T}^n$.

To make this fully precise, we should explain what we mean by the *system of $n+1$ values*. So let $\mathbf{u} \in \mathbb{R}^{n+1}$. Then we define the "ordered vector of its coordinates" $T\mathbf{u} = \mathbf{v}$ the following way: $v_i := \min\{t \in \mathbb{R} \ : \ \exists\, i \,\text{coordinates}\ u_{k_1}, \ldots, u_{k_i} \leq t\}$. This mapping $T : \mathbf{u} \to \mathbf{v}$ is clearly continuous, and maps $\mathbb{R}^{n+1}$ or $[0, 2\pi)^{n+1}$ to its subset of the vectors with coordinates in non-decreasing ordering: $v_0 \leq v_1 \leq \cdots \leq v_n$. It is a permutation of the coordinates, except that we do not care the indexing of equal coordinate values: only the number of coordinates, equal to a certain number t, must remain the same in $\mathbf{v}$ than it was in $\mathbf{u}$. Naturally, the permutation or indexing, which rearranges the coordinates of $\mathbf{u}$ to those of $\mathbf{v}$, is not unique (if there are equal values), and changes from one vector to another vector. Not the indexing, but the set of coordinate values is what is preserved—or, in other words, we do not map individual coordinates, but the sequence of coordinate values to its non-decreasingly ordered listing.

Basically, all what we are saying when stating continuity of the system of arcs $\mathcal{I}$ is that in a small neighborhood of a given point $\mathbf{y} \in \mathbb{T}^n$, fixing a point c as above and representing the system of arcs as $\{\hat{I}_i = [(T\mathbf{x})_i, (T\mathbf{x})_{i+1}] \ : \ i = 0, 1, \ldots, n\}$ ($\forall \|\mathbf{x}-\mathbf{y}\|_\infty < \delta$) is continuous from that small neighborhood of $\mathbf{y}$ to $([c, c+2\pi)^2)^{n+1}$, which follows from continuity of T (i.e., its coordinate functions T_i defined by $T_i(\mathbf{y}) := (T\mathbf{y})_i$). Clearly, this holds true for the small enough neighborhood, and then re-injecting the (real) representations of intervals and arcs into the torus will of course keep continuity. The good thing is, however, that while the indexing of $(T\mathbf{x})_i$ depended on the choice of c, whence also the point $\mathbf{y}$, after re-injecting the arcs into $\mathbb{T}$ the system of resulting arc decomposition of $\mathbb{T}$ will no longer preserve the real ordering, whence indexing—only the system, tiling (i.e., covering, with only a few endpoints multiply covered) the torus, can now be seen.

If condition (∞) holds, then it is evident that $z_j(\mathbf{y})$ belongs to the interior of $I_j(\mathbf{y})$ (if this latter is non-empty). However, we obtain the same even under the weaker assumption (∞').

Proposition 3.9. *Suppose that $K_0, \ldots, K_n$ are concave with at least one of them strictly, and continuous (in the extended sense).*

(a) *For each $\mathbf{y} \in \mathbb{T}$ and $j = 0, \ldots, n$ there is a unique maximum point $z_j(\mathbf{y})$ of $m_j(\mathbf{y})$ in $I_j(\mathbf{y})$.*
(b) *If condition (∞'_+) holds for each $j = 0, \ldots, n$, then $z_j(\mathbf{y})$ is different from the left endpoint of $I_j(\mathbf{y})$ whenever this interval is non-degenerate.*
(c) *If condition (∞'_-) holds for each $j = 0, \ldots, n$, then $z_j(\mathbf{y})$ is different from the right endpoint of $I_j(\mathbf{y})$ whenever this interval is non-degenerate.*

(d) *If condition* (∞') *holds for each* $j = 0, \ldots, n$, *then* $z_j(\mathbf{y})$ *belongs to the interior of* $I_j(\mathbf{y})$ *whenever it is non-degenerate.*

Proof. (a) Uniqueness of a maximum point, i.e., the definition of $z_j(\mathbf{y})$ has been already discussed in Remark 3.7.

(b) Suppose Let now the arc $I_j(\mathbf{y}) = [y_j, y_i]$ be non-degenerate. We are to prove $z_j(\mathbf{y}) \neq y_j$.

Obviously, in case $K_j(0) = -\infty$, we also have $F(\mathbf{y}, y_j) = -\infty$ and y_j cannot be a maximum point on I_j. So we may assume $K_j(0) \in \mathbb{R}$, in which case $F(\mathbf{y}, \cdot)$ is finite, continuous and concave on $[y_j, y_i)$.

In view of condition (∞') we have $D_+F(\mathbf{y}, y_j) = \infty$. Hence there is $\delta > 0$ such that for every t with $y_j < t < y_j + \delta < y_i$,

$$\frac{F(\mathbf{y}, t) - F(\mathbf{y}, y_j)}{t - y_j} > 1,$$

i.e., $F(\mathbf{y}, t) > t - y_j + F(\mathbf{y}, y_j) > F(\mathbf{y}, y_j)$, entailing $z_j(\mathbf{y}) \neq y_j$.

The proof of (c) is entirely the same as that of (b). Assertion (d) follows from (b) and (c) □

For the next lemma we need that the function z_j is well-defined for each $j = 0, \ldots, n$, so we need $F(\mathbf{y}, \cdot)$ to be strictly concave, in order to which it suffices if at least one of the kernels is strictly concave.

Lemma 3.10. *Suppose that the kernels are concave and continuous, with at least one of them strictly concave, so that the maximum point* $z_j(\mathbf{y})$ *of* $F(\mathbf{y}, \cdot)$ *in* I_j *is unique for every* $j = 0, \ldots, n$. *For each* $j = 0, \ldots, n$ *and for each simplex* $S = S_\sigma$ *the mapping*

$$z_j : \overline{S} \to \mathbb{T}, \quad \mathbf{y} \mapsto z_j(\mathbf{y})$$

is continuous. Moreover, for a given $\mathbf{y}_0 \in \mathbb{T}^n$ *consider a cut of the torus (cf. Remark 3.1). The mapping*

$$\mathbf{y} \mapsto \hat{z}_i(\mathbf{y})$$

is continuous at $\mathbf{y}_0$.

Proof. Let $(\overline{S} \ni)\mathbf{y}_n \to \mathbf{y} \in \overline{S}$. Then, by Proposition 3.3, $m_j(\mathbf{y}_n) \to m_j(\mathbf{y}) \in [-\infty, \infty)$. Let $x \in \mathbb{T}$ be any accumulation point of the sequence $z_j(\mathbf{y}_n)$, and by passing to a subsequence assume $z_j(\mathbf{y}_n) \to x$.

By definition of z_j, we have $F(\mathbf{y}_n, z_j(\mathbf{y}_n)) = m_j(\mathbf{y}_n) \to m_j(\mathbf{y})$, and by continuity of F also $F(\mathbf{y}_n, z_j(\mathbf{y}_n)) \to F(\mathbf{y}, x)$, so $F(\mathbf{y}, x) = m_j(\mathbf{y})$. But we have already remarked that by strict concavity there is a *unique* point, where $F(\mathbf{y}, \cdot)$ can attain its maximum on I_j—this provided us the definition of $z_j(\mathbf{y})$ as a uniquely defined point in I_j. Thus we conclude $z_j(\mathbf{y}) = x$. The second assertion follows from this in an obvious way, or can be proved similarly. □

Proposition 3.11. *For a simplex* S *we always have* $M(S) = M(\overline{S})$ *and* $m(S) = m(\overline{S})$. *Furthermore, both minimax problems* (3) *and* (4) *have finite extremal values, and both have an extremal node system, i.e., there are* $\mathbf{w}^*, \mathbf{w}_* \in \overline{S}$ *such that*

$$\overline{m}(\mathbf{w}^*) = M(S) := \inf_{\mathbf{y} \in S} \overline{m}(\mathbf{y}) = M(\overline{S}) = \min_{\mathbf{y} \in \overline{S}} \overline{m}(\mathbf{y}) \in \mathbb{R},$$

$$\underline{m}(\mathbf{w}_*) = m(S) := \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y}) = m(\overline{S}) = \max_{\mathbf{y} \in \overline{S}} \underline{m}(\mathbf{y}) \in \mathbb{R}.$$

Proof. By Proposition 3.3 the functions $\underline{m}$ and $\overline{m}$ are continuous (in the extended sense), whence we have $m(S) = m(\overline{S})$ and $M(S) = M(\overline{S})$. Since $\overline{S}$ is compact, the function $\underline{m}$ has a maximum on $\overline{S}$, i.e., (3) has an extremal node system $\mathbf{w}_*$. Similarly, $\overline{m}$ has a minimum, meaning that (4) has an extremal node system $\mathbf{w}^*$.

Both of these extremal values, however, must also be *finite*, according to Corollary 2.3. □

As a consequence, we obtain the following.

Corollary 3.12. *Both minimax problems* (1) *and* (2) *have an extremal node system.*

To decide whether the extremal node systems belong to S or to the boundary ∂S is the subject of the next sections.

4. Approximation of kernels

In what follows we shall consider a sequence $K_j^{(k)}$ of kernel functions converging to K_j as $k \to \infty$ for $j = 0, \ldots, n$ (in some sense or another). The corresponding values of local maxima and related quantities will be denoted as $m_j^{(k)}(\mathbf{x})$, $\underline{m}^{(k)}(\mathbf{x})$, $\overline{m}^{(k)}(\mathbf{x})$, $m^{(k)}(S)$, $M^{(k)}(S)$, and we study the limit behavior of these as $k \to \infty$. Of course, one has here a number of notions of convergence for the kernels, and we start with the easiest ones.

Let K be a compact space and let $f_n, f \in \mathrm{C}(K; \bar{\mathbb{R}})$ (the space of continuous functions with values in $\bar{\mathbb{R}}$). We say that $f_n \to f$ *uniformly* (in the extended sense) if $\arctan f_n \to \arctan f$ uniformly in the ordinary sense (as real valued functions). We say that $f_n \to f$ *strongly uniformly* if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$f(x) - \varepsilon \le f_n(x) \le f(x) + \varepsilon \quad \text{for every } x \in \mathbb{T} \text{ and } n \ge n_0.$$

Lemma 4.1. *Let $f, f_n \in \mathrm{C}(K; \bar{\mathbb{R}})$ be uniformly bounded by $C \in \mathbb{R}$ from above. We then have $f_n \to f$ uniformly if and only if for each $R > 0, \eta > 0$ there is $n_0 \in \mathbb{N}$ such that for all $x \in K$ and all $n \ge n_0$*

$$\begin{aligned} f_n(x) &< -R + \eta \quad \textit{whenever } f(x) < -R \textit{ and} \\ f(x) - \eta \le f_n(x) &\le f(x) + \eta \quad \textit{whenever } f(x) \ge -R. \end{aligned} \tag{7}$$

Proof. Suppose first that $f_n \to f$ uniformly, and let $\eta > 0$, $R > 0$ be given. The set $L := \arctan[-R-1, C+1]$ is compact in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and tan is uniformly continuous thereon. Therefore there is $\varepsilon \in (0, 1]$ sufficiently small such that $\tan(s) - \eta \le \tan(t) \le \tan(s) + \eta$ whenever $|s - t| \le \varepsilon$, $s \in \arctan[-R, C]$, and such that $\tan(\arctan(-R) + \varepsilon) \le -R + \eta$. Let $n_0 \in \mathbb{N}$ be so large that $\arctan f(x) - \varepsilon \le \arctan f_n(x) \le \arctan f(x) + \varepsilon$ holds for every $n \ge n_0$. Apply the tan function to this inequality to obtain that $f(x) - \eta \le f_n(x) \le f(x) + \eta$ for $x \in K$ with $f(x) \in [-R, C]$, and $f_n(x) \le \tan(\arctan f(x) + \varepsilon) < \tan(\arctan(-R) + \varepsilon) < -R + \eta$ for $x \in K$ with $f(x) < -R$.

Suppose now that the condition involving η and R is satisfied, and let $\varepsilon > 0$ be arbitrary. Take $R > 0$ so large that $\arctan(t) < -\frac{\pi}{2} + \varepsilon$ whenever $t < -R + 1$. For $\varepsilon > 0$ take $1 > \eta > 0$ according to the uniform continuity of arctan. By assumption there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have (7). Let $x \in K$ be arbitrary. If $f(x) < -R$, then

$$\begin{aligned} \arctan f(x) - \varepsilon < -\frac{\pi}{2} &\le \arctan f_n(x) \\ &\le \arctan(-R + \eta) < -\frac{\pi}{2} + \varepsilon < \arctan f(x) + \varepsilon. \end{aligned}$$

On the other hand, if $f(x) \ge -R$, then by the choice of η we immediately obtain $\arctan f(x) - \varepsilon < \arctan f_n(x) \le \arctan f(x) + \varepsilon$. □

The previous lemma has an obvious version for sequences that are not uniformly bounded from above, this is, however a bit more technical an will not be needed. It is now also clear that strong uniform convergence implies uniform convergence. Also the next assertions follow immediately from the corresponding classical results about real-valued functions.

Lemma 4.2. *Let $f_n, g_n, f, g \in \mathrm{C}(K; \bar{\mathbb{R}})$ for $n \in \mathbb{N}$.*

(a) *If $f_n, g_n \le C < \infty$ and $f_n \to f$ and $g_n \to g$ uniformly (in the extended sense), then $f_n + g_n \to f + g$ uniformly (in the extended sense).*
(b) *If $f_n \downarrow f$ pointwise, i.e., if $f_n(x) \to f(x)$ decreasingly for each $x \in K$, then $f_n \to f$ uniformly.*
(c) *If $f_n \to f$ uniformly, then $\sup f_n \to \sup f$ in $[-\infty, \infty]$.*

Proof. (a) The proof can be based on Lemma 4.1.

(b) This is a consequence of Dini's theorem.

(c) Follows from standard properties of arctan and tan, and from the corresponding result for real-valued functions. □

Proposition 4.3. *Suppose the sequence of kernel functions $K_j^{(k)} \to K_j$ uniformly for $k \to \infty$ and $j = 0, 1, \ldots, n$. Then for each simplex $S := S_\sigma$ we have that $m_j^{(k)} \to m_j$ uniformly on $\bar{S}$ $(j = 0, 1, \ldots, n)$. As a consequence, $m^{(k)}(S) \to m(S)$ and $M^{(k)}(S) \to M(S)$ as $k \to \infty$.*

Proof. The functions $F^{(k)}(\mathbf{x}, t) = \sum_{j=0}^n K_j^{(k)}(t - x_j)$ are continuous on $\mathbb{T}^{n+1}$ and converge uniformly to $F(\mathbf{x}, t) = \sum_{j=0}^n K_j(t - x_j)$ by (a) of Lemma 4.2. So that we can apply part (b) of the same lemma, to obtain the assertion. □

We now relax the notion of convergence of the kernel function but, contrary to the above, we shall make essentially use of the concavity of kernel functions. We say that a sequence of functions over a set Ω converges *locally uniformly*, if this sequence of functions converges uniformly on each compact subset of Ω

Recall the notation

$$d_{\mathbb{T}}(x, y) = \min\bigl\{|x - y|, 2\pi - |x - y|\bigr\} \quad (x, y \in [0, 2\pi]),$$

and

$$d_{\mathbb{T}^m}(\mathbf{x}, \mathbf{y}) = \max_{j=1,\ldots,m} d_{\mathbb{T}}(x_j, y_j) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{T}^m).$$

Define the compact set

$$D := \bigl\{(\mathbf{x}, t) \ : \ \exists i \in \{0, 1, \ldots, n\}, \text{ such that } t = x_i\bigr\} = \bigcup_{i=0}^n \bigl\{(\mathbf{x}, t) \ : \ t = x_i\bigr\} \subseteq \mathbb{T}^{n+1}.$$

Lemma 4.4. *Suppose the sequence of kernel functions $K_j^{(k)}$ converges to the kernel function K_j locally uniformly on $(0, 2\pi)$. Then $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ locally uniformly on $\mathbb{T}^{n+1} \setminus D$, i.e., for every compact subset $H \subseteq \mathbb{T}^{n+1} \setminus D$ one has $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ uniformly on H as $k \to \infty$.*

Note that in general F can attain $-\infty$, and that convergence in 0 of the kernels is not postulated.

Proof. Because of compactness of H and D we have $0 < \rho := d(H, D)$.

Now take $0 < \delta < \rho$ arbitrarily and consider for any $(\mathbf{x}, t) \in H$ the defining expression $F^{(k)}(\mathbf{x}, t) := \sum_{i=0}^n K_i^{(k)}(t - x_i)$. In this sum for points of H we surely have $|t - x_i| = d((\mathbf{x}, t), (\mathbf{x}, x_i)) \ge \rho > \delta$. In other words, $\Phi_i(H) \subset [\delta, 2\pi - \delta]$

for $i = 0, 1, \ldots, n$, where $\Phi_i(\mathbf{x}, t) := t - x_i$ is continuous—whence also uniformly continuous—on the whole $\mathbb{T}^{n+1}$.

