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**CONSTRUCTIVE APPROXIMATION**


# Uniqueness of Markov-Extremal Polynomials on Symmetric Convex Bodies

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**Abstract.** For a compact set $K \subset \mathbb{R}^d$ with nonempty interior, the Markov constants $M_n(K)$ can be defined as the maximal possible absolute value attained on $K$ by the gradient vector of an $n$-degree polynomial $p$ with maximum norm 1 on $K$.

It is known that for convex, symmetric bodies $M_n(K) = n^2/r(K)$, where $r(K)$ is the "half-width" (i.e., the radius of the maximal inscribed ball) of the body $K$. We study extremal polynomials of this Markov inequality, and show that they are essentially unique if and only if $K$ has a certain geometric property, called flatness. For example, for the unit ball $B^d(\mathbf{0}, 1)$ we do not have uniqueness, while for the unit cube $[-1, 1]^d$ the extremal polynomials are essentially unique.

## 1.

Denote by $K \subset \mathbb{R}^d$ a convex body ($K$ is bounded and $\operatorname{int} K \neq \emptyset$), symmetric with respect to the origin. The Markov constant of $K$ is

$$M_n(K) := \sup\left\{ \frac{\| |\operatorname{grad} p| \|_{C(K)}}{\|p\|_{C(K)}} : p \in \pi_n^{d,1} \right\}, \tag{1}$$

where $\pi_k^{\ell,m}$ denotes the set of polynomials with $\ell$ variables and $m$ coordinates having maximal total degree not exceeding $k$. For polynomials mapping the variable space into $\mathbb{R}$ we use $\pi_n^d = \pi_n^{d,1}$ as well. Note that for $p \in \pi_n^{d,1}$ the derivative or gradient polynomial $\operatorname{grad} p = (\partial p/\partial x_1, \dots, \partial p/\partial x_d)$ lies in $\pi_{n-1}^{d,d}$ and the absolute value $|\operatorname{grad} p|$ is the usual Euclidean norm in $\mathbb{R}^d$. Also $C(K)$ norm is the usual sup norm. Since $K$ is compact, both $C(K)$ norms in (1) are actually a maximum, and also the sup in the definition of $M_n(K)$ is attained for some $p \in \pi_n^{d,1}$. We also know the exact value of this constant.

**Theorem** (Sarantopoulos [4], Baran [1]). *$M_n(K) = n^2/r(K)$, where $r(K) := \inf\{|\mathbf{y}| : \mathbf{y} \in \partial K\} = \sup\{r : B(\mathbf{0}, r) \subset K\}$ and $B(\mathbf{0}, r)$ denotes the ball centered at the origin and having radius $r$.*

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Naturally for $K$ being compact the definition of $r(K)$ can be written with min and max in place of inf and sup. That is, we can always find points $\mathbf{a} \in \partial K$ with $|\mathbf{a}| = r(K)$. For any such point $\mathbf{a} \in K^0 := B(\mathbf{0}, r(K)) \cap \partial K$ one can take the following "canonical" extremal polynomial or Chebyshev polynomial

$$g := g_{n,\mathbf{a}}(\mathbf{x}) := T_n\left(\frac{\langle \mathbf{x}, \mathbf{a}\rangle}{r^2(K)}\right) = T_n\left(\frac{\langle \mathbf{x}, \mathbf{a}\rangle}{|\mathbf{a}|^2}\right), \tag{2}$$

where $T_n$ denotes the usual one-dimensional Chebyshev polynomial

$$T_n(t) = \cos(n \arccos t) = 2^{n-1}\prod_{j=1}^{n}(t - z_j) \qquad \left(z_j := z_{j,n} := \cos\left(\frac{2j-1}{2n}\pi\right)\right). \tag{3}$$

Plainly, $\|g\|_{C(K)} = 1$ and

$$\operatorname{grad} g(\mathbf{x}) = T_n'\left(\frac{\langle \mathbf{x}, \mathbf{a}\rangle}{r^2(K)}\right) \cdot \frac{\mathbf{a}}{r^2(K)}, \tag{4}$$

hence

$$|\operatorname{grad} g(\mathbf{x})| = \frac{1}{r(K)} \cdot \left|T_n'\left(\frac{\langle \mathbf{x}, \mathbf{a}\rangle}{r^2(K)}\right)\right|$$

and

$$\||\operatorname{grad} g(\mathbf{x})|\|_{C(K)} = \frac{1}{r(K)} \cdot \left\|T_n'\left(\frac{\langle \mathbf{x}, \mathbf{a}\rangle}{r^2(K)}\right)\right\|_{C(K)} = \frac{T_n'(1)}{r(K)} = \frac{n^2}{r(K)}. \tag{5}$$