As the locally uniform convergence of $K_i^{(k)}$ (to K_i) on $(0, 2\pi)$ entails uniform convergence on $[\delta, 2\pi - \delta]$, we have uniform convergence of $f_i^{(k)} := K_i^{(k)} \circ \Phi_i$ on the compact set H (to the function $K_i \circ \Phi_i$). It follows that $F^{(k)} = \sum_{i=0}^n f_i^{(k)}$ converges uniformly (to $F = \sum_{i=0}^n f_i$) on H, whence the assertion. □

Lemma 4.5. *Let $K : (0, 2\pi) \to \mathbb{R}$ be a concave function (so K has limits, possibly $-\infty$, at 0 and 2π). For each $u, v \in [0, 1]$ we have*

$$K(u) \le K(u + v) - v\big(K(\pi + 1/2) - K(\pi - 1/2)\big),$$
$$K(2\pi - u) \le K(2\pi - u - v) + v\big(K(\pi + 1/2) - K(\pi - 1/2)\big).$$

Proof. It is sufficient to prove the statement for $u > 0$ only, the case $u = 0$ follows from that by passing to limit. Also we may suppose $v > 0$ otherwise the inequalities are trivial. By concavity of K for any system of four points $0 < a < b < c < d < 2\pi$ we clearly have the inequality

$$\frac{K(b) - K(a)}{b - a} \ge \frac{K(d) - K(c)}{d - c}.$$

Specifying $a := u$, $b := u + v \le 2$, $c := \pi - 1/2$ and $d := \pi + 1/2$ yields the first inequality, while for $a := 2\pi - u$, $b := 2\pi - u - v \le 2$, $c := \pi + 1/2$ and $d := \pi - 1/2$ we obtain the second one. □

Theorem 4.6. *Suppose that $z \ne x_j$, $j = 0, \ldots, n$, whenever $F(\mathbf{x}, z) = \overline{m}(\mathbf{x})$. If the sequence of kernel functions $K_j^{(k)} \to K_j$ locally uniformly on $(0, 2\pi)$, then $\overline{m}^{(k)}(\mathbf{x}) \to \overline{m}(\mathbf{x})$ uniformly on $\mathbb{T}^n$.*

Proof. Let us define the set $H := \{(\mathbf{x}, z) \ : \ F(\mathbf{x}, z) = \overline{m}(\mathbf{x})\} \subset \mathbb{T}^{n+1}$, which is obviously closed by virtue of the continuity of the occurring functions. By assumption $H \subseteq \mathbb{T}^{n+1} \setminus D$, so the condition of Lemma 4.4 is satisfied, whence $F^{(k)} \to F$ uniformly on H.

Let now $\mathbf{x} \in \mathbb{T}^n$ be arbitrary, and take any $z \in \mathbb{T}$ such that $F(\mathbf{x}, z) = \overline{m}(\mathbf{x})$ (such a z exists by compactness and continuity). Now, $\overline{m}^{(k)}(\mathbf{x}) \ge F^{(k)}(\mathbf{x}, z) > F(\mathbf{x}, z) - \varepsilon = \overline{m}(\mathbf{x}) - \varepsilon$ whenever $k > k_0(\varepsilon)$, whence $\liminf_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) \ge \overline{m}(\mathbf{x})$ is clear, moreover, according to the above, this holds uniformly on $\mathbb{T}$, as $\overline{m}^{(k)}(\mathbf{x}) > \overline{m}(\mathbf{x}) - \varepsilon$ for each $\mathbf{x} \in \mathbb{T}$ whenever $k > k_0(\varepsilon)$.

It remains to see that, given $\varepsilon > 0$, there exists $k = k_0(\varepsilon)$ such that $\overline{m}^{(k)}(\mathbf{x}) < \overline{m}(\mathbf{x}) + \varepsilon$ for all $k > k_0$. Let us define the constant

$$C := \max_{j=0,1,\ldots,n} \max_{k\in\mathbb{N}} |K_j^{(k)}(\pi + 1/2) - K_j^{(k)}(\pi - 1/2)|.$$

The inner expression is indeed a finite maximum, as $K_j^{(k)}(\pi \pm 1/2) \to K_j(\pi \pm 1/2)$ for $k \to \infty$. By Lemma 4.5 for all $u, v \in [0, 1]$

$$K_j^{(k)}(u) \le K_j^{(k)}(u + v) + Cv, \quad K_j^{(k)}(2\pi - u) \le K_j^{(k)}(2\pi - u - v) + Cv. \tag{8}$$

For the given $\varepsilon > 0$ choose $\delta \in (0, 1/2)$ such that $\overline{m}(\mathbf{y}) \le \overline{m}(\mathbf{x}) + \frac{\varepsilon}{3}$ holds for all $\mathbf{y}$ with $d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) < \delta$ (use Corollary 3.5, the uniform continuity of $\overline{m} : \mathbb{T}^n \to \mathbb{R}$). Fix moreover $0 < h < \min\{\delta/2, \varepsilon/(6C(n + 1))\}$ and define

$$H := \big\{(\mathbf{y}, w) \in \mathbb{T}^{n+1} \ : \ d_{\mathbb{T}}(y_i, w) \ge h \ (i = 0, 1, \ldots, n)\big\}.$$

Let $\mathbf{x} \in \mathbb{T}^n$ and let $z_k \in \mathbb{T}$ be any point with $F^{(k)}(\mathbf{x}, z_k) = \overline{m}(\mathbf{x})$.

For an arbitrarily given point $(\mathbf{x}, z) \in \mathbb{T}^{n+1}$ we construct another one $(\mathbf{y}, w) \in \mathbb{T}^{n+1}$, which we will call "approximating point", in two steps as follows. First, we shift them (even x_0 which was assumed to be 0 all the time), and then correct them. So we set for $i = 0, 1, \ldots, n$

$$x_i' := \begin{cases} x_i & \text{if} \quad d_{\mathbb{T}}(x_i, z) \geq h, \\ x_i \pm h & \text{if} \quad d_{\mathbb{T}}(x_i, z) \leq h, \end{cases}$$

where we add h or $-h$ such that $d_{\mathbb{T}}(x_i \pm h, z) \geq h$. Then we set $y_i := x_i' - x_0'$ $(i = 0, 1, \ldots, n)$ and $w := z - x_0'$. This new approximating point $(\mathbf{y}, w)$ has the following properties:

$$d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) = \max_{i=1,\ldots,n} d_{\mathbb{T}}(x_i, y_i) \leq 2h < \delta, \qquad d_{\mathbb{T}}(z, w) \leq h < \delta. \tag{9}$$

Moreover, we have $(\mathbf{y}, w) \in H$, since $d_{\mathbb{T}}(y_i, w) = d_{\mathbb{T}}(x_i', z_i) \geq h$ for $i = 0, 1, \ldots, n$.

By construction of $(\mathbf{y}, w)$ we have

$$\begin{aligned} y_i - w &= x_i - z && \text{if} \quad d_{\mathbb{T}}(x_i, z) \geq h, \\ y_i - w &= x_i - z \pm h && \text{if} \quad d_{\mathbb{T}}(x_i, z) \leq h, \end{aligned} \tag{10}$$

So by using the inequalities in Lemma 4.5 we conclude

$$K_j^{(k)}(x_j - z) \leq K_j^{(k)}(y_j - w) + 2Ch \qquad (j = 0, 1, \ldots, n),$$

providing us

$$F^{(k)}(\mathbf{x}, z) = \sum_{j=0}^{n} K_j^{(k)}(x_j - z) \leq \sum_{j=0}^{n} (K_j^{(k)}(y_j - w) + 2Ch) = F^{(k)}(\mathbf{y}, w) + 2(n+1)Ch.$$

Now for given $\mathbf{x} \in \mathbb{T}^n$ let $z_k \in \mathbb{T}$ be any point with $F^{(k)}(\mathbf{x}, z_k) = \overline{m}^{(k)}(\mathbf{x})$, and let $(\mathbf{y}^{(k)}, w_k) \in H$ be the corresponding approximating point. So that we have

$$\overline{m}^{(k)}(\mathbf{x}) = F^{(k)}(\mathbf{x}, z_k) \leq F^{(k)}(\mathbf{y}^{(k)}, w_k) + 2(n+1)Ch. \tag{11}$$

Since $(\mathbf{y}^{(k)}, w_k) \in H \subseteq \mathbb{T}^n \setminus D$ we can invoke Lemma 4.4 to get $F^{(k)} \to F$ uniformly on H. Therefore, for the given $\varepsilon > 0$ there exists $k_0(\varepsilon)$ with

$$F^{(k)}(\mathbf{y}^{(k)}, w_k) \leq \max\bigl\{F(\mathbf{y}, w) \ : \ (\mathbf{y}, w) \in H, d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) \leq \delta, d_{\mathbb{T}}(z, w) \leq \delta\bigr\} + \tfrac{\varepsilon}{3}$$

for all $k \geq k_0(\varepsilon)$. Extending further the maximum on the right hand side to arbitrary $w \in \mathbb{T}$ we are led to

$$F^{(k)}(\mathbf{y}^{(k)}, w_k) \leq \max\bigl\{\overline{m}(\mathbf{y}) \ : \ d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) \leq \delta\bigr\} + \tfrac{\varepsilon}{3} \qquad (k > k_0). \tag{12}$$

From (11), (12) and by the choices of $h, \delta > 0$ we conclude

$$\overline{m}^{(k)}(\mathbf{x}) \leq F^{(k)}(\mathbf{y}^{(k)}, w_k) + 2C(n+1)h \leq (\overline{m}(\mathbf{x}) + \tfrac{\varepsilon}{3}) + \tfrac{\varepsilon}{3} + 2C(n+1)h < \overline{m}(\mathbf{x}) + \varepsilon$$

for all $k > k_0(\varepsilon)$. So that we get that uniformly on $\mathbb{T}^n$ $\limsup_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) \leq \overline{m}(\mathbf{x})$ holds.

Since $k_0(\varepsilon)$ does not depend on $\mathbf{x}$, by the first part we obtain $\lim_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) = \overline{m}(\mathbf{x})$ uniformly on $\mathbb{T}^n$. □

5. Elementary properties

In this section we record some elementary properties of the function m_j that are useful in the study of minimax and maximin problems and constitute also a substantial part of the abstract framework of [15]. Our aim is here also to reveal the structural connections between these notions.

Proposition 5.1. *Suppose that the kernels $K_0, \ldots, K_n$ satisfy (∞). Let $S = S_\sigma$ be a simplex. Then*

$$\lim_{\mathbf{y}\to\partial S} \max_{j=0,\ldots,n-1} \bigl| m_{\sigma(j)}(\mathbf{y}) - m_{\sigma(j+1)}(\mathbf{y}) \bigr| = \infty. \tag{13}$$

Proof. Without loss of generality we may suppose that $\sigma = \mathrm{id}$, i.e., $\sigma(j) = j$. Let $\mathbf{y}_i \in S$ be convergent to some $\mathbf{y}_0 \in \partial S$ as $i \to \infty$. This means that some arcs determined by the nodes $\mathbf{y}_i$ and $y_0 = 0$, $y_{n+1} = 2\pi$ shrink to a singleton. On any such arc $I_j(\mathbf{y}_i)$ we obviously have, with the help of (∞),

$$m_j(\mathbf{y}_i) \to -\infty \quad \text{as } i \to \infty.$$

Of course, there is at least one such arc, say with index j_0, that has a neighboring arc with index $j_0 \pm 1$ which is not shrinking to a singleton as $i \to \infty$. This means

$$\left|m_{j_0}(\mathbf{y}_i) - m_{j_0 \pm 1}(\mathbf{y}_i)\right| \to \infty \quad \text{as } i \to \infty,$$

and the proof is complete. □

The properties introduced below have nothing to do with the conditions we pose on the kernel functions $K_0, \ldots, K_n$ (concavity and some type of singularity at 0 and 2π), so we can formulate them in whole generality. (Note that m_j (in contrast to z_j) is well-defined even if the kernels are not strictly concave).

Definition 5.2. Let $S = S_\sigma$ be a simplex.

(a) **Jacobi Property:**
The functions $m_0, \ldots, m_n$ are C^1 on $S = S_\sigma$ and

$$\det\Big(\partial_i m_{\sigma(j)}\Big)_{i=1,j=0,j\neq k}^{n,n} \neq 0 \quad \text{for each } k \in \{0, \ldots, n\}.$$

(b) **Difference Jacobi Property:**
The functions $m_0, \ldots, m_n$ are C^1 on S and

$$\det\Big(\partial_i (m_{\sigma(j)} - m_{\sigma(j+1)})\Big)_{i=1,j=0}^{n,n-1} \neq 0.$$

Remark 5.3. Shi [15] proved that under the condition (13) (which is now a consequence of the assumption (∞)) the Jacobi Property implies the Difference Jacobi Property.

Definition 5.4. Let $S = S_\sigma$ be a simplex.

(a) **Equioscillation Property:**
There exists an **equioscillation point** $\mathbf{y} \in S$, i.e.,

$$\overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = \cdots = m_n(\mathbf{y}).$$

(b) **(Lower) Weak Equioscillation Property:**
There exists a **weak equioscillation point** $\mathbf{y} \in \overline{S}$, i.e.,

$$m_j(\mathbf{y}) \begin{cases} = \overline{m}(\mathbf{y}) & \text{if } I_j \text{ is non-degenerate,} \\ < \overline{m}(\mathbf{y}) & \text{if } I_j \text{ is degenerate.} \end{cases}$$

Remark 5.5. For given $S = S_\sigma$ the Equioscillation Property implies the inequality $M(S) \leq m(S)$.

Proof. Let $\mathbf{y} \in S$ be an equioscillation point. Then for this particular point $\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) = \min_{j=0,\ldots,n} m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$, whence

$$M(S) \leq \overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) \leq m(S).$$

□

Proposition 5.6. *Given a simplex $S = S_\sigma$ the following are equivalent:*

(i) $M(S) \geq m(S)$.
(ii) *For every* $\mathbf{x} \in S$ *one has* $\underline{m}(\mathbf{x}) = \min_{j=0,\ldots,n} m_j(\mathbf{x}) \leq M(S)$.
(iii) *For every* $\mathbf{y} \in S$ *one has* $\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) \geq m(S)$.

(iv) *There exists a value* $\mu \in \mathbb{R}$ *such that for each* $\mathbf{y} \in S$

$$\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) \geq \mu \geq \underline{m}(\mathbf{y}) = \min_{j=0,\ldots,n} m_j(\mathbf{y})$$

Proof. Recalling the inequalities

$$\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) \geq M(S) = \inf_S \overline{m}, \quad \sup_S \underline{m} = m(S) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\ldots,n} m_j(\mathbf{x})$$

being true for each $\mathbf{x}, \mathbf{y} \in S$, the equivalence of (i), (ii) and (iii) is obvious. Suppose (i) and take $\mu \in [m(S), M(S)]$. Then (iv) is evident. From (iv) assertion (i) follows trivially. □

Definition 5.7. Let $S = S_\sigma$ be a simplex. We say that the **Sandwich Property** is satisfied if any of the equivalent assertions in Proposition 5.6 holds true, i.e., if for each $\mathbf{x}, \mathbf{y} \in S$

$$\max_{j=0,\ldots,n} m_j(\mathbf{y}) = \overline{m}(\mathbf{y}) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\ldots,n} m_j(\mathbf{x}).$$

Remark 5.8. For given $S = S_\sigma$ the Equioscillation Property and the Sandwich Property together imply that $M(S) = m(S)$.

Definition 5.9. We say that $\mathbf{x}$ *majorizes* (or *strictly majorizes*) $\mathbf{y}$—and $\mathbf{y}$ *minorizes* (or *strictly minorizes*) $\mathbf{x}$—if $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ (or if $m_j(\mathbf{x}) > m_j(\mathbf{y})$) for all $j = 0, \ldots, n$.

Let $S = S_\sigma$ be a simplex. We define the following properties on S_σ.

(a) **Local (Strict) Comparison Property at z:** There exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{y} \in B(\mathbf{z}, \delta)$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there are no two different $\mathbf{x} \neq \mathbf{y} \in B(\mathbf{z}, \delta)$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.
(b) **Local (Strict) Non-Majorization Property at y:** There exists $\delta > 0$ such that there is no $\mathbf{x} \in (S \cap B(\mathbf{y}, \delta)) \setminus \{\mathbf{y}\}$ which (strictly) majorizes $\mathbf{y}$.
(c) **Local (Strict) Non-Minorization Property at y:** There exists $\delta > 0$ such that there is no $\mathbf{x} \in (S \cap B(\mathbf{y}, \delta)) \setminus \{\mathbf{y}\}$ which (strictly) minorizes $\mathbf{y}$.