Thus $g = g_{n,\mathbf{a}}$ furnishes an extremal polynomial to the theorem of Sarantopoulos. Clearly, for geometrically different bodies, the set $K^0$ can be small in some cases and can be large for other cases. For many bodies $K^0$ consists of only $\mathbf{a}$ and $-\mathbf{a}$, while for $K = B(\mathbf{0}, 1)$ all points of $\partial K$ belong to $K^0$ and the corresponding Chebyshev polynomials will furnish essentially different examples of extremal polynomials. Moreover, in this case also $T_{n/2}(\langle \mathbf{x}, \mathbf{x}\rangle)$ can be a totally different example whenever $n$ is even.

Thus we can see that the polynomials (2) do not necessarily exhaust all the possible types of the extremal polynomials. However, they are typical in the following sense:

**Theorem A** (Kroó [2]). *For any $K$ (symmetric, convex body in $\mathbb{R}^d$) and any Markov-extremal polynomial $p_n \in \pi_n^d$ satisfying $\||\operatorname{grad} p_n|\|_{C(K)} = n^2/r(K)$ and also $\|p_n\|_{C(K)} = 1$, there exists $\mathbf{a} \in \partial K$ with $|\mathbf{a}| = r(K)$, so that*

$$\pm p_n(t\mathbf{a}) \equiv T_n(t) \equiv g_{n,\mathbf{a}}(t\mathbf{a}) \qquad (\forall\, t \in \mathbb{R}). \tag{6}$$

Denoting $\ell_{\mathbf{a}} := \{t\mathbf{a} : t \in \mathbb{R}\}$ the line spanned by $\{-\mathbf{a}, \mathbf{a}\}$, this result says that at least on a line $\ell = \ell_{\mathbf{a}}$ the otherwise arbitrary extremal polynomial $p$ coincides with a canonical example.

Note that one particular $p_n$ can coincide with several different $g_{n,\mathbf{a}}$ on different lines $\ell_{\mathbf{a}}$, simultaneously. See, e.g., the case $K = B(\mathbf{0}, 1)$, $n$ even, and $p_n = T_{n/2}(\langle \mathbf{x}, \mathbf{x}\rangle))$.

Still, the connection between $p_n$ and (any of the occurring) $g_{n,\mathbf{a}}$ is even stronger. That will be clear from our Theorem 1 below.

**Theorem 1.** *Suppose $K$ is a symmetric convex body in $\mathbb{R}^d$, $p \in \pi_n^d$ is a Markov-extremal polynomial, and $\mathbf{a} \in \partial K$ is such that*

$$p_n|_{\ell_{\mathbf{a}}} \equiv g_{n,\mathbf{a}}|_{\ell_{\mathbf{a}}}.$$

*Then we also have* (*with the very same* $\mathbf{a}$) *the relation*

$$\text{grad}\, p_n|_{\ell_{\mathbf{a}}} \equiv \text{grad}\, g_{n,\mathbf{a}}|_{\ell_{\mathbf{a}}}. \tag{7}$$

The proof of Theorem 1 will be given in the next section. In the remaining part of this introduction we present the idea of the paper and describe the organization of it.

Our goal is to find the connection between the set of Markov extremal polynomials and the geometric features of the body $K$. This geometric characterization of unicity of Markov extremal polynomials was first investigated by Kroó in [2]. Also, similar questions were studied for Chebyshev-type extremal polynomials by Kroó in [3]. In particular, Kroó was the first to notice the intrinsic connection between unicity on one hand, and second-order approximation of $K$ at $\mathbf{a}$ by its tangential hyperplane on the other hand. However, he could completely settle the case $d = 2$ only. To formulate his result, let us quote the following simple notion, which was used by Kroó only for dimension $d = 2$, i.e., for $K \subset \mathbb{R}^2$.

**Definition.** We call a point $\mathbf{y} \in \partial K$ *flat*, if no disk touching the tangential line $L_{\mathbf{y}}$ of $K$ at $\mathbf{y}$ contains $K$.

With this definition and the above notation of $K^0$ Kroó has proved:

**Theorem B** (Kroó [2])**.** *In order that every extremal polynomial for Markov inequality on $K$ coincides with $cT_n(\langle \mathbf{x}, \mathbf{y}\rangle / r^2(K))$ for some $\mathbf{y} \in K^0$ and $c \in \mathbb{R}$ it is necessary and sufficient that each $\mathbf{y} \in K^0$ is flat.*

The present work continues investigation in this direction and extends the results of Kroó to arbitrary dimension. In retrospect, we see that "jumping up" to higher dimensions needed two additions to the previous work: the corresponding geometric property had to be found, and a somewhat more complicated analysis (allowing sufficiently exact treatment of partial derivatives) had to be carried out. Those are the novel technical features of the present paper.

In Section 2 the proof of Theorem 1, handling first partial derivatives, will be given. In Section 3 we proceed to explore the second-order partial derivatives' behavior. We present a series of lemmas on this question. Then to formulate the needed geometric property, we select a particular type of second-order smooth support of surfaces, and define the notion of "parabolically close support" and "parabolical separation" (see Definitions 1 and 2). Points on the boundary with this parabolically close supporting property will again be called "parabolically flat."

Finally, in Section 4, we present the main results of the paper. Our Theorems 3 and 4, when read together, lead to the following description of the necessary and sufficient condition for unicity of extremal polynomials in Markov's inequality.

**Theorem 2.** *Suppose $K$ is a symmetric convex body in $\mathbb{R}^d$ and $\mathbf{a} \in K^0$ is a given boundary point of minimal length. All Markov-extremal polynomials $p_n \in \pi_n^{d,1}$, normalized by*

(i) $$\|p_n\|_{C(K)} = 1,$$

*and also satisfying*

(ii) $$p_n|_{\ell_\mathbf{a}} \equiv g_{n,\mathbf{a}}|_{\ell_\mathbf{a}},$$

*must necessarily coincide with $g_{n,\mathbf{a}}$ if and only if $\mathbf{a}$ is parabolically flat.*

**Corollary 1.** *In order that every extremal polynomial for Markov inequality on $K$ coincide with $cg_{n,\mathbf{a}}$ for some $\mathbf{a} \in K^0$ and $c \in \mathbb{R}$, it is necessary and sufficient that each $\mathbf{a} \in K^0$ is parabolically flat.*

In dimension 1 the Markov extremal polynomials form a one-dimensional set. In higher dimensions the size of this extremal set depends on the geometry of the particular set $K$. Similarly to dimension 2, explored earlier by Kroó, we have:

**Corollary 2.** *The extremal polynomial in Markov's inequality for $K$ is unique up to a constant factor if and only if $K^0 = \{\pm\mathbf{a}\}$ for some parabolically flat point $\mathbf{a}$.*

Concerning the proofs we point out that the proof of the "if part," Theorem 3, closely follows the corresponding argument of Kroó from [2]. The new elements of this work are built in the proof of Theorem 4, which provides us with the "only if" part of the characterization.

## 2. Proof of Theorem 1

Note that considering $-p_n$ if necessary, we can take the plus sign from the two possible signs in (6), and thus by Kroó's theorem we always have at least one line $\ell_\mathbf{a}$ satisfying the conditions.

First we show that with $g = g_{n,\mathbf{a}}$ defined by (2) we have

(1) $$\operatorname{grad} p_n(\mathbf{a}) = \operatorname{grad} g(\mathbf{a}).$$

The right-hand side is known (see (4)) to be

(2) $$\operatorname{grad} g(\mathbf{a}) = \frac{T_n'(1)}{r^2(K)}\mathbf{a} = \frac{n^2}{r^2(K)} \cdot \mathbf{a}.$$

For the left-hand side of (8) we have with $\mathbf{a}^* := \mathbf{a}/|\mathbf{a}| = \mathbf{a}/r(K)$:

$$\left|\frac{\partial p_n}{\partial \mathbf{a}^*}(\mathbf{a})\right| = |\langle \operatorname{grad} p_n(\mathbf{a}), \mathbf{a}^*\rangle| \le |\operatorname{grad} p_n(\mathbf{a})| \le M_n(K) \cdot \|p_n\|_{C(K)} = \frac{n^2}{r(K)},$$

while from (6) we get

$$\frac{\partial p_n}{\partial \mathbf{a}^*}(\mathbf{a}) = \left.\frac{d}{du} p_n(\mathbf{a} + u \cdot \mathbf{a}^*)\right|_{u=0} = \left.\frac{d}{du} T_n\left(1 + \frac{u}{r(K)}\right)\right|_{u=0} = \frac{T_n'(1)}{r(K)} = \frac{n^2}{r(K)}.$$

Hence $|\langle \text{grad } p_n(\mathbf{a}), \mathbf{a}^*\rangle| = |\text{grad } p_n(\mathbf{a})| = n^2/r(K)$ and so

$$\text{grad } p_n(\mathbf{a}) = \frac{\partial p_n}{\partial \mathbf{a}^*}(\mathbf{a}) \cdot \mathbf{a}^* = \frac{n^2}{r(K)} \cdot \mathbf{a}^*. \tag{3}$$

Now comparing (9) and (10) we get (8).