Further, we will pick the following special cases as important.

(A) **(Strict) Comparison Property:** If $\mathbf{x}$, $\mathbf{y} \in S$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there exists no two different $\mathbf{x} \neq \mathbf{y} \in S$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.
(B) **Local (Strict) Comparison Property on S:** At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Comparison Property holds.
(C) **Local (Strict) Non-Majorization Property on S:** At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Non-Majorization Property holds.
(D) **Local (Strict) Non-Minorization Property on S:** At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Non-Minorization Property holds.
(E) **Singular (Strict) Comparison Property on S:** At each *equioscillation point* $\mathbf{y} \in S$ the Local (Strict) Comparison Property holds.
(F) **Singular (Strict) Non-Majorization Property:** At each *equioscillation point* $\mathbf{y} \in S$ the Local (Strict) Non-Majorization Property holds.
(G) **Singular (Strict) Non-Minorization Property:** At each *equioscillation point* $\mathbf{y} \in S$ the Local (Strict) Non-Minorization Property holds.

Remark 5.10. The comparison properties are symmetric in $\mathbf{x}$ and $\mathbf{y}$, while the non-majorization and non-minorization properties are not. One has the following relations between the previously defined properties: (a)⇒(b) and (c), (A)⇒(B)⇒(E), (B)⇒(C) and (D), (E)⇒(F) and (G), (C)⇒(F), (D)⇒(G). It will be proved in Corollary 8.1 that for strictly concave kernels all comparison, non-majorization and non-minorization properties (A), (B), (C), (D) (with their strict version as well) are equivalent to each other.

Remark 5.11. Shi [15] proved that (under condition (13)) the Jacobi Property implies the Comparison Property, the Sandwich Property, and that the Difference Jacobi Property implies the Equioscillation Property. Example 5.12 shows that the Comparison Property (even the Local Strict Non-Majorization Property) fails in general, even though the Difference Jacobi Condition is fulfilled. In Proposition 9.2 we will show that in our setting the Difference Jacobi Condition is always fulfilled (as long as the kernels are at least C^2) and we have the Equioscillation Property.

Example 5.12. Let $n = 1$ and $K_0 : (0, 2\pi) \to \mathbb{R}$ be a strictly concave C^∞ kernel function satisfying (∞) and such that the maximum of K_0 is 0, K_0 is increasing in $(0, \alpha)$ with $\alpha \in (0, \pi)$ and is decreasing in $(\alpha, 2\pi)$, and let $K_1(t) := K_0(2\pi - t)$. For $\mathbf{y} := y \in (0, 2\pi)$ we have $F(\mathbf{y}, t) = K_0(t) + K_1(t - y) = K_0(t) + K_0(2\pi + y - t)$, so by symmetry we obtain $z_0(\mathbf{y}) = \frac{y}{2}$ and $z_1(\mathbf{y}) = \frac{2\pi+y}{2}$. So that

$$m_0(\mathbf{y}) = F(\mathbf{y}, z_0(\mathbf{y})) = K_0(\tfrac{y}{2}) + K_0(2\pi + y - \tfrac{y}{2}) = 2K_0(\tfrac{y}{2}),$$
$$m_1(\mathbf{y}) = F(\mathbf{y}, z_1(\mathbf{y})) = K_0(\tfrac{2\pi+y}{2}) + K_0(2\pi + y - \tfrac{2\pi+y}{2}) = 2K_0(\tfrac{2\pi+y}{2}).$$

Whence we conclude that

$$m_0(\mathbf{y} + h) < m_0(\mathbf{y}) \quad \text{and} \quad m_1(\mathbf{y} + h) < m_1(\mathbf{y}),$$

whenever $y \in (2\alpha, 1)$ and $h > 0$ with $y + h \in (2\alpha, 1)$. This shows that the Non-Majorization Property does not hold in general. Since $m_0'(2\alpha) = 0$, the Jacobi Property fails for this example (which anyway follows from Remark 5.3). Notice also that

$$m_0'(\mathbf{y}) - m_1'(\mathbf{y}) = K_0'(\tfrac{y}{2}) - K_0'(\tfrac{2\pi+y}{2}) > 0,$$

since K_0' is strictly decreasing, meaning that the Difference Jacobi Condition holds (this holds in general, see Proposition 9.2). Finally, we remark that we have the Singular Non-Majorization Property. Indeed, $\mathbf{y}$ is an equioscillation point if and only if

$$2K_0(\tfrac{y}{2}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = 2K_0(\tfrac{2\pi+y}{2}),$$

i.e., at the corresponding points in the graph of K_0 there is a horizontal chord of length π. This implies that $y/2$ falls in the interval where K_0 is strictly increasing, whereas $\pi + y/2$ belongs to the interval where K_0 is strictly decreasing. Hence if we move $\mathbf{y} = y$ slightly, m_0 and m_1 will change in different directions.

This example shows that Shi's results are not applicable in this general setting, even if we supposed the kernels to be C^∞.

6. Distribution of local minima of $\overline{m}$

As a first step we look at how small perturbation of the node system $\mathbf{y}$ affects the values of $m_j(\mathbf{y})$. The a slight difficulty arises if during the perturbation the order of the nodes, i.e., the indexing of the functions m_j changes.

Remark 6.1. Suppose f_j are (strictly) concave functions for $j = 0, \dots, n$ and let $f = \sum_{j=0}^n f_j$. Let μ_j be the slope of a tangent line of f_j at some point t. Then $\mu := \sum_{j=0}^n \mu_j$ is the slope of a tangent line of f at the same point t. Conversely, if μ is given as a slope of a tangent line at some point t, then it is not hard to find μ_j, $j = 0, \dots, n$ being the slope of the corresponding tangent line of f_j at t, with $\mu = \sum_{j=0}^n \mu_j$.

Lemma 6.2 (Perturbation lemma). *Suppose that $K_0, \dots, K_n$ are strictly concave. Let $\mathbf{y} \in \mathbb{T}^n$ be a node system, and for $k \in \mathbb{N}$, $1 \le k \le n$ let $t_1, \dots, t_k \in (0, 2\pi)$ be all different from the nodes in $\mathbf{y}$. Let*

$$\delta := \tfrac{1}{2} \min\bigl\{ |t_i - y_j| : i = 1, \dots, k,\ j = 0, \dots, n \bigr\}$$

For $i = 1, \dots, k$ let $\mu^{(i)}$ be the slope of a tangent line to the graph of $F(\mathbf{y}, \cdot)$ at the point t_i. Finally, let $\mathbf{x}_1, \dots, \mathbf{x}_{n-k} \in \mathbb{R}^n$ be fixed arbitrarily.

(a) *Then there is $\mathbf{a} \in [-1, 1]^n \setminus \{\mathbf{0}\}$ such that $\mathbf{x}_\ell^\top \mathbf{a} = 0$ for $\ell = 1, \dots, n-k$ and for all $0 < h < \delta$ we have*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i)$$

for all s_i with $|s_i - t_i| < \delta$, $i = 1, \dots, k$.

(b) *If $F(\mathbf{y}, \cdot)$ has local maximum in t_i for some $i \in \{1, \dots, k\}$, i.e., when $t_i = z_j(\mathbf{y}) \in \operatorname{int} I_j(\mathbf{y})$ for some $j \in \{0, \dots, n\}$, then*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, z_j(\mathbf{y})) = m_j(\mathbf{y}) \quad \text{for all } s_i \text{ with } |s_i - z_j(\mathbf{y})| < \delta.$$

(c) *For the fixed node system $\mathbf{y}$ consider a corresponding cut of the torus (cf. Remark 3.1). Let $i_1, \dots, i_k \in \{0, \dots, n\}$ be pairwise different, and suppose that $\hat{I}_{i_1}(\mathbf{y}), \dots, \hat{I}_{i_k}(\mathbf{y})$ are non-degenerate and $\hat{z}_{i_j} \in \operatorname{int} \hat{I}_{i_j}$ for each $j = 1, \dots, k$. Then there is $\eta > 0$ such that for all $0 < h < \eta$*

$$\hat{m}_{i_j}(\mathbf{y} + h\mathbf{a}) < \hat{m}_{i_j}(\mathbf{y}) \quad j = 1, \dots, k.$$

Proof. (a) By Remark 6.1 for $i = 1, \dots, k$ and $j = 0, \dots, n$ there are μ_{ij} each of them being the slope of a tangent line to the graph of K_j at $t_i - y_j$, i.e., a tangent slope of $K_j(\cdot - y_j)$ at t_i with

$$\mu^{(i)} = \sum_{j=0}^{n} \mu_{ij}.$$

Take a vector $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$ with $a_j \in [-1, 1]$ $(j = 1, \dots, n)$ and such that

$$\sum_{j=1}^{n} a_j \mu_{ij} \ge 0 \quad \text{for } i = 1, \dots, k$$

and

$$\mathbf{x}_\ell^\top \mathbf{a} = \sum_{j=1}^{n} a_j x_{\ell j} = 0 \quad \text{for } \ell = 1, \dots, n-k.$$

Such a vector does exist by standard linear algebra. We set $a_0 := 0$.

Since K_j is concave, it follows

$$K_j(s_i - (y_j + ha_j)) \le K_j(t_i - y_j) + \mu_{ij}(s_i - t_i - ha_j)$$

for s_i with $|s_i - t_i| < \delta$ and $0 \le h < \delta$, because then $|s_i - t_i - ha_j| < \delta + |a_j|h < 2\delta$ and $|t_i - y_j| \ge 2\delta$ guarantees that the full interval between the points $t_i - y_j$ and $s_i - (y_j + ha_j)$ stays on the same side of 0, i.e., the continuous change of $t_i - y_j$ to $s_i - (y_j + ha_j)$ happens within the concavity interval of K_j.

Observe that here in view of strict concavity equality holds for some i, j if and only if $s_i - t_i - ha_j = 0$. However, for any given value of i, this cannot occur for all $j = 0, \dots, n$. Indeed, if this were so, then $a_0 = 0$ would imply $s_i = t_i$ and, by $h > 0$, it would follow that $\mathbf{a} = 0$, which was excluded.

Summing for all j, with at least one of the inequalities being strict, we obtain

$$\sum_{j=0}^{n} K_j(s_i - (y_j + ha_j)) < \sum_{j=0}^{n} K_j(t_i - y_j) + \sum_{j=0}^{n} \mu_{ij}(s_i - t_i - ha_j)$$

for $|s_i - t_i| < \delta$, $i = 1, \dots, k$, i.e., dropping also $a_0 = 0$

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i) - h \sum_{j=1}^{n} \mu_{ij} a_j.$$

Now, by the choice of $\mathbf{a}$, the last sum is non-negative, and since $h > 0$ the last term can be estimated from above by 0, and we obtain the first statement.

(b) In the case when $t_i = z_j(\mathbf{y})$ for some j (and only then) the tangent line can be chosen horizontal, i.e., $\mu^{(i)} = 0$. Therefore, with this choice the already proven result directly implies the second statement.

(c) Take a fixed $\mathbf{y}$ and a corresponding cut of torus at some c (cf. Remark 3.1). For sufficiently small η we have $\hat{z}_{i_j} \in \hat{I}_{i_j}(\mathbf{y} + h\mathbf{a})$ for all $0 < h < \eta$ and $j = 1, \dots, k$. Since $\mathbf{x} \mapsto \hat{z}_{i_j}(\mathbf{x})$ is continuous at $\mathbf{y}$, for even smaller $\eta > 0$ we have $|\hat{z}_{i_j}(\mathbf{y}) - \hat{z}_{i_j}(\mathbf{y} + h\mathbf{a})| < \delta$, whenever $0 < h < \eta$. From this we conclude, by the already proven part (b), that

$$\hat{m}_{i_j}(\mathbf{y} + h\mathbf{a}) = F(\mathbf{y} + h\mathbf{a}, \hat{z}_{i_j}(\mathbf{y} + h\mathbf{a})) < \hat{m}_{i_j}(\mathbf{y}).$$

□

Lemma 6.3. *Let the kernel functions $K_0, \dots, K_n$ be concave. Suppose that $I_j(\mathbf{y}) = [y_j, y_{j'}]$ is degenerate, i.e., a singleton.*

(a) *Suppose the kernels satisfy condition (∞'_-). Then there exists $\varepsilon > 0$ such that for all $0 < y_j - t < \varepsilon$ we have $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.*
(b) *Suppose the kernels satisfy condition (∞'_+). Then there exists $\varepsilon > 0$ such that for all $0 < t - y_j < \varepsilon$ we have $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.*
(c) *Suppose the kernels are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Then there exists $\varepsilon > 0$ such that either for all $t \in (y_j - \varepsilon, y_j)$ or for all $t \in (y_j, y_j + \varepsilon)$ we have $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.*

Proof. Let $I_j(\mathbf{y}) = \{y_j\} = \{y_{j'}\} = \{z_j(\mathbf{y})\}$ and let $\varepsilon < \min_{k=0,\dots,n,\, y_k \neq y_j} |y_k - y_j|$. Then in the intervals $(y_j - \varepsilon, y_j)$ and $(y_j, y_j + \varepsilon)$ there are no nodes and therefore the functions $K_k(\cdot - y_k)$ are all finite and concave thereon. In particular, for a point t in one of these intervals $F(\mathbf{y}, t) \in \mathbb{R}$ finite, so in case $K_j(0) = -\infty$ we also have $F(\mathbf{y}, z_j(\mathbf{y})) = -\infty < F(\mathbf{y}, t)$ and there is nothing to prove.

(a) Let now K_j have a *finite* cusp and assume (∞'_-). Then for the fixed $\mathbf{y}$ and the function $F(\mathbf{y}, \cdot)$ we have for any fixed $t \in (y_j - \varepsilon, y_j)$ that

$$D_- f(y_j) = \lim_{s \uparrow y_j} \sum_{k=0}^{n} D_- K_k(s - y_k) \leq \sum_{k=0, k \neq j}^{n} D_- K_k(t - y_k) + \lim_{s \uparrow y_j} D_- K_j(s - y_j) = -\infty$$

since $D_- K_k(\cdot - y_k)$ is decreasing by concavity (on the small interval). Therefore, choosing ε even smaller, we find that $D_- F(\mathbf{y}, \cdot) < 0$ in the interval $(y_j - \varepsilon, y_j)$, which implies that $F(\mathbf{y}, \cdot)$ is decreasing in this interval. That is, for any $t \in (y_j - \varepsilon, y_j)$, $F(\mathbf{y}, t) > F(\mathbf{y}, z_j(\mathbf{y}))$, as needed.

(b) Under condition (∞'_+) the proof is similar for the right neighborhood $(y_j, y_j + \varepsilon)$ of y_j.

(c) Finally, let us assume that the kernel functions are C^1 and non-constant, and let the left and right neighboring (non-degenerate) arcs to I_j be I_ℓ and I_r, respectively (they can be the same). Let us write $y_\ell < y_{j_1} = \dots = y_{j_\nu} < y_k$ with $j_1 = j$ and $j_\nu = r$, $I_\ell = [y_\ell, y_j]$, $I_r = [y_j, y_k]$. In view of the already settled case, we can assume $K_{j_\mu} > -\infty$ for all $\mu = 1, \dots, \nu$. So summing up, $F(\mathbf{y}, \cdot)$ is concave and C^1 both on $\overline{I_\ell} = [y_\ell, y_j]$ and on $\overline{I_r} = [y_j, y_k]$, and continuous on $[y_\ell, y_k]$. Assume for a contradiction that there exists no $\varepsilon > 0$ with the required property. Therefore, e.g., on the left hand side of y_j it is possible to converge to y_j by some sequence $x_m \uparrow y_j$ satisfying $F(\mathbf{y}, x_m) \le F(\mathbf{y}, y_j)$.

Since $F(\mathbf{y}, \cdot)$ is concave, there is some maximum point $z_\ell \in [y_\ell, y_j]$ (which, however, need not be unique if F is not strictly concave), and $F(\mathbf{y}, \cdot)$ is increasing on $[y_\ell, z_\ell]$ and decreasing on $[z_\ell, y_j]$. If $z_\ell \ne y_j$, then by the indirect assumption $F(\mathbf{y}, x_m) = F(\mathbf{y}, y_j)$ for sufficiently large m. But then by concavity the function $F(\mathbf{y}, \cdot)$ is constant $F(\mathbf{y}, y_j)$ on $[z_\ell, y_j]$. So that $F(\mathbf{y}, y_j) = \max_{t\in[y_\ell,y_j]} F(\mathbf{y}, t)$. On the other hand, if $z_\ell = y_j$, then evidently $F(\mathbf{y}, y_j) = \max_{t\in[y_\ell,y_j]} F(\mathbf{y}, t)$. By the same reasoning we obtain the same for $[y_j, y_k]$. So altogether $F(\mathbf{y}, y_j) = \max_{t\in[y_\ell,y_k]} F(\mathbf{y}, t)$.