Denote now

$$h := h_{n,\mathbf{a}}(\mathbf{x}) := p_n(\mathbf{x}) - g(\mathbf{x}). \tag{4}$$

In view of the condition of the theorem we now have

$$h|_{\ell_{\mathbf{a}}} \equiv 0, \tag{5}$$

while

$$\text{grad}\, h(\mathbf{a}) = \mathbf{0} \tag{6}$$

according to (8). We put

$$\mathbf{a}_j := t_j \cdot \mathbf{a}, \qquad t_j := \cos\left(\frac{j\pi}{n}\right) \qquad (j = 0, 1, \ldots, n). \tag{7}$$

As $B(\mathbf{0}, r(K)) \subset K$, we easily see that

$$B(\mathbf{a}_j, \delta) \subset K \qquad \text{with} \quad \delta := 2\sin^2 \frac{\pi}{2n}\, r(K) \qquad (j = 1, \ldots, n-1). \tag{8}$$

Let $H := \mathbf{a}^\perp := \{\mathbf{b} \in \mathbb{R}^d : \langle \mathbf{b}, \mathbf{a}\rangle = 0\}$ and consider $H_j := \mathbf{a}_j + H$. Plainly $g$ is constant on any hyperplanes orthogonal to $\mathbf{a}$, thus $g|_{H_j} \equiv g(\mathbf{a}_j) = (-1)^j$. On the other hand, (15) entails

$$|p_n(\mathbf{a}_j + \mathbf{b})| \le \|p_n\|_{C(K)} = 1 \qquad (\mathbf{b} \in H,\ |\mathbf{b}| \le \delta,\ j = 1, \ldots, n-1) \tag{9}$$

whence

$$(-1)^j p_n(\mathbf{a}_j + \mathbf{b}) \le 1 = (-1)^j g(\mathbf{a}_j + \mathbf{b}) \qquad (\mathbf{b} \in H,\ |\mathbf{b}| \le \delta,\ 1 \le j \le n-1)$$

or, equivalently,

$$(-1)^j h(\mathbf{a}_j + \mathbf{b}) \le 0 \qquad (\mathbf{b} \in H,\ |\mathbf{b}| \le \delta,\ 1 \le j \le n-1). \tag{10}$$

Since $h(\mathbf{a}_j) = 0$ $(j = 1, \ldots, n-1)$, (17) entails

$$\frac{\partial h}{\partial \mathbf{b}^*}(\mathbf{a}_j) = 0 \qquad (\mathbf{b}^* \in H,\ |\mathbf{b}^*| = 1). \tag{11}$$

In other words, $\text{grad}\, h(\mathbf{a}_j)$ is orthogonal to (all vectors of) $H$, i.e.,

$$\text{grad}\, h(\mathbf{a}_j) \| \mathbf{a}, \quad \text{i.e.,} \quad \text{grad}\, h(\mathbf{a}_j) = \frac{\langle \text{grad}\, h(\mathbf{a}_j), \mathbf{a}\rangle \mathbf{a}}{r^2(K)}. \tag{12}$$

However, (12) entails $(\partial h/\partial \mathbf{a})|_{\ell_{\mathbf{a}}} = 0$, hence $(\partial h/\partial \mathbf{a})(\mathbf{a}_j) = 0$, and so

$$\text{grad}\, h(\mathbf{a}_j) = \mathbf{0} \qquad (j = 1, \ldots, n-1). \tag{13}$$

Consider now the polynomial $\varphi(t) := \text{grad}\, h(t\mathbf{a}) \in \pi_{n-1}^{1,d}$. $(\varphi(t) = \varphi_1(t), \ldots, \varphi_d(t))$, where $\varphi_\ell(t) = (\partial/\partial x_\ell) h(t\mathbf{a}) \in \pi_{n-1}$). In view of (13) and (20) we find $\varphi(t_j) = \mathbf{0}$ for $j = 0, 1, \ldots, n-1$, and being a polynomial of degree less than $n$, $\varphi$ can have these $n$ zeros only if $\varphi \equiv \mathbf{0}$. Thus $\partial h|_{\ell_{\mathbf{a}}} \equiv \mathbf{0}$ and the proof is complete. ∎