It follows that $F(\mathbf{y}, \cdot)$ stays below $F(\mathbf{y}, y_j)$ on $[y_\ell, y_k]$, and hence we find $D_-F(\mathbf{y}, y_j) \ge 0 \ge D_+F(\mathbf{y}, y_j)$. However, using the non-constancy of the kernel functions K_i in the form that $D_-K_i(0) < D_+K_i(0)$, we find

$$\begin{aligned} D_-F(\mathbf{y}, y_j) &= \lim_{t\uparrow y_j} \sum_{i=0}^{n} K_i'(t-y_i) = \sum_{\substack{\mu=0\\ \mu\ne j_1,\dots,j_\nu}}^{n} K_\mu'(y_j - y_\mu) + \sum_{\mu=1}^{\nu} D_-K_{j_\mu}(0) \\ &< \sum_{\substack{\mu=0\\ \mu\ne j_1,\dots,j_\nu}}^{n} K_\mu'(y_j - y_\mu) + \sum_{\mu=1}^{\nu} D_+K_{j_\mu}(0) = \lim_{t\downarrow y_j} \sum_{i=0}^{n} K_i'(t-y_i) \\ &= D_+F(\mathbf{y}, y_j), \end{aligned}$$

which furnishes the required contradiction. Whence the statement follows. □

Corollary 6.4. *Let the kernel functions $K_0, \dots, K_n$ be concave. Suppose that $I_j(\mathbf{y})$ is degenerate.*

(a) *Suppose the kernels satisfy condition (∞'_+) (or all satisfy (∞'_-)). Then for at least one neighboring, non-degenerate arc I_ℓ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*
(b) *Suppose the kernels satisfy condition (∞'). Then for any neighboring, non-degenerate arc I_ℓ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*
(c) *Suppose the kernels are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Then for at least one neighboring, non-degenerate arc I_ℓ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*

Corollary 6.5. *If $K_0, \dots, K_n$ are non-constant, concave kernel functions and either all satisfy (∞'_+), or all satisfy (∞'_-), or all belong to $\mathrm{C}^1(0, 2\pi)$, then an equioscillation point $\mathbf{e} \in \mathbb{T}^n$ must belong to the interior of some simplex $S = S_\sigma$, i.e., we have $\mathbf{e} \in X = \bigcup_\sigma S_\sigma$.*

Proof. Let $\mathbf{y} \in \mathbb{T} \setminus X$ be arbitrary. Then there exists some j with $I_j(\mathbf{y})$ being degenerate. According to the above, there exists some $\ell \ne j$ with $m_j(\mathbf{y}) < m_\ell(\mathbf{y})$, so there is no equioscillation at $\mathbf{y}$. □

Example 6.6. It can happen that an equioscillation point falls on the boundary of a simplex S, and that maximum points of non-degenerate arcs lie on the endpoints. Indeed, let $K_0 := -4\pi^3/|x|$ on $[-\pi, \pi)$, extended periodically, and let $K_1(x) := K_2(x) := -(x-\pi)^2$ on $(0, 2\pi)$, again extended periodically. Observe that K_0 satisfies (∞') (and is in $\mathrm{C}^1((0,\pi) \cup (\pi, 2\pi))$, and $K_1, K_2 \in \mathrm{C}^1(0, 2\pi)$. Still, for the node system $y_1 = y_2 = \pi$, we have $\mathbf{y} \in \partial S = \partial S_{\mathrm{Id}}$, $F(\mathbf{y}, x) = F(\mathbf{y}, 2\pi - x) = -4\pi^3/x - 2x^2$ $(0 \le x \le \pi)$, whence $z_0 = z_1 = z_2 = \pi$ and $m_0 = m_1 = m_2 = F(\mathbf{y}, \pi) = -6\pi^2$, showing that $\mathbf{y}$ is in fact an equioscillation point.

Lemma 6.7. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy (∞'_+), or all satisfy (∞'_-), or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $j \in \{0, \dots, n\}$ and $\mathbf{w} \in \mathbb{T}^n$ be such that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$. Then $z_j(\mathbf{w})$ belongs to the interior of $I_j(\mathbf{w})$.*

Proof. By Corollary 6.4 it follows that the arc $I_j(\mathbf{w}) = [w_j, w_r]$ is non-degenerate. Suppose first that all kernels satisfy (∞'_+) (the case of (∞'_-) is similar). By Proposition 3.9 we have $z_j(\mathbf{w}) \neq w_j$, and as matter fact for each $i \in \{0, \dots, n\}$ we have $F(\mathbf{w}, w_i) < F(\mathbf{w}, z_i(\mathbf{w})) = m_i(\mathbf{w})$ for each non-degenerate arc $I_i(\mathbf{w})$. So that if $z_j(\mathbf{w}) = w_r$ were true, then there would exist a neighboring non-degenerate arc $I_i(\mathbf{w})$ such that $m_i(\mathbf{w}) > F(\mathbf{w}, w_r) = \overline{m}(\mathbf{w})$. This is impossible, so $z_j(\mathbf{w}) \neq w_r$ follows.

Next, let us suppose that the kernels are C^1. Let us consider now $F(\mathbf{w}, \cdot)$ on the non-degenerate arc $I_j(\mathbf{w}) = [w_j, w_r]$ and on the left neighboring non-degenerate arc $I_\ell(\mathbf{w}) = [w_\ell, w_j]$. By assumption we have that $F'(\mathbf{w}, \cdot)$ is continuous and decreasing on (w_ℓ, w_j) and on (w_j, w_r), while at w_j there is a *strictly positive* (maybe even infinite) jump, due to the jump(s) of K'_j (and of other K'_i with $w_j = w_i$) at 0. So we find $D_-F(\mathbf{w}, w_j) < D_+F(\mathbf{w}, w_j)$.

Now assume for a contradiction that $z_j(\mathbf{w}) = w_j$. It follows that $F(\mathbf{w}, w_j) > -\infty$ is finite and that $D_+F(\mathbf{w}, w_j) \leq 0$, hence by the above $D_-F(\mathbf{w}, w_j) < 0$. Therefore, $F(\mathbf{w}, \cdot)$ is strictly decreasing at least in some small left neighborhood of w_j, and so $m_\ell(\mathbf{w}) > m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$, a contradiction. With entirely the same proof we can exclude the possibility of $z_j(\mathbf{w}) = w_r$. So the proof is complete. □

Proposition 6.8. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy either all satisfy (∞'_+), or all satisfy (∞'_-), or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $\mathbf{w}^* \in \mathbb{T}^n$ be a local minimum point of $\overline{m}$, i.e., such that for some $\eta > 0$*

$$\overline{m}(\mathbf{w}^*) = \min_{|\mathbf{y} - \mathbf{w}^*| < \eta} \overline{m}(\mathbf{y}).$$

Then $\mathbf{w}^*$ is an equioscillation point, i.e.,

$$m_j(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) \quad \textit{for all } j = 0, \dots, n.$$

As a consequence, such a local minimum point belongs to $X = \bigcup_\sigma S_\sigma$.

Proof. Consider an appropriate cut of the torus (cf. Remark 3.1). Suppose for a contradiction that $i_1, \dots, i_k \in \{0, \dots, n\}$ with $k \leq n$ are precisely the indices i with $\hat{m}_i(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) =: M_0$. By Lemma 6.7 $t_j := z_{i_j}(\mathbf{w}^*)$ (for $j = 1, \dots, k$) belong to the interior of non-degenerate arcs. With this choice we can use the Perturbation Lemma 6.2 to slightly move $\mathbf{w}^* = (w_1, \dots, w_n)$ to $\mathbf{w}' = (w'_1, \dots, w'_n)$, $|\mathbf{w}' - \mathbf{w}^*| < \eta$ and achieve

$$\max_{j=1,\dots,k} \hat{m}_{i_j}(\mathbf{w}') < M_0,$$

while at the same time $\hat{m}_i(\mathbf{w}')$ for $i \neq i_j$, $j = 1, \dots, k$ do not increase too much (because by Proposition 3.3 the functions $\hat{m}_i$ are continuous), i.e.,

$$\max_{p=0,\dots,n} m_p(\mathbf{w}') = \max_{j=1,\dots,k} \hat{m}_{i_j}(\mathbf{w}') < M_0,$$

a contradiction.

The last assertion follows now immediately from Corollary 6.5. □

Corollary 6.9. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave, and either all satisfy (∞'_+), or all satisfy (∞'_-), or all belong to $\mathrm{C}^1(0, 2\pi)$.*

Let $S = S_\sigma$ be a simplex, and let $\mathbf{w}^* \in \overline{S}$ be an extremal node system for (3). Then the following assertions hold.

(a) *If $\mathbf{w}^* \in S$, then $\mathbf{w}^*$ is an equioscillation point.*
(b) *Even in case $\mathbf{w}^* \in \partial S$ we have that $\mathbf{w}^*$ is a weak equioscillation point.*
(c) *Furthermore, if also (∞) holds, then we have $\{m_0(\mathbf{w}^*), \dots, m_n(\mathbf{w}^*)\} \subseteq \{-\infty, M(S)\}$, with $m_j(\mathbf{w}^*) = -\infty$ iff $I_j(\mathbf{y})$ is degenerate.*

(d) *If* (∞') *holds, and* $\mathbf{w}^* \in \partial S$*, then there exists another simplex* $S' = S_\pi$ *with* $\mathbf{w}^* \in \overline{S} \cap \overline{S'}$ *with* $M(S') < M(S)$*, moreover* $\mathbf{w}^*$ *is not even a local (conditional) minimum within* $\overline{S'}$*.*

Proof. (a) When the extremal node system $\mathbf{w}^*$ lies in the interior of the simplex S, it is necessarily a local minimum point, hence the previous Proposition 6.8 applies.

(b) Let $\mathbf{w}^* = (w_1, \dots, w_n) \in \partial S$ and assume that $0 = w_0 = \dots = w_{i_0} < w_{i_0+1} = \dots = w_{i_0+i_1} < w_{i_0+i_1+1} = \dots < w_{i_0+\dots i_s+1} = w_{i_0+\dots i_s+2} = \dots = w_{i_0+\dots+i_s+i_{s+1}} = 2\pi (\equiv 0 = w_0)$ is the listing of nodes with the number of equal ones being exactly $i_0, i_1, \dots, i_s$. Thus $i_0 + \dots i_{s+1} = n$ with i_0, i_{s+1} maybe 0 but the other i_j at least 1, and the number of distinct nodes strictly in $(0, 2\pi)$ being s.

In between the equal nodes there are degenerate arcs I_k, where —in view of Corollary 6.4—the respective maximum m_k of the function $F(\mathbf{w}^*, \cdot)$ is strictly smaller, than one of the maximums on the neighboring non-degenerate arcs, whence m_k is also smaller than $\overline{m}(\mathbf{w}^*)$.

So in particular if $s = 0$ and there is only one non-degenerate arc $I_{i_0} = [0, 2\pi]$, with all the nodes merging to 0 and 2π, then weak equioscillation (of this one value m_{i_0}) trivially holds.

Next, assume that there exists at least one node $0 < w_k < 2\pi$, and let us now define a new system of s $(1 \leq s < n)$ nodes $\mathbf{y} = (y_1, \dots, y_s)$ with $y_j = w_{i_0+\dots+i_j}$ $(j = 1, \dots, s)$ extended the usual way by $y_0 = 0$. Note that we will thus have $0 = y_0 < y_1 < \dots < y_s < 2\pi$, and the arising s arcs between these nodes are exactly the same as the non-degenerate arcs determined by the node system $\mathbf{w}^*$.

Further, let us define new kernel functions $L_j := K_{i_0+\dots i_{j-1}+1} + \dots + K_{i_0+\dots+i_j}$ $(j = 1, \dots, s)$, and $L_0 = (K_0 + K_1 + \dots + K_{i_0}) + (K_{i_0+\dots+i_s+1} + \dots + K_{i_0+\dots+i_{s+1}})$. Obviously, the new $s+1$-element system $L_0, L_1, \dots, L_s$ consists of strictly concave kernels, either all satisfying (∞'), or all belonging to $\mathrm{C}^1(0, 2\pi)$, and now the node system $\mathbf{y}$ belongs to the interior of the respective s-dimensional simplex $\tilde{S}$.

Observe that by construction we now have

$$\tilde{F}(\mathbf{y}, t) = \sum_{j=0}^{s} L_j(t - y_j) = \sum_{i=0}^{n} K_i(t - w_i) = F(\mathbf{w}^*, t),$$

and so from the assumption that $\overline{m}(\mathbf{w}^*)$ is minimal within the simplex S, it also follows that $\sup_{t \in \mathbb{T}} \tilde{f}(\mathbf{y}, t)$ is minimal within $\tilde{S}$. Therefore, by part (a) the maximum values $\tilde{m}_j$ of the function $\tilde{F}$ on these non-degenerate arcs are all equal, and this was to be proven.

(c) is obvious once we have the weak equioscillation in view of (b).

(d) If we had $\mathbf{w}^*$ being a local conditional minimum point in each of the simplexes to the boundary of which it belongs, then altogether it would even be a local minimum point on $\mathbb{T}^n$. Then, by a similar argument as in (a), Proposition 6.8 would give $\mathbf{w}^* \in X$, contrary to the assumption. So there has to be some simplex S', containing $\mathbf{w}^*$ in $\partial S'$, where $\mathbf{w}^*$ is not a local conditional minimum point. Consequently, $M(S') < \overline{m}(\mathbf{w}^*) = M(S)$, whence the assertion follows. □

Corollary 6.10. *Suppose the kernels* $K_0, \dots, K_n$ *are strictly concave and either all satisfy* (∞'_+)*, or all satisfy* (∞'_-)*, or all belong to* $\mathrm{C}^1(0, 2\pi)$*. If* $\mathbf{w}$ *is an extremal node system for* (1)*, i.e.,*

$$\overline{m}(\mathbf{w}) = \min_{\mathbf{y} \in \mathbb{T}^n} \overline{m}(\mathbf{y}) = M,$$

then the nodes w_j $(j = 0, \dots, n)$ are pairwise different (i.e., $\mathbf{w} \in X$) and, moreover, $\mathbf{w}$ is an equioscillation point, i.e., we have

$$m_j(\mathbf{w}) = M \quad \text{for } j = 0, \dots, n.$$

7. Distribution of local maxima of $\underline{m}$

Lemma 7.1. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave. Let $S = S_\sigma$ be a simplex. Then $F(\mathbf{y}, s) : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$ restricted to the convex open set*

$$\mathcal{D} := \mathcal{D}_{\sigma,i} := \big\{(\mathbf{y}, s) : \mathbf{y} \in S \text{ and } s \in \operatorname{int} I_i(\mathbf{y})\big\}$$

is strictly concave.

Proof. First, note that the set $\mathcal{D} := \mathcal{D}_{\sigma,i}$ is a convex subset of $\mathbb{T}^{n+1}$. Indeed, let $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ and $t \in [0, 1]$. Then $x_i < x_\ell$ and $y_i < y_\ell$ implies $tx_i + (1-t)y_i < tx_\ell + (1-t)y_\ell$, and $x_i < r < x_\ell$, $y_i < s < y_\ell$ entails also $tx_i + (1-t)y_i < tr + (1-t)s < tx_\ell + (1-t)y_\ell$.

Now consider the sum representation of F and concavity of each K_ℓ to conclude

$$\begin{aligned} F(t(\mathbf{x}, r) + (1-t)(\mathbf{y}, s)) &= \sum_{\ell=0}^{n} K_\ell(tr + (1-t)s - (tx_i + (1-t)y_i)) \\ &\geq \sum_{\ell=0}^{n} tK_\ell(r - x_i) + (1-t)K_\ell(s - (1-t)y_i) \\ &= tF(\mathbf{x}, r) + (1-t)F(\mathbf{y}, s). \end{aligned} \tag{14}$$

This shows concavity of F. To see strict concavity suppose $t \neq 0, 1$ and that $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ are different points. If $r \neq s$, then using strict concavity of K_0 we must have $K_0(tr + (1-t)s) > tK_0(r) + (1-t)K_0(s)$, and if $r = s$, but $x_\ell \neq y_\ell$ for some $1 \leq \ell \leq n$, then using strict concavity of K_ℓ (and also that $r = s$) it follows that $K_\ell(tr + (1-t)s - (tx_\ell + (1-t)y_\ell)) = K_\ell(s - (tx_\ell + (1-t)y_\ell)) > tK_\ell(s - x_\ell) + (1-t)K_\ell(s - y_\ell)$. Altogether we obtain strict inequality in (14). □

Proposition 7.2. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave. Then for all $i = 0, 1, \dots, n$, the functions $m_i(\mathbf{y}) : S \to \mathbb{R}$ are also strictly concave. As a consequence,*

$$\underline{m} : S \to [-\infty, \infty), \quad \underline{m}(\mathbf{y}) := \min_{j=0,\dots,n} m_j(\mathbf{y})$$

is a strictly concave function.

Proof. Take $\mathbf{x}, \mathbf{y} \in S$ and abbreviate $w_i := z_i(\mathbf{x})$, $v_i := z_i(\mathbf{y})$ (the unique maximum points of $F(\mathbf{x}, \cdot)$ and $F(\mathbf{y}, \cdot)$ in $I_i(\mathbf{x})$ and $I_i(\mathbf{y})$, respectively, i.e., $m_i(\mathbf{y}) = F(\mathbf{y}, v_i)$, $m_i(\mathbf{x}) = F(\mathbf{x}, w_i)$). Let $\zeta(t) := z_i(t\mathbf{x} + (1-t)\mathbf{y})$, $\zeta(0) = v_i$, $\zeta(1) = w_i$. According to the previous Lemma 7.1 the function F is strictly concave, hence for different $\mathbf{x} \neq \mathbf{y}$ we necessarily have

$$F(t(\mathbf{x}, w_i) + (1-t)(\mathbf{y}, v_i)) > tF(\mathbf{x}, w_i)) + (1-t)F(\mathbf{y}, v_i) = tm_i(\mathbf{x}) + (1-t)m_i(\mathbf{y}).$$

Here the left hand side can be written as $F(t\mathbf{x} + (1-t)\mathbf{y}, \omega(t))$ with $\omega(t) = tw_i + (1-t)v_i \in I_i(t\mathbf{x} + (1-t)\mathbf{y})$. Thus by the definition of m_i we have

$$m_i(t\mathbf{x} + (1-t)\mathbf{y}) = \max_{s \in I_i(t\mathbf{x}+(1-t)\mathbf{y})} F(t\mathbf{x} + (1-t)\mathbf{y}, s) \geq F(t(\mathbf{x}, w_i) + (1-t)(\mathbf{y}, v_i)).$$

Hence, the previous considerations yield even $m_i(t\mathbf{x} + (1-t)\mathbf{y}) > tm_i(\mathbf{x}) + (1-t)m_i(\mathbf{y})$, whence the first assertion follows. Since minimum of strictly concave functions is strictly concave, the last assertion follows, too. □

Remark 7.3. It is easy to see then that the functions $\underline{m}, m_0, \dots, m_n$ are all concave on any convex subset of $\overline{S}$ on which they are finite valued.

Corollary 7.4. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave, and let $S := S_\sigma$ be a simplex.*

(a) *In $\overline{S}$ the function $\underline{m}$ has a* unique *global maximum point $\mathbf{y}_*$, and no local minimum point in S.*
(b) *If the kernels satisfy (∞), then $\mathbf{y}_* \in S$.*
(c) *There is no other point in $\overline{S}$ majorizing $\mathbf{y}_*$ than $\mathbf{y}_*$ itself.*

Proof. (a) Since $\underline{m}$ is strictly concave on S and continuous on $\overline{S}$ the assertion is evident.

(b) Under condition (∞) we have $\underline{m}|_{\partial S} = -\infty$, whence the assertion follows.

(c) If $\mathbf{x} \in \overline{S}$ with $m_j(\mathbf{x}) \geq m_j(\mathbf{y}_*)$ for all $j = 0, 1, \ldots, n$, then for the minimum $\underline{m} := \min_{j=0,\ldots,n} m_j$ we also have $\underline{m}(\mathbf{x}) \geq \underline{m}(\mathbf{y}_*)$, whence $\mathbf{x}$ is also a maximum point, and by uniqueness (part (a)) this entails $\mathbf{x} = \mathbf{y}_*$. □

8. Local properties of concave potentials

Corollary 8.1. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave.*

(a) *Let $\mathbf{y} \in S = S_\sigma$, $\mathbf{x} \in \overline{S}$, $\mathbf{x} \neq \mathbf{y}$ be such that $\mathbf{x}$ majorizes $\mathbf{y}$, i.e., $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ for each $j = 0, \ldots, n$. Then there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ such that for every $j = 0, \ldots, n$*
$$m_j(\mathbf{y} - t\mathbf{a}) < m_j(\mathbf{y}) \qquad \big(t \in (0, \delta)\big),$$
$$m_j(\mathbf{y} + t\mathbf{a}) > m_j(\mathbf{y}) \qquad \big(t \in (0, \delta)\big).$$
In particular, the Local Strict Non-Majorization and Non-Minorization Properties fail at $\mathbf{y}$.
(b) *The Local Non-Majorization Property, the Local Non-Minorization Property, the Local Comparison Property and the Comparison Property are all equivalent, also together with their strict versions.*
(c) *If one has the Local Strict Non-Minorization Property at an* interior *point $\mathbf{y} \in S$, then also the Local Non-Majorization Property holds at the same point $\mathbf{y}$.*

Proof. (a) Take $\mathbf{a} := \mathbf{x} - \mathbf{y}$ and let
$$\mathbf{y}_t := \mathbf{y} + t\mathbf{a} = (1-t)\mathbf{y} + t\mathbf{x}.$$
For sufficiently small $\delta > 0$ we have $\mathbf{y}_t \in S$ for every $(-\delta, 1]$ (since S is convex and open). By the strict concavity of m_j we obtain for $t \in [0, 1]$ that
$$m_j(\mathbf{y}_t) > (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \geq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y})$$
and for $t \in (-\delta, 0)$
$$m_j(\mathbf{y}_t) < (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \leq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y}).$$
This proves the first assertion.

(b) The Comparison Property evidently implies the Local Comparison Property and that implies further the Local Non-Minorization and Non-Majorization Properties. The already established first assertion (a) provides the converse implications even if we start with the even weaker Local Strict Non-Minorization or Non-Majorization Properties.

(c) Follows directly from (a) by contraposition. □

Proposition 8.2. *Suppose that the kernel functions $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a fixed simplex and let $\mathbf{e}, \mathbf{f} \in \overline{S}$ be two different equioscillation points.*

(a) *Then we have $M(S) < m(S)$, and the Sandwich Property (see Definition 5.7 and Remark 5.6) fails.*
(b) *If $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$ and $\mathbf{e} \in S$, then the Strict Local Non-Majorization and Non-Minorization Properties fail to hold at $\mathbf{e}$.*
(c) *If the kernels either all satisfy (∞'_+), or all satisfy (∞'_-), or are all C^1, then the Comparison Property fails (see Definition 5.9).*

Proof. For definiteness let us assume, as we may, that $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$.

(a) If $\overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f})$, then we obviously have $M(S) \le \overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f}) = \underline{m}(\mathbf{f}) \le m(S)$. If, on the other hand, $\overline{m}(\mathbf{e}) = \overline{m}(\mathbf{f})$, then for the point $\mathbf{g} := \frac{1}{2}(\mathbf{e}+\mathbf{f}) \in \overline{S}$ by the strict concavity we find $m_j(\mathbf{g}) > \frac{1}{2}(m_j(\mathbf{e}) + m_j(\mathbf{f}) = \overline{m}(\mathbf{e})$ for all $j = 0, \dots, n$, whence also $\underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e})$ and thus also $m(S) \ge \underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e}) \ge M(S)$. In both cases the Sandwich Property must fail, because by Remark 5.6 this property is equivalent to $M(S) \ge m(S)$.

(b) If $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$, then $\mathbf{f}$ majorizes $\mathbf{e}$, so Corollary 8.1 (a) finishes the proof.

(c) The assertion follows from part (b) and Remark 5.10, since under the conditions equioscillation points must belong to S. □

Corollary 8.3. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave. Let $S := S_\sigma$ be a simplex and let $\mathbf{y}^* \in S$ be a local minimum of $\overline{m}$.*
(a) *Then there exists no other point different from $\mathbf{y}^*$ in $\overline{S}$ majorizing $\mathbf{y}^*$.*
(b) *Suppose the kernels either all satisfy (∞'_+), or all satisfy (∞'_-), or are all C^1. Then in $\overline{S}$ there is no other local minimum point of $\overline{m}$ than $\mathbf{y}^*$. In particular, $\overline{m}$ has a global minimum point at $\mathbf{y}^*$ in $\overline{S}$.*

Proof. (a) Suppose $\mathbf{x} \in \overline{S}$ majorizes $\mathbf{y}^*$ and $\mathbf{x} \ne \mathbf{y}$. Then by Corollary 8.1 there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ with $m_j(\mathbf{y}^* - t\mathbf{a}) < m_j(\mathbf{y}^*)$ for every $t \in (0, \delta)$ and $j = 0, \dots, n$. Hence $\mathbf{y}^*$ cannot be a local minimum point for $\overline{m}$.

(b) Under condition (∞'_+) (or (∞'_-)) local minimum points of $\overline{m}$ are also equioscillation points according to Proposition 6.8, so at least one of two such points majorizes the other. But then by part (a) the two point must be equal. □

To sum up our findings we can state:

Proposition 8.4. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy (∞'_+), or all satisfy (∞'_-), or all are C^1. Let $S := S_\sigma$ be a simplex. If $\overline{m}$ has a local minimum point $\mathbf{y}^* \in S$, then $\mathbf{y}^*$ is a unique point of equioscillation in $\overline{S}$, and $\underline{m}$ has there its (unique, global) maximum. In particular, then $M(S) = m(S)$. Moreover, the Sandwich Property holds true in S. Furthermore, the Singular Non-Majorization and the Non-Minorization Properties hold on S.*

Proof. Let $\mathbf{y}_* \in \overline{S}$ be the (unique, global) maximum point of $\underline{m}$. Obviously, $\min_{j=0,\dots,n} m_j(\mathbf{y}_*) = \underline{m}(\mathbf{y}_*) \ge \underline{m}(\mathbf{y}^*)$. By assumption we can apply Proposition 6.8 to conclude that $\mathbf{y}^*$ is an equioscillation point, i.e., $\underline{m}(\mathbf{y}^*) = \overline{m}(\mathbf{y}^*) = m_j(\mathbf{y}^*)$ for $j = 0, \dots, n$. Thus we find that $\mathbf{y}_*$ majorizes the point $\mathbf{y}^*$. According to Corollary 8.3 (a) this is not possible unless $\mathbf{y}_* = \mathbf{y}^*$. Therefore we obtain $M(S) = m(S)$, and Remark 5.6 yields the Sandwich Property. If $\mathbf{e} \in \overline{S}$ is another equioscillation point, then $\overline{m}(\mathbf{e}) \ge \overline{m}(\mathbf{y}^*)$ (since $\mathbf{y}^*$ is a minimum point). By Corollary 8.2 this would imply $M(S) < m(S)$, which is nonsense. Since $\mathbf{y}^*$ is the unique equioscillation point in $\overline{S}$, by Corollary 8.3 there is no point majorizing it. But also $\mathbf{y}^*$ is the unique global minimum point of $\overline{m}$, so there is no point in $\overline{S}$ minorizing it. □

9. The Difference Jacobi Property

Proposition 9.1. *Suppose that $K_0, \ldots, K_n$ are C^2 with $K_j'' < 0$ ($j = 0, \ldots, n$), and let S be a simplex. For $j = 0, \ldots, n$ the functions $m_j(\mathbf{y})$ are continuously differentiable in S and*

$$\frac{\partial m_j}{\partial y_r}(\mathbf{y}) = -K_r'\big(z_j(\mathbf{y}) - y_r\big) \quad \text{for } r = 1, \ldots, n. \tag{15}$$

Proof. Let $\mathbf{y} \in S$ be fixed. Recall that $t = z_j(\mathbf{y})$ is the unique maximum point in $I_j(\mathbf{y})$, i.e., with $F'(\mathbf{y}, t) = 0$. Since

$$F''(\mathbf{y}, t) = K_0''(t) + \sum_{j=1}^{n} K_j''(t - y_j) < 0$$

by the implicit function theorem, for a suitable neighborhood $U \times V \subseteq S \times I_j(\mathbf{y})$ we have that $z_j : U \to V$ is continuously differentiable. Since $m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$ we obtain that m_j, too is continuously differentiable and

$$\begin{aligned}\frac{\partial m_j}{\partial y_r}(\mathbf{y}) &= \frac{\partial}{\partial y_r}\Big(F\big(\mathbf{y}, z_j(\mathbf{y})\big)\Big) = \frac{\partial F}{\partial y_r}\big(\mathbf{y}, z_j(\mathbf{y})\big) + \frac{\partial}{\partial t}F\big(\mathbf{y}, t\big)|_{t=z_j(\mathbf{y})}\frac{\partial}{\partial y_r}z_j(\mathbf{y}) \\ &= -K_r'\big(z_j(\mathbf{y}) - y_r\big).\end{aligned}$$

□

As a consequence, the Jacobian $D\mathbf{m}$ of $\mathbf{m} = (m_0, \ldots, m_n)^\top$ is

$$D\mathbf{m} = \begin{array}{cc} & r \\ j & \begin{pmatrix} & \vdots & \\ \cdots & -K_r'\big(z_j(\mathbf{y}) - y_r\big) & \cdots \\ & \vdots & \end{pmatrix}\end{array} \tag{16}$$

where $r = 1, \ldots, n$ and $j = 0, \ldots, n$.

For a given permutation σ of $\{1, \ldots, n\}$ let us consider the mapping Δ_σ defined by

$$\Delta_\sigma(\mathbf{y}) := \big(m_{\sigma(1)}(\mathbf{y}) - m_{\sigma(0)}(\mathbf{y}), \ldots, m_{\sigma(n)}(\mathbf{y}) - m_{\sigma(n-1)}(\mathbf{y})\big)^\top. \tag{17}$$

Its Jacobian $D\Delta_\sigma$ is

$$D\Delta_\sigma(\mathbf{y}) = \begin{array}{cc} & r \\ j & \begin{pmatrix} & \vdots & \\ \cdots & -K_r'\big(z_{\sigma(j)}(\mathbf{y}) - y_r\big) + K_r'\big(z_{\sigma(j-1)}(\mathbf{y}) - y_r\big) & \cdots \\ & \vdots & \end{pmatrix}\end{array} \tag{18}$$

where $r = 1, \ldots, n$ and $j = 1, \ldots, n$.

Proposition 9.2. *Suppose that for each $j = 0, \ldots, n$ the kernel K_j is C^2 with $K_j'' < 0$. Let $S = S_\sigma$ be a simplex and let $\mathbf{y} \in S$. The Jacobian A of*

$$\Delta_\sigma(\mathbf{y}) := \big(m_{\sigma(1)}(\mathbf{y}) - m_{\sigma(0)}(\mathbf{y}), \ldots, m_{\sigma(n)}(\mathbf{y}) - m_{\sigma(n-1)}(\mathbf{y})\big)^\top. \tag{19}$$

is non-singular. That is, on S, we have the Difference Jacobi Property.

Proof. For the sake of brevity we may suppose $\sigma = \mathrm{id}$, i.e., $\sigma(j) = j$, otherwise we can relabel the kernels K_j accordingly. We abbreviate $z_j := z_j(\mathbf{y})$ and have

$$z_{r-1} < y_r < z_r \quad \text{for } r = 1, \ldots, n.$$

First, we show that $-A$ is a so-called Z-matrix, that is, the entries are non-negative on the diagonal and are non-positive off the diagonal. On the diagonal the entries are $-K_r'(z_r - y_r) + K_r'(z_{r-1} - y_r)$, $r = 1, \ldots, n$. Since $z_{r-1} < y_r < z_r$, $z_r < 2\pi$

and $z_{r-1} > 0$, we obtain $z_{r-1} - y_r < 0 < z_r - y_r$ and $z_r - y_r < 2\pi + z_{r-1} - y_r$. Now using the 2π periodicity of K_r' and that K_r' is strictly monotone decreasing, we obtain $K_r'(z_{r-1} - y_r) < K_r'(z_r - y_r)$, that is, $-K_r'(z_r - y_r) + K_r'(z_{r-1} - y_r) < 0$.

For $i < r$ we have $z_{i-1} < z_i \le z_{r-1} < y_r$. Therefore, $-2\pi < z_{i-1} - y_r < z_i - y_r < 0$ and using that K_r' is strictly monotone decreasing and 2π periodic, we can write

$$-K_r'(z_i - y_r) + K_r'(z_{i-1} - y_r) > 0.$$

Therefore the elements above the diagonal are strictly positive.

If $i > r$, then $y_r < z_r \le z_{i-1} < z_i$. As above, $0 < z_{i-1} - y_r < z_i - y_r < 2\pi$ and using that K_r' is strictly monotone decreasing, we can write

$$-K_r'(z_i - y_r) + K_r'(z_{i-1} - y_r) > 0,$$

meaning that the entries below the diagonal are strictly positive, too.