**3.**

In the following we are going to make repeated use of the following well-known property of polynomials:

**Lemma** (Weak Sign Changes Property). *Let $P \in \pi_n$ be a univariate polynomial of degree at most $n$. Suppose that for a sequence of $n+2$ distinct points $x_0 < x_1 < \cdots < x_n < x_{n+1}$, $P$ changes sign weakly, i.e., we have $(-1)^j P(x_j) \geq 0$ $(j = 0, \ldots, n+1)$. Then $P \equiv 0$ identically.*

In the following lemmas we keep the notations already introduced in the proof of Theorem 1. Namely, $\mathbf{a} \in \partial K$ is a (fixed) boundary point of $K$, $p := p_n$ is a (fixed) Markov-extremal polynomial, and we also use the difference polynomial $h$ as defined in (11). We also keep the notation $H := \mathbf{a}^\perp$ as well. With these notations fixed, now we proceed to analyze the second derivative $\partial^2 p \in \pi_{n-2}^{d,d\times d}$.

For a smooth function $f : \mathbb{R}^d \to \mathbb{R}$, $\partial^2 f$ can be represented as a square matrix $\left(\cdots \partial^2 f/\partial x_k\, \partial x_\ell \cdots\right)_{k,\ell=1,\ldots,d}$, which is symmetric in view of the commutativity of the partial derivation. For a fixed point $\mathbf{z} \in \mathbb{R}^d$ the symmetric matrix $B = \partial^2 f(\mathbf{z})$ is a bilinear mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$, mapping the vector pair $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ to $B(\mathbf{u}, \mathbf{v}) = \langle B\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, B\mathbf{v}\rangle$. (Here the symmetry of $B$ is exploited as $B^\top = B$ iff $B$ is symmetric.) Now the "diagonal" of this bilinear mapping is a quadratic form $Q(\mathbf{x}) := B(\mathbf{x}, \mathbf{x})$, which defines the symmetric matrix (bilinear form) uniquely as $B(\mathbf{x}, \mathbf{y}) = \frac{1}{4}(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}-\mathbf{y}))$. Standard calculation yields

$$\frac{\partial^2 f}{\partial \mathbf{b}^2}(\mathbf{z}) = Q(\mathbf{b}) \qquad (\mathbf{b} \in \mathbb{R}^d)$$

representing the connection between the quadratic form and successive directional derivation. Note that restricting $f$ to the line $\ell_{\mathbf{b}}(\mathbf{z})$ through $\mathbf{z}$ and in direction $\mathbf{b}$, an application of the usual Taylor formula with the Lagrange remainder term of the second order will give, with $\mathbf{b}^* := \mathbf{b}/|\mathbf{b}|$, $|\mathbf{b}^*| = 1$, and $\rho \in \mathbb{R}$:

$$f(\mathbf{z} + \rho\mathbf{b}^*) = f(\mathbf{z}) + \frac{\partial f}{\partial \mathbf{b}^*}(\mathbf{z}) \cdot \rho + \frac{1}{2}\,\frac{\partial^2 f}{\partial \mathbf{b}^{*2}}(\mathbf{z} + \xi\rho\mathbf{b}^*) \cdot \rho^2 \tag{1}$$

with some suitable $0 \leq \xi \leq 1$ depending on all quantities involved. This Taylor formula can be applied to deduce

**Lemma 1.** *With $\mathbf{a}$, $H$, and $h$ as above, we have for any $\mathbf{b} \in H$ (i.e., $\mathbf{b} \perp \mathbf{a}$) $(\partial^2 h/\partial \mathbf{b}^2)(\mathbf{a}) \geq 0$. Moreover, if it is zero, then $(\partial^2 h/\partial \mathbf{b}^2)|_{\ell_{\mathbf{a}}} \equiv 0$ identically.*