We now show that the column sums of $-A$ are strictly positive, i.e., with $\mathbf{x} = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$ we have $-A^\top \mathbf{x}$ is a strictly positive vector. This means then that $-A^\top$ satisfies condition I27 in [3] (see page 136). Hence by [3, Theorem 2.3] it will follow that $-A^\top$ is an M-matrix and is non-singular, this yielding also the non-singularity of A.

Now indeed, the sum of the r^{th} column of A is telescopic

$$\sum_{i=1}^{n} -K_r'(z_i - y_r) + K_r'(z_{i-1} - y_r) = -K_r'(z_n - y_r) + K_r'(z_0 - y_r).$$

Since $0 < z_0 < y_r < z_n < 2\pi$, we have $0 < z_n - y_r < 2\pi + z_0 - y_r < 2\pi$. Since K_r' is strictly decreasing and 2π periodic, it follows $-K_r'(z_n - y_r) + K_r'(z_0 - y_r) < 0$. The proof is hence complete. □

Corollary 9.3. *Suppose that for each $j = 0, \dots, n$ the kernel K_j is C^2 with $K_j'' < 0$ and satisfies (∞). Let $S = S_\sigma$ be a simplex. The mapping $\Delta_\sigma : S \to \mathbb{R}^n$ is then a homeomorphism.*

Proof. By Proposition 9.2 the mapping Δ_σ is locally a homeomorphism (onto its image), and by Proposition 5.1 it carries the boundary ∂S onto the boundary of the one-point compactified $\mathbb{R}^n$. By a well-known result, see e.g., [13, Chap. VI], Δ_σ is a homeomorphism. □

Here is a proof of existence (and even uniqueness) of equioscillation points in a given simplex under the special conditions of this section.

Corollary 9.4. *Suppose that for each $j = 0, \dots, n$ the kernel K_j is C^2 with $K_j'' < 0$ and satisfies (∞). Then all equioscillation points belong to some (open) simplex, and in each simplex $S = S_\sigma$ there is a unique equioscillation point.*

Proof. An equioscillation point must belong to X according to Corollary 6.5.

In a fixed simplex S_σ, an equioscillation point is the inverse image of $\mathbf{0} \in \mathbb{R}^n$ under the homeomorphism Δ_σ from Corollary 9.3. □

10. Equioscillation points

In this section we prove the existence of equioscillation points in each simplex $S = S_\sigma$, and discuss the uniqueness of such points. The main tool will be the approximation of kernels by a sequence of kernel functions having more satisfactory properties, so the arguments rely on the results of Section 4.

Lemma 10.1. *Suppose that $K_0, \dots, K_n$ are strictly concave kernel functions and a sequence of strictly concave kernel functions $(K_j^{(k)})_{k\in\mathbb{N}}$ converges uniformly (in the extended sense) to K_j as $k \to \infty$. Let $\mathbf{e}^{(k)} \in S$ be equioscillation points for the system of kernels $K_j^{(k)}$, $j = 0, \dots, n$. Then any accumulation point $\mathbf{e} \in \overline{S}$ of the sequence $(\mathbf{e}^{(k)})_{k\in\mathbb{N}}$ is an equioscillation point of the system K_j, $j = 0, \dots, n$.*

Proof. By passing to a subsequence we may assume that $\mathbf{e}^{(k)} \to \mathbf{e} \in \overline{S}$. By assumption and by Proposition 4.3 $m_j^{(k)} \to m_j$ uniformly on $\overline{S}$ as $k \to \infty$. It follows that $m_j^{(k)}(\mathbf{e}_k) \to m_j(\mathbf{e})$ as $k \to \infty$, so $\mathbf{e} \in \overline{S}$ is an equioscillation point. □

We need another lemma in order to apply the previous result.

Lemma 10.2. *Let $f : [0, 1) \to \mathbb{R}$ be a strictly convex, increasing function. Then to each $\varepsilon > 0$ there exists another strictly convex increasing function $g : [0, 1) \to \mathbb{R}$ such that $g \in \mathrm{C}^2[0, 1)$, $g'' > 0$ on $[0, 1)$, and $f(x) \le g(x) \le f(x) + \varepsilon$ all over $[0, 1)$.*

Proof. The proof is fairly standard, therefore we skip it now. □

Theorem 10.3. *Suppose that for each $j = 0, \dots, n$ the kernels K_j are strictly concave. Then for each simplex $S = S_\sigma$ there exists an equioscillation point in $\overline{S}$. Moreover, if the kernels are either all in $\mathrm{C}^1(0, 2\pi)$ or all satisfy (∞'_+), or all satisfy (∞'_-), then any equioscillation point is in the open simplex S.*

Proof. Before starting the proof, first let us make it clear that to any function K, which is strictly concave on $(0, 2\pi)$, and to any $\varepsilon > 0$, there exists an approximating function $k(x)$ with the properties that k is strictly concave on $(0, 2\pi)$, $K > k > K - \varepsilon$ on $(0, 2\pi)$, and $k(0) > K(0) - \varepsilon$ and $k(2\pi) > K(2\pi) - \varepsilon$ (whether these limits are finite or $-\infty$) and $k \in \mathrm{C}^2(0, 2\pi)$ with $k''(x) < 0$ on $(0, 2\pi)$.

This approximation is indeed possible, for given $\varepsilon > 0$ and a given (strictly) concave function $K : (0, 2\pi) \to \mathbb{R}$ satisfying (∞), we can chose a maximum point $c \in (0, 2\pi)$, and consider the intervals ($[c, 2\pi)$ and $(0, c]$ separately: applying Lemma 10.2 right next for $-K((x - c)/(2\pi - c))$ and $-K((c - x)/c)$ separately provides an approximating strictly concave kernel function $k \in \mathrm{C}^2((0, 2\pi \setminus \{c\})$ with $k'' < 0$ and $K - \varepsilon < k < K$. (The simple discontinuity possibly arising at c can then easily be cured). We split the proof of Theorem 10.3 into several steps.

Step 1. First, let us suppose that all the kernel functions $K_0, \dots, K_n$ satisfy (∞). By Lemma 10.2 and by what is said above we can take a sequence $(K_i^{(k)})_{k\in\mathbb{N}}$ of C^2 strictly concave functions satisfying $K_i'' < 0$ and converging strongly uniformly to the functions K_i, see Section 4. Note that hence we also require that $K_j^{(k)}$ satisfy (∞). According to Corollary 9.4 each system $K_j^{(k)}$,$j = 0, \dots, n$, has an (unique) equioscillation point $\mathbf{e}^{(k)}$. By Lemma 10.1 any accumulation point $\mathbf{e}$ of this sequence (and, by compactness, there is one) is an equioscillation point. Finally, by Corollary 6.5 an equioscillation point is necessarily inside S. This concludes the proof for the special case when all the kernels satisfy (∞).

Step 2. Now let us consider the case when the kernels are strictly concave but satisfy (∞') only. Let us fix the auxiliary functions $L_k(x) := \log_-(k|x|)$, which is a concave function on $(0, \pi)$ and on $(-\pi, 0)$ with singularity at 0 and takes non-positive values. We extend these functions to $\mathbb{R}$ periodically. For $k \in \mathbb{N}$ and $j = 0, \dots, n$ define $K_i^{(k)} := L_k + K_i$. Then $K_j^{(k)} \uparrow K_j$ on $\mathbb{T} \setminus \{0\}$. By Step 1, for each $k \in \mathbb{N}$ there is an equioscillation point $\mathbf{e}^{(k)}$ for the system $K_j^{(k)}$, $j = 0, \dots, n$. By passing to a subsequence we can assume $\mathbf{e}^{(k)} \to \mathbf{e} \in \overline{S}$. For $j \in \{0, \dots, n\}$ we have

$$m_j^{(k)}(\mathbf{e}^{(k)}) = \max_{t\in I_j(\mathbf{e}^{(k)})} F^{(k)}(\mathbf{e}^{(k)}, t) \le \max_{t\in I_j(\mathbf{e}^{(k)})} F(\mathbf{e}^{(k)}, t) = m_j(\mathbf{e}^{(k)}).$$

Since m_j is continuous on $\overline{S}$, we obtain

$$\limsup_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \le m_j(\mathbf{e}). \tag{20}$$

Suppose that the arc $I_j(\mathbf{e})$ is non-degenerate. Then

$$m_j^{(k)}(\mathbf{e}^{(k)}) = \max_{t\in I_j(\mathbf{e}^{(k)})} F^{(k)}(\mathbf{e}^{(k)},t) \ge F^{(k)}(\mathbf{e}^{(k)}, z_j(\mathbf{e})) = F(\mathbf{e}^{(k)}, z_j(\mathbf{e})),$$

if $k \ge k_0$ sufficiently large, since by construction $K_j(t) = K_j^{(k)}(t)$ for $t \notin [-\frac{1}{k}, \frac{1}{k}]$. This implies

$$\liminf_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \ge \liminf_{k\to\infty} F(\mathbf{e}^{(k)}, z_j(\mathbf{e})) = F(\mathbf{e}, z_j(\mathbf{e})) = m_j(\mathbf{e}). \tag{21}$$

So the proof of Step 2 is complete if $\mathbf{e} \in S$. Finally, we show that $\mathbf{e} \in \partial S$ is impossible. Indeed, if there is a degenerate arc in $I_j(\mathbf{e})$, then by Corollary 6.4 there is a neighboring non-degenerate arc $I_i(\mathbf{e})$ such that $m_i(\mathbf{e}) > m_j(\mathbf{e})$. But then by (20) and (21)

$$m_j(\mathbf{e}) \ge \limsup_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \ge \liminf_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) = \liminf_{k\to\infty} m_i^{(k)}(\mathbf{e}^{(k)}) \ge m_i(\mathbf{e}),$$

a contradiction.

Step 3. Finally, we suppose only that $K_0, \dots, K_n$ are strictly concave kernel functions. We now take the functions $L_k(x) := (\sqrt{|x|} - 1/k)_-$, which is negative only for $-1/k^2 < x < 1/k^2$ and zero otherwise, and converges uniformly to zero. Restricting L_k to $[-\pi, \pi)$ and then extending it periodically we thus obtain a function on $\mathbb{T}$ which is concave on $(0, 2\pi)$ and converges to 0 uniformly. Note that $\lim_{x\to 0\pm 0} L_k'(x) = \mp\infty$, hence the perturbed system of kernels $K_j^{(k)} := K_j + L_k$, $j = 0, \dots, n$, satisfies (∞'). Again, in view of the already proven Case 2, there exist some equioscillation points $\mathbf{e}^{(k)}$ for the system $K_j^{(k)}$, $j = 0, \dots, n$, and by compactness, there exists an accumulation point $\mathbf{e} \in \overline{S}$ of the sequence $(\mathbf{e}^{(k)})_{k\in\mathbb{N}}$. By uniform convergence of the kernels we can apply Lemma 10.1 to conclude that $\mathbf{e}$ is an equioscillation point of the system K_j, $j = 0, \dots, n$.

It remains to prove that $\mathbf{e} \in S$ if the additional assumptions are fulfilled, but this has been done in Corollary 6.5. □

Corollary 10.4. *Let the kernel functions $K_0, \dots, K_n$ be strictly concave. Then in any simplex $S = S_\sigma$ the Equioscillation Property holds, and we have $M(S) \le m(S)$.*

Corollary 10.5. *Let the kernel functions $K_0, \dots, K_n$ be strictly concave and let $S = S_\sigma$ be a simplex. Suppose that $M(S) = m(S)$. Then there is $\mathbf{w}_* \in \overline{S}$ with $m(S) = \underline{m}(\mathbf{w}_*)$ and $\mathbf{w}_*$ is the unique equioscillation point in $\overline{S}$.*

Proof. Let $\mathbf{e} \in \overline{S}$ be an equioscillation point (Corollary 10.4), and let $\mathbf{w}_* \in \overline{S}$ be such that $\underline{m}(\mathbf{w}_*) = m(S)$ (Proposition 3.11). Since $\underline{m}(\mathbf{e}) = \overline{m}(\mathbf{e}) \ge M(S) = m(S) = \underline{m}(\mathbf{w}_*)$. We have that $\mathbf{e}$ is also a maximum point of $\underline{m}$, and that $\underline{m}(\mathbf{e}) = M(S)$. By Corollary 7.4 $\mathbf{e} = \mathbf{w}_*$, and by Proposition 8.2 the equioscillation point is unique. □

11. Summary and conclusions

Theorem 11.1. *Suppose the kernel functions $K_0, K_1, \dots, K_n$ are strictly concave and either all satisfy (∞'_+), or all satisfy (∞'_-), or all are C^1. Then there is $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \dots, w_n)$ with*

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y}, t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) $\mathbf{w}^*$ *is an equioscillation point, i.e.,* $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.
(b) $\mathbf{w}^* \in S := S_\sigma$ *for some simplex, i.e., the nodes in* $\mathbf{w}^*$ *are different, and*

$$\inf_{\mathbf{y}\in S} \max_{j=0,\dots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y},t) = M = \sup_{\mathbf{y}\in S} \min_{j=0,\dots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y},t).$$

(c) *We have the Sandwich Property on* S*, i.e., for each* $\mathbf{x},\mathbf{y} \in S$

$$\underline{m}(\mathbf{x}) \le M \le \overline{m}(\mathbf{y}).$$

Below we present examples showing that on different simplexes we may have different $\overline{m}$ values, and show that the previous result is sharp.

Example 11.2. Consider the functions

$$K(x) := \pi - |x-\pi| \qquad \text{for } x \in [0,2\pi]$$
$$Q(x) := |x|(2\pi - |x|) \qquad \text{for } x \in [0,2\pi],$$

and extend them periodically to $\mathbb{R}$. We take $K_0 = K_1 = K$ and $K_2 = K_3 = \varepsilon Q$ where $\varepsilon > 0$ to be determined later.

Case 1. We consider the simplex S_σ for $\sigma = (213)$, and the node system $y_0 = 0$, $y_1 = \pi$, $y_2 = \frac{\pi}{2}$, $y_3 = \frac{3\pi}{2}$. Then we have

$$F(\mathbf{y},t) = K_0(t) + K_1(t-y_1) + K_2(t-y_2) + K_3(t-y_3) = \pi + \varepsilon Q(t-\tfrac{\pi}{2}) + \varepsilon Q(t-\tfrac{3\pi}{2}).$$

It is easy to see that

$$m_0(\mathbf{y}) = F(\mathbf{y},0) = \max_{t\in[0,\frac{\pi}{2}]} F(\mathbf{y},t) = \pi + 3\varepsilon\tfrac{\pi^2}{2},$$

and by symmetry $m_0(\mathbf{y}) = m_1(\mathbf{y}) = m_2(\mathbf{y}) = m_3(\mathbf{y})$, i.e., $\mathbf{y}$ is an equioscillation point. Now, let $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ be strictly concave, symmetric, satisfying the condition (∞') and

$$K_j^{(k)} \to K_j \quad \text{uniformly as } k\to\infty \text{ for } j = 0,1,2,3.$$

For example one can take

$$K_j^{(k)}(x) = K_j(x) + \frac{1}{k}\sqrt{\pi^2 - (x-\pi)^2},$$

then by Dini's theorem one has the desired uniform convergence. Since the configuration of the kernel functions for this simplex is symmetric and the node system $\mathbf{y}$ is symmetric, it is easy to see that $\mathbf{y}$ is an equioscillation point in S also in the case of the kernels $K_j^{(k)}$. By symmetry considerations it is also evident that for the minimum point $\mathbf{w}^* = (w_0^*, w_1^*, w_2^*, w_3^*)$ of $\overline{m}^{(k)}$ in $\overline{S}$ we must have either $w_0^* = 0$, $w_1^* = \pi$, $w_2^* = \frac{\pi}{2}$, $w_3^* = \frac{3\pi}{2}$, i.e., $\mathbf{w}^* = y$, or $w_0^* = 0$, $w_1^* = \pi$, $w_2^* = 0$, $w_3^* = \pi$. The latter possibility can be excluded by simple calculation, so only $\mathbf{w}^* = \mathbf{y}$ is possible. Since this is an interior point of S, by Proposition 8.4 we then have $M^{(k)}(S) = \overline{m}^{(k)}(\mathbf{y})$, and by Proposition 4.3 we have $M^{(k)}(S) \to M(S)$ and $m_j^{(k)}(\mathbf{y}) \to m_j(\mathbf{y})$ as $k\to\infty$. So that $M(S) = m_j(\mathbf{y})$ for all $j = 0,1,2,3$.

Case 2. We consider the simplex $S_{\sigma'} = (123)$ and the node system $w_0 = 0$, $w_1 = \pi + (3-2\sqrt{2})\varepsilon\pi^2$, $w_2 = (2\sqrt{2}-2)\pi$, $w_3 = 2\pi$. We calculate $m_j(\mathbf{w})$ for $j = 0,1,2,3$. We write $s := w_1 - \pi = (3-2\sqrt{2})\varepsilon\pi^2$, $u = w_2/2 = (\sqrt{2}-1)\pi$, $v = (w_2+w_3)/2 = (\sqrt{2}+1)\pi$ and $H(t) = \varepsilon(Q(t-w_2) + Q(t-w_3))$. We have

$$\begin{aligned} m_0(\mathbf{w}) &= F(\mathbf{w},u) = \pi + s + H(u) \\ &= 2\pi + \varepsilon(3-2\sqrt{2})\pi^2 + 2\varepsilon Q((\sqrt{2}-1)\pi) \\ &= \pi + \varepsilon\pi^2\big(3 - 2\sqrt{2} + 2(\sqrt{2}-1)(3-\sqrt{2})\big) = \pi + \varepsilon\pi^2(6\sqrt{2}-7). \end{aligned}$$

$$\begin{aligned}
m_1(\mathbf{w}) = F(\mathbf{w}, \pi) &= \pi + s + H(\pi) = \\
&= \pi + (3 - 2\sqrt{2})\varepsilon\pi^2 + \varepsilon Q(\pi - (2 - 2\sqrt{2})\pi) + \varepsilon Q(\pi) \\
&= \pi + \varepsilon\pi^2\big(3 - 2\sqrt{2} + (3 - 2\sqrt{2})(2\sqrt{2} - 1) + 1\big) = \varepsilon\pi^2(6\sqrt{2} - 7).
\end{aligned}$$

$$\begin{aligned}
m_2(\mathbf{w}) = F(\mathbf{w}, v) &= \pi - s + H(v) = \\
&= \pi - (3 - 2\sqrt{2})\varepsilon\pi^2 + \varepsilon\pi^2(\sqrt{2} + 1)(1 - \sqrt{2}) + \varepsilon\pi^2(3 - \sqrt{2})(\sqrt{2} - 1) \\
&= \pi + \varepsilon\pi^2\big(-3 + 2\sqrt{2} + 1 + 4\sqrt{2} - 5\big) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7).
\end{aligned}$$

It is easy to see from construction that $m_3(\mathbf{w}) \le m_j(\mathbf{w})$ for $j = 0, 1, 2$, but the precise value of $m_3(\mathbf{w})$ can also be calculated:

$$\begin{aligned}
m_3(\mathbf{w}) = F(\mathbf{w}, 0) &= \pi - s + H(0) = \pi - (3 - 2\sqrt{2})\varepsilon\pi^2 + \varepsilon Q((2\sqrt{2} - 2)\pi) \\
&= \pi + \varepsilon\pi^2\big(-3 + 2\sqrt{2} + (4 - 2\sqrt{2})(2\sqrt{2} - 2)\big) \\
&= \pi + \varepsilon\pi^2(14\sqrt{2} - 19).
\end{aligned}$$

From the previous considerations we conclude

$$M(S_\sigma) = \overline{m}(\mathbf{y}) = \pi + \varepsilon\pi^2\frac{3}{2} > \pi + \varepsilon\pi^2(6\sqrt{2} - 7) = \overline{m}(\mathbf{w}) \ge M(S_{\sigma'}).$$

The phenomenon observed in the previous example can be present also for strictly concave kernels with the (∞) property. To see this one can start with the example from the above and approximate by kernels that have the mentioned properties and use the results of Sections 4.

Example 11.3. Consider the kernel functions K_0, K_1, K_2, K_3 from Example 11.2, and let $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ be strictly concave, symmetric, satisfying the condition (∞') and

$$K_j^{(k)} \to K_j \quad \text{uniformly as } k \to \infty \text{ for } j = 0, 1, 2, 3.$$

For example one can take

$$K_j^{(k)}(x) = K_j(x) + \frac{1}{k}\sqrt{\pi^2 - (x - \pi)^2},$$

then by Dini's theorem one has the desired uniform convergence. By Proposition 4.3 we have $M^{(k)}(S) \to M(S)$ for any simplex. So for sufficiently large $k \in \mathbb{N}$ we must have

$$M^{(k)}(S_\sigma) \ne M^{(k)}(S_{\sigma'}) \tag{22}$$

for the two different simplexes corresponding to the permutations $\sigma = (213)$ and $\sigma' = (123)$.

Example 11.4. Let K_0, K_1, K_2, K_3 be kernel functions satisfying (∞') with

$$M^{(k)}(S_\sigma) \ne M^{(k)}(S_{\sigma'})$$

for different simplexes S_σ and $S_{\sigma'}$ (see the preceding example). Let $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ be C^2, strictly concave, symmetric and satisfying the condition (∞) such that

$$K_j^{(k)} \to K_j \quad \text{locally uniformly on } (0, 2\pi)$$

as $k \to \infty$ for $j = 0, 1, 2, 3$ (use Lemma 10.2). By Theorem 4.6 we have $M^{(k)}(S) \to M^{(k)}(S)$ for any simplex. So for sufficiently large $k \in \mathbb{N}$ we must have

$$M^{(k)}(S_\sigma) \ne M^{(k)}(S_{\sigma'}).$$

Remark 11.5. Let K_0, K_1, K_2, K_3 be strictly concave C^2 kernel functions satisfying (∞) with

$$M(S_\sigma) > M(S_{\sigma'})$$

for different simplexes S_σ and $S_{\sigma'}$. Consider, for example, the situation of the preceding Example 11.4.

(a) Let $\mathbf{w}^* \in \mathbb{T}^3$ be a global minimum point of $\overline{m}$ on $\mathbb{T}^3$. Let S denote the simplex in which $\mathbf{w}^*$ lies. We have

$$M(\mathbb{T}^3) = m(S) = M(S) < M(S_{\sigma'}) \leq m(S_{\sigma'})$$

by Theorem 11.1 and by Corollary 10.4. This implies that

$$M(\mathbb{T}^3) < m(\mathbb{T}^3)$$

holds.

(b) By Corollary 7.4 $m(S_{\sigma'})$ is attained as the maximum of $\underline{m}$ in the interior of $S_{\sigma'}$. If $M(S_{\sigma'})$ is attained as a minimum of $\overline{m}$ in $S_{\sigma'}$, then $M(S_{\sigma'}) = m(S_{\sigma'})$ by Proposition 8.4. If $M(S_{\sigma'})$ is attained only on the boundary $\partial S_{\sigma'}$, then it is a weak equioscillation point and the unique equioscillation point in $S_{\sigma'}$ majorizes it.

Next, let us discuss the case when all kernel function are the same. Then the phenomenon in the previous example is not present anymore.

Theorem 11.6. *Suppose the kernel functions L, K are strictly concave and either both satisfy (∞'_+), or both satisfy (∞'_-), or they belong to $\mathrm{C}^1(0, 2\pi)$. Set*

$$F(\mathbf{y}, t) := L(t) + \sum_{j=1}^{n} K(t - y_j).$$

Then there is a unique $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y}, t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) *$\mathbf{w}^*$ is an equioscillation point, i.e., $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.*

(b) *The nodes $w_0, \ldots, w_n$ are different and $\mathbf{w}^*$ is an equioscillation point, i.e.,*

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

(c) *We have*

$$\inf_{\mathbf{y}\in\mathbb{T}^n} \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t) = M = \sup_{\mathbf{y}\in\mathbb{T}^n} \min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

(d) *We have the Sandwich Property on $\mathbb{T}$, i.e., for each $\mathbf{x}, \mathbf{y} \in \mathbb{T}^n$*

$$\underline{m}(\mathbf{x}) \leq M \leq \overline{m}(\mathbf{y}).$$

(e) *If $K = L$, then the points $w_0 = 0, w_1, \ldots, w_n$ lie equidistantly in $\mathbb{T}$.*

12. An application: a minimax problem on the torus

Similar results as in the next lemma appeared already in [8] and [9].

Lemma 12.1. *Let K be strictly concave an let $a, b > 0$, $x \in (0, 2\pi)$ be given. Then for each $x \in (0, 2\pi)$ for sufficiently small $\delta > 0$ we have that*

$$\frac{1}{a}K(t - (x + ah)) + \frac{1}{b}K(t - (x - bh)) < \frac{1}{a}K(t - x) + \frac{1}{b}K(t - x)$$

for each $t \in (0, x - b\delta) \cup (x + a\delta, 2\pi)$ and each $0 < h < \delta$.

Proof. Let $\delta > 0$ be so small that for $h \in (0, \delta)$ we have $x - bh > 0$ and $x + ah < 2\pi$. Then $K(\cdot - x)$ is strictly concave on the intervals $(0, x)$ and $(x, 2\pi)$. By strict concavity the difference quotients are strictly decreasing in both variables, so that for all $h \in (0, \delta)$ and $t \in (0, x - b\delta)$ and $t \in (x + a\delta, 2\pi)$

$$\frac{K(t-x) - K(t-(x-bh))}{-bh} < \frac{K(t-x) - K(t-(x+ah))}{ah}.$$

But this inequality is equivalent to the assertion. □

The next result generalizes that of Hardin, Kendall and Saff [9] in that extent that we do not assume the kernels to be symmetric about 0.

Corollary 12.2. *Let K be any concave kernel function, and let $0 = e_0 < e_1 < \cdots < e_n$ be the equidistant node system in $\mathbb{T}$. Consider*

$$F(\mathbf{y}, t) := K(t) + \sum_{j=1}^{n} K(t - y_j).$$

(a) *For $\mathbf{e} = (e_1, \ldots, e_n)$ we have*

$$\max_{t \in \mathbb{T}} F(\mathbf{e}, t) = M = \inf_{\mathbf{y} \in \mathbb{T}^n} \max_{t \in \mathbb{T}} F(\mathbf{y}, t),$$

i.e., $\mathbf{e}$ is a minimum point of $\overline{m}$. Moreover,

$$\inf_{\mathbf{y} \in \mathbb{T}^n} \max_{j=0,\ldots,n} m_j(\mathbf{y}) = M = m = \sup_{\mathbf{y} \in \mathbb{T}^n} \min_{j=0,\ldots,n} m_j(\mathbf{y}).$$

(b) *If K is strictly concave, then $\mathbf{e}$ is the unique (up to permutation of the coordinates) maximum point of $\underline{m}$ and the unique minimum point of $\overline{m}$.*

Proof. Since the permutation of the nodes is irrelevant we may restrict the consideration to the simplex $S := S_{\mathrm{id}}$, where id is the identical permutation.

(a) Approximate K uniformly by kernel functions $K^{(k)}$ satisfying (∞') (cf. Example 11.3). By Theorem 11.6, $M^{(k)} = \overline{m}^{(k)}(\mathbf{e})$ and $M^{(k)} = m^{(k)}$ and obviously $M^{(k)} = M^{(k)}(S)$, $m^{(k)} = m^{(k)}(S)$. By Proposition 4.3 we have $M^{(k)}(S) \to M(S) = M$, $m^{(k)}(S) \to m(S)$, $\underline{m}^{(k)}(\mathbf{e}) \to \underline{m}(\mathbf{e})$ and $\overline{m}^{(k)}(\mathbf{e}) \to \overline{m}(\mathbf{e})$. So that $\overline{m}(\mathbf{e}) = M = m(S) = M(S)$.

(b) Let $\mathbf{w}^* \in \overline{S}$ be a minimum point of $\overline{m}$. Suppose that $I_j(\mathbf{w}^*) = [y_j, y_i]$ is a degenerate arc. If $m_j(\mathbf{w}^*) < \overline{m}(\mathbf{w}^*) = M(S)$ held, then by an application of Lemma 12.1 and Corollary 3.6 we could arrive at a new node system $\mathbf{w}'$ with $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w}^*)$, which is impossible. So that for each $j \in \{0, 1, \ldots, n\}$ with $I_j(\mathbf{w}^*)$ degenerate we have $m_j(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) = M(S)$. Again by Lemma 12.1 we can conclude that on any non-degenerate arc $I_k(\mathbf{w}^*)$ we have $M(S) = m_k(\mathbf{w}^*)$. So that altogether $\mathbf{w}^*$ is an equioscillation point. Since by part (a) we have $m(S) = M(S)$, the equioscillation point is unique by Corollary 10.5. We conclude $\mathbf{w}^* = \mathbf{e}$, hence uniqueness follows. □

13. An application: generalized polynomials and Bojanov's result

We use the following form of our main theorem.

Theorem 13.1. *Suppose the kernel function K is strictly concave and either satisfies (∞'_+), or (∞'_-), or is C^1. Let $a_0, a_1, \ldots, a_n > 0$, set $K_j := a_j K$ and*

$$F(\mathbf{y}, t) := K_0(t) + \sum_{j=1}^{n} K_j(t - y_j) = a_0 K(t) + \sum_{j=1}^{n} a_j K(t - y_j).$$

Let $S = S_\sigma$ be a simplex. Then there is a unique $\mathbf{w}^* \in S$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with

$$M(S) := \inf_{\mathbf{y}\in S} \sup_{t\in\mathbb{T}} F(\mathbf{y}, t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) *The nodes $w_0, \ldots, w_n$ are different and $\mathbf{w}^*$ is an equioscillation point, i.e.,*

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

(b) *We have*

$$\inf_{\mathbf{y}\in S} \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t) = M(S) = m(S) = \sup_{\mathbf{y}\in S} \min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

(c) *We have the Sandwich Property on $\mathbb{T}$ in $\overline{S}$, i.e., for each $\mathbf{x}, \mathbf{y} \in \overline{S}$*

$$\underline{m}(\mathbf{x}) \le M(S) \le \overline{m}(\mathbf{y}).$$

Proof. There is $\mathbf{w} \in \overline{S}$ with $M(S) = \sup_{t\in\mathbb{T}} F(\mathbf{w}, t)$. By Proposition 8.4 we only need to prove that belongs to the interior of the simplex, i.e., $\mathbf{w} \in S$. Suppose by contradiction that $w_{\sigma(\ell-1)} < w_{\sigma(\ell)} = w_{\sigma(\ell+1)} = \cdots = w_{\sigma(r)} < w_{\sigma(n+1)}$ with $\ell \ne r$, $\ell \in \{1, \ldots, n\}$ (the case $\ell = 0$ will be considered below separately). Then we can apply Lemma 12.1 with $a = \frac{1}{a_{\sigma(r)}}$, $b = \frac{1}{a_{\sigma(\ell)}}$ and $x = w_{\sigma(\ell)}$, and move the two nodes $w_{\sigma(\ell)}$ and $w_{\sigma(r)}$ away from each other, such that new node system $\mathbf{w}'$ still belongs to S, and we conclude

$$\begin{aligned} &F(\mathbf{w}', t) - F(\mathbf{w}, t) \\ &= K_{\sigma(\ell)}(t - w'_{\sigma(\ell)}) + K_{\sigma(r)}(t - w'_{\sigma(r)}) - K_{\sigma(\ell)}(t - w_{\sigma(\ell)}) - K_{\sigma(r)}(t - w_{\sigma(r)}) < 0 \end{aligned}$$

for all $t \in \mathbb{T} \setminus [w'_{\sigma(\ell)}, w'_{\sigma(r)}]$. Hence we obtain

$$m_j(\mathbf{w}') < m_j(\mathbf{w}) \quad \text{for each } j \in \{0, \ldots, n\} \setminus \{\sigma(\ell-1), \ldots, \sigma(r)\}.$$

Since by Proposition 6.4 $m_{\sigma(\ell)}(\mathbf{w}) = m_{\sigma(\ell+1)}(\mathbf{w}) = \cdots = m_{\sigma(r-1)}(\mathbf{w}) < \overline{m}(\mathbf{w})$, if we move the two nodes $w_{\sigma(\ell)}$ and $w_{\sigma(r)}$ to a sufficiently small amount, by Corollary 3.6 we can achieve

$$m_{\sigma(\ell)}(\mathbf{w}') = m_{\sigma(\ell+1)}(\mathbf{w}') = \cdots = m_{\sigma(r-1)}(\mathbf{w}') < \overline{m}(\mathbf{w}).$$

Altogether we would obtain $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which is in contradiction with choice of $\mathbf{w}$. If w_0 happens to coincide with some $w_{\sigma(r)}$, then we can move w_0 and $w_{\sigma(r)}$ away from each other as above and obtain a new node system $\mathbf{w}'$ with $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, and then we need to rotate back by w'_0.

We have seen that $\mathbf{w}^* := \mathbf{w} \in S$, therefore the proof is complete. □

Bojanov proved in [4] the following

Theorem 13.2. *Let $\nu_1, \ldots, \nu_n$ be fixed positive integers. Fix $[a, b] \subset \mathbb{R}$. Then, there exists a unique system of points $\mathbf{w}^* = (w_1, \ldots, w_n)$, $a < w_1 < \ldots < w_n < b$ such that*

$$\|(x - w_1)^{\nu_1} \ldots (x - w_n)^{\nu_n}\| = \inf_{a\le x_1<\ldots<x_n\le b} \|(x - x_1)^{\nu_1} \ldots (x - x_n)^{\nu_n}\|$$

where $\|\cdot\|$ denotes the sup-norm over $[a, b]$. The extremal polynomial $P^*(x) := (x-w_1)^{\nu_1} \ldots (x-w_n)^{\nu_n}$ has the property that there exist $a = z_0 < z_1 < \ldots < z_{n-1} < z_n = b$ such that $|P^*(z_j)| = \|P^*\|$ for $j = 0, 1, \ldots, n$ and $P^*(z_{j+1}) = -P^*(z_j)$ for $j = 0, 1, \ldots, n-1$.

Now we are going to establish a similar result for trigonometric polynomials and relate this new result to Bojanov's theorem.

It is well known (see e.g. [5] p. 19) that a trigonometric polynomial $T(t) = a_0 + \sum_{k=1}^{m} a_k \cos(kt) + b_k \sin(kt)$ where $|a_m| + |b_m| > 0$, can be written in the form

$T(t) = c\prod_{j=1}^{2m} \sin\frac{t-x_j}{2}$ where c, $x_1, \ldots, x_{2m}$ are numbers. More precisely, if $T(t') = 0$, $t' \in \mathbb{C}$, $\Re t' \in [0, 2\pi)$, then t' appears in $x_1, \ldots, x_{2m}$ and if $a_0, a_1, b_1, \ldots, a_m, b_m \in \mathbb{R}$ and $T(t') = 0$, $t' \in \mathbb{C} \setminus \mathbb{R}$, $\Re t' \in [0, 2\pi)$, then the conjugate of t' is also a zero, $T(\overline{t'}) = 0$ and both appear among $x_1, \ldots, x_{2m}$.

The following class of functions is called generalized trigonometric polynomials (GTP for short), see e.g. [5] Appendix 4:

$$a\prod_{j=1}^{m} |\sin\frac{t-z_j}{2}|^{r_j} \text{ where } a, r_j > 0,\ z_j \in \mathbb{C}$$

for all $j = 1, \ldots, m$ and $\frac{1}{2}\sum_{j=1}^{m} r_j$ is usually called the degree of this GTP.

In the next theorem, we describe Chebyshev type (having minimal sup norm and fixed leading coefficient) GTPs when the multiplicities of the zeros are fixed and the zeros are real.

Theorem 13.3. *Let $r_0, r_1, \ldots, r_n > 0$ be fixed. Then, there exists a unique system of points $\mathbf{w}^* = (w_0, w_1, \ldots, w_n)$, $0 = w_0 < w_1 < \ldots < w_n < 2\pi$ such that*

$$\Big\| \Big|\sin\frac{t-w_0}{2}\Big|^{r_0} \cdots \Big|\sin\frac{t-w_n}{2}\Big|^{r_n} \Big\| = \inf_{0=y_0\le y_1<\ldots<y_n<2\pi} \Big\| \Big|\sin\frac{t-y_0}{2}\Big|^{r_0} \cdots \Big|\sin\frac{t-y_n}{2}\Big|^{r_n} \Big\|$$

where $\|\cdot\|$ denotes the sup-norm over $[0, 2\pi]$. The extremal GTP

$$T^*(t) := \Big|\sin\frac{t-w_0}{2}\Big|^{r_0} \cdots \Big|\sin\frac{t-w_n}{2}\Big|^{r_n}$$

has the properties that there exists $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ such that w_j's and z_j's interlace, i.e., $0 = w_0 < z_0 < w_1 < \ldots < w_n < z_n < w_0 + 2\pi = 2\pi$, and $T^*(z_j) = \|T^*\|$ for $j = 0, 1, \ldots, n$.

Proof. Let $K(x) := \log|\sin(x/2)|$ for $-\pi \le x \le \pi$, then extend K 2π-periodically to $\mathbb{R}$. Then K is a C^2 kernel with $K'' < 0$. Let $K_j(x) := r_j K(x)$, $j = 0, 1, 2, \ldots, n$ be the kernels and consider the simplex $S := S_{\mathrm{id}}$. Further, let $T(\mathbf{y}, t) := \prod_{j=0}^{n} |\sin\frac{t-y_j}{2}|^{r_j}$ where $\mathbf{y} \in S$ and $F(\mathbf{y}, t) := \log|T(\mathbf{y}, t)|$ is a potential. Then

$$F(\mathbf{y}, t) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j) = \sum_{j=0}^{n} r_j K(x - y_j).$$

Applying Theorem 13.1, we obtain $M(S) = \inf_{\mathbf{y}\in S}\sup_{t\in[0,2\pi)} F(\mathbf{y}, t)$ is attained at exactly one configuration $\mathbf{w}^* = (w_1, \ldots, w_n) \in S$, that is $M(S) = \sup_{t\in[0,2\pi)} F(\mathbf{w}^*, t)$ and $\sup_{t\in[0,2\pi)} F(\mathbf{y}, t) > M(S)$ when $\mathbf{y} \ne \mathbf{w}^*$. Moreover, $\mathbf{w}^*$ is an equioscillation point, i.e., there exist $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ such that $F(\mathbf{w}^*, z_j) = M(S)$. The interlacing property obviously follows. Rewriting these properties for to $T^*(t) := \exp F(\mathbf{w}^*, t)$, we obtain the assertions of this theorem. □

We turn to the interval case.

Suppose there are n fixed positive real numbers $r_1, r_2, \ldots, r_n > 0$, and consider $|x - x_1|^{r_1} \ldots |x - x_n|^{r_n}$. Such functions are sometimes called generalized algebraic polynomials (GAP, see, for instance, [5] Appendix 4). Now fix $[a, b] \subset \mathbb{R}$ and consider the following problem

$$\inf_{a\le x_1<\ldots<x_n\le b} \big\| |x - x_1|^{r_1} \ldots |x - x_n|^{r_n} \big\| \tag{23}$$

where $\|\cdot\|$ denotes the sup-norm over $[a, b]$.

Based on this, we will investigate the problem

$$\inf \| |\sin \frac{t-y_1}{2}|^{r_n} \dots |\sin \frac{t-y_n}{2}|^{r_1} |\sin \frac{t-y_{n+1}}{2}|^{r_1} \dots |\sin \frac{t-y_{2n}}{2}|^{r_n} \| \tag{24}$$

where the infimum is taken for $0 \le y_1 < \dots < y_n < y_{n+1} < \dots < y_{2n} < 2\pi$.

Theorem 13.4. *Using the notations introduced above, (24) has unique solution $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n})$ with $w_1 + (w_{2n} - 2\pi) = 0$ and $0 < w_1 < \dots < w_{2n} < 2\pi$. Further, $\mathbf{w}^*$ is symmetric: $w_k = 2\pi - w_{2n+1-k}$ for $k = 1, 2, \dots, n$.*

This theorem follows from the next more general, symmetry theorem.

Theorem 13.5. *Let $K_1, \dots, K_n$ be strictly concave kernels such that K_j is even: $K_j(t) = K_j(-t)$ for all $j = 1, \dots, n$. Assume that the kernels are either all in $\mathrm{C}^1(0, 2\pi)$ or all satisfy (∞'_+), or all satisfy (∞'_-). Take the simplex $S := \{0 \le y_1 < y_2 < \dots < y_{2n} < 2\pi\}$. Define the symmetric potential*

$$\begin{aligned} F_{\mathrm{symm}}(\mathbf{y}, t) := {} & K_1(t - y_1) + \dots + K_{n-1}(t - y_{n-1}) + K_n(t - y_n) \\ & + K_n(t - y_{n+1}) + K_{n-1}(t - y_{n+2}) + \dots + K_1(t - y_{2n}) \end{aligned} \tag{25}$$

and consider the "doubled" trigonometric problem

$$M_{\mathrm{symm}} := \inf_{\mathbf{y} \in S} \sup_{t \in [0, 2\pi)} F_{\mathrm{symm}}(\mathbf{y}, t). \tag{26}$$

Then there is a unique solution $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n}) \in S$ with $w_1 + (w_{2n} - 2\pi) = 0$. Further, $\mathbf{w}^*$ is symmetric: $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$ and there are exactly $2n$ points: $0 = z_1 < z_2 < \dots < z_{n+1} = \pi < \dots < z_{2n}$ where $F_{\mathrm{symm}}(\mathbf{w}^*, .)$ attains its supremum. Moreover, z_j's and w_j's interlace and z_j's are symmetric too: $z_k = 2\pi - z_{2n+1-k}$ $(k = 1, 2, \dots, n)$.

Proof. Following the symmetric definition, we denote $K_{n+k}(t) := K_{n+1-k}(-t)$ where $k = 1, 2, \dots, n$, and by symmetry,

$$K_{n+k}(t) = K_{n+1-k}(t) \text{ for } k = -n+1, \dots, n. \tag{27}$$

Hence $F_{\mathrm{symm}}(\mathbf{y}, t) = \sum_{j=1}^{2n} K_j(t - y_j)$.

The existence and uniqueness follow from Theorem 13.1. That is, there exists a unique $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n}) \in S$ (unique with the normalization $w_1 = 0$ such that $M(S) = \overline{m}(\mathbf{w}^*)$). Furthermore, $M(S) = m(S)$ and $F(\mathbf{w}^*, \cdot)$ equioscillates, hence $m(S) = \underline{m}(\mathbf{w}^*)$. Using rotation, we can assume that $w_1 > 0$ is such that $w_1 + (w_{2n} - 2\pi) = 0$.

Now we establish $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$. By the assumption, it holds for $k = 1$, i.e., $w_1 = 2\pi - w_{2n}$. Reflect the w_k's: $v_k := 2\pi - w_{2n+1-k}$, $k = 1, \dots, 2n$ and write $\mathbf{v} := (v_1, \dots, v_{2n})$. Then $v_1 = w_1$ and $v_{2n} = w_{2n}$. Furthermore, put $L_k(t) := K_{2n+1-k}(-t)$ and consider

$$\tilde{F}(\mathbf{v}, t) := \sum_{k=-n+1}^{n} L_{n+k}(t - v_{n+k})$$

the potential of the reflected configuration. We can write, using (27) and the even property of kernels,

$$\begin{aligned} L_{n+k}(t - v_{n+k}) &= K_{n+1-k}(-t + v_{n+k}) = K_{n+1-k}(t - v_{n+k}) \\ &= K_{n+1-k}(t - 2\pi + w_{n+1-k}) = K_{n+1-k}((2\pi - t) - w_{n+1-k}) \end{aligned}$$

for all $k = -n+1, \dots, n$. Hence

$$\tilde{F}(\mathbf{v}, t) = \sum_{k=-n+1}^{n} L_{n+k}(t - v_{n+k}) = \sum_{k=-n+1}^{n} K_{n+1-k}((2\pi - t) - w_{n+1-k})$$
$$= F_{\mathrm{symm}}(\mathbf{w}^*, 2\pi - t) = F_{\mathrm{symm}}(\mathbf{w}^*, -t).$$

Obviously $\mathbf{v} \in S$. By definition, $m_j(\mathbf{w}^*) = \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_j \le t \le w_{j+1}\}$, $j = 1, 2, \dots, 2n-1$ and $m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*) = \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_{2n} - 2\pi \le t \le w_1\}$ and similarly for $\mathbf{v}$, $m_j(\mathbf{v}) = \sup\{\tilde{F}(\mathbf{v}, t) : v_j \le t \le v_{j+1}\}$, $j = 1, 2, \dots, 2n-1$ and $m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = \sup\{\tilde{F}(\mathbf{v}, t) : v_{2n} - 2\pi \le t \le v_1\}$. Hence, we also have for $j = 1, 2, \dots, 2n-1$

$$\begin{aligned}
&m_j(\mathbf{w}^*) = \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_j \le t \le w_{j+1}\} \\
&= \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, -t) : -w_{j+1} \le t \le -w_j\} = \sup\{\tilde{F}(\mathbf{v}, t) : -w_{j+1} \le t \le -w_j\} \\
&= \sup\{\tilde{F}(\mathbf{v}, t) : 2\pi - w_{j+1} \le t \le 2\pi - w_j\} = \sup\{\tilde{F}(\mathbf{v}, t) : v_{2n-j} \le t \le v_{2n+1-j}\} \\
&\qquad = m_{2n-j}(\mathbf{v})
\end{aligned}$$

and obviously $m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*)$. This implies that $\underline{m}(\mathbf{v}) = \underline{m}(\mathbf{w}^*)$. Indirectly, suppose $\mathbf{v} \ne \mathbf{w}^*$. We use Proposition 7.2 hence the strict concavity of $\underline{m}$ implies that there is an $\mathbf{a} = (a_1, \dots, a_{2n}) \in S$ such that $\underline{m}(\mathbf{a}) > \underline{m}(\mathbf{w}^*)$. But this contradicts that $\underline{m}(\mathbf{w}^*) = m(S) = \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y})$. Therefore $\mathbf{v} = \mathbf{w}^*$, hence $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$.

The symmetry of w_k's implies the remaining assertions (interlacing and symmetry of z_j's). □

We connect the interval problem (23) and (24) using a classical idea transferring with $x = \cos(t)$ in the following (see e.g. [16])

Theorem 13.6. *We use the notations introduced above. Consider the algebraic problem* (23) *and the associated "doubled" trigonometric problem* (24)*. Denote the unique solution of* (24) *by* $\mathbf{w}^* = (w_1, \dots, w_{2n})$ *and that of* (23) *by* $\mathbf{x} = (x_1, \dots, x_n)$*, and let* $L(x) := \frac{b-a}{2}x + \frac{b+a}{2}$.

Then we can obtain $\mathbf{x}$ *from* $\mathbf{w}^*$ *:* $x_j = L(\cos w_{n+1-j})$*,* $j = 1, \dots, n$.

Proof. For simplicity, assume that $a = -1$, $b = 1$, hence $L(x) = x$. Recall

$$\sin\frac{t-\alpha}{2}\sin\frac{t+\alpha-2\pi}{2} = \frac{1}{2}(\cos(t) - \cos(\alpha)) \tag{28}$$

hence

$$\begin{aligned}
&\left|\sin\frac{t-t_1}{2}\right|^{r_n} \cdots \left|\sin\frac{t-t_n}{2}\right|^{r_1} \left|\sin\frac{t+t_n-2\pi}{2}\right|^{r_1} \cdots \left|\sin\frac{t+t_1-2\pi}{2}\right|^{r_n} \\
&\qquad = \frac{1}{2^{\sum_{j=1}^n r_j}} |\cos(t) - \cos(t_1)|^{r_n} \cdots |\cos(t) - \cos(t_n)|^{r_1}
\end{aligned} \tag{29}$$

Therefore, for every GAP $P(x) = |x - x_1|^{r_1} \dots |x - x_n|^{r_n}$ there is a GTP $T(t)$ of the form as in (24) such that $P(\cos t) = 2^{-\sum_{j=1}^n r_j} T(t)$. Now consider the problem (24). According to Theorem 13.4, there is a unique GTP $T^*(t)$ with minimal sup-norm and

$$T^*(t) = \left|\sin\frac{t-w_1}{2}\right|^{r_n} \cdots \left|\sin\frac{t-w_n}{2}\right|^{r_1} \left|\sin\frac{t+w_n-2\pi}{2}\right|^{r_1} \cdots \left|\sin\frac{t+w_1-2\pi}{2}\right|^{r_n}.$$

In view of (29), $P^*(x) := 2^{\sum_{j=1}^n r_j} T^*(\arccos x) = |x - \cos(w_1)|^{r_n} \dots |x - \cos(w_n)|^{r_1}$ is the unique solution of (23) and we can also write $P^*(x) = |x - x_1|^{r_1} \dots |x - x_n|^{r_n}$ where $x_j = \cos w_{n+1-j}$, $j = 1, 2, \dots, n$, hence $-1 \le x_1 < \dots < x_n \le 1$. □

This last theorem implies and generalizes Bojanov's result.

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Bálint Farkas
Bergische Universität Wuppertal, School of Mathematics and Natural Sciences
Gaußstraße 20, 42119 Wuppertal, Germany.
email address: `farkas@uni-wuppertal.de`

Béla Nagy
MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged, Szeged, Aradi vértanuk tere 1, H-6720, Hungary.
email address: `nbela@math.u-szeged.hu`

Szilárd Gy. Révész
Alfréd Rényi Institute of Mathematics, Budapest, Reáltanoda utca 13-15, H-1053, Hungary, and
Institute of Mathematics, Faculty of Sciences, Budapest University of Technology and Economics, Budapest, Műegyetem rkp. 3-9, Hungary.
email address: `revesz.szilard@renyi.mta.hu`