**Proof.** We can restrict ourselves to $\mathbf{b}^* \perp \mathbf{a}$, $|\mathbf{b}^*| = 1$, since for $\mathbf{b} = \beta \cdot \mathbf{b}^*$, $(\partial^2 h/\partial \mathbf{b}^2)(\mathbf{a}) = \beta^2 \cdot (\partial^2 h/\partial \mathbf{b}^{*2})(\mathbf{a})$. We refer to (17) and the above Taylor formula to deduce for any $|\rho| \leq \delta$ and $1 \leq j < n$:

$$\begin{aligned} 0 &\geq (-1)^j h(\mathbf{a}_j + \rho\mathbf{b}^*) \\ &= (-1)^j \left\{ h(\mathbf{a}_j) + \frac{\partial h}{\partial \mathbf{b}^*}(\mathbf{a}_j) \cdot \rho + \frac{1}{2}\,\frac{\partial^2 h}{\partial \mathbf{b}^{*2}}(\mathbf{a}_j + \xi\rho\mathbf{b}^*) \cdot \rho^2 \right\} \end{aligned} \tag{2}$$

or, using that $h(\mathbf{a}_j) = 0$, grad $h(\mathbf{a}_j) = \mathbf{0}$, and $\frac{1}{2}\rho^2 > 0$ ($\rho \neq 0$):

$$0 \geq (-1)^j \frac{\partial^2 h}{\partial \mathbf{b}^{*2}}(\mathbf{a}_j + \xi\rho\mathbf{b}^*).$$

Now for $\rho \to 0$ continuity of $\partial^2 h/\partial \mathbf{b}^{*2} \in \pi_{n-2}^{d,1}$ leads to

$$0 \geq (-1)^j \frac{\partial^2 h}{\partial \mathbf{b}^{*2}}(\mathbf{a}_j) \qquad (1 \leq j \leq n-1). \tag{3}$$

If $(\partial^2 h/\partial \mathbf{b}^{*2})(\mathbf{a}) \leq 0$, i.e., (23) holds also for $\mathbf{a}_0 = \mathbf{a}$, i.e., $j = 0$, we obtain $n$ weak sign changes of the polynomial $\psi(t) := (\partial^2 h/\partial \mathbf{b}^{*2})(t\mathbf{a}) \in \pi_{n-2}$. Hence by the Weak Sign Changes Property of univariate polynomials we are led to $\psi \equiv 0$, i.e., $(\partial^2 h/\partial \mathbf{b}^{*2})|_{\ell_\mathbf{a}} = 0$. That is, $(\partial^2 h/\partial \mathbf{b}^{*2})(\mathbf{a}) \leq 0$ entails that it is identically zero on $\ell_\mathbf{a}$, and this proves Lemma 1. ■

**Lemma 2.** *Suppose that for some* $\mathbf{b} \in H$ $(\partial^2 h/\partial \mathbf{b}^2)(\mathbf{a}) = 0$. *Then we have*

$$h(t\mathbf{a} + s\mathbf{b}) \equiv 0 \qquad (t, s \in \mathbb{R}). \tag{4}$$

**Proof.** As in Lemma 1, we can restrict ourselves to the case $\mathbf{b}^* \perp \mathbf{a}$, $|\mathbf{b}^*| = 1$. Put

$$F(t, s) := h(t\mathbf{a} + s\mathbf{b}^*) \qquad (t, s \in \mathbb{R}). \tag{5}$$

Theorem A of Kroó states $F(t, 0) \equiv 0$ ($t \in \mathbb{R}$), and Theorem 1 states grad $h_n|_{\ell_\mathbf{a}} = \mathbf{0}$, hence $(\partial F/\partial s)(t, 0) = \langle \text{grad}\, h_n(t\mathbf{a}), \mathbf{b}^* \rangle \equiv 0$ ($t \in \mathbb{R}$). Moreover, the condition $(\partial^2 h/\partial \mathbf{b}^2)(\mathbf{a}) = 0$ and Lemma 1 furnishes $(\partial^2 h/\partial b^{*2})|_{\ell_\mathbf{a}} \equiv 0$, i.e., $(\partial^2 F/\partial s^2)(t, 0) \equiv 0$ ($t \in \mathbb{R}$). Hence, for $F \in \pi_n^2$, we get

$$F(t, s) = s^3 \hat{F}(t, s) \qquad (t, s \in \mathbb{R}) \tag{6}$$

with some $\hat{F} \in \pi_{n-3}^2$. We argue as in the proof of Theorem 1 recalling (17) in the form

$$(-1)^j h(\mathbf{a}_j + s\mathbf{b}^*) \leq 0 \qquad (|s| \leq \delta,\ 1 \leq j \leq n-1). \tag{7}$$

Plainly (26) and (27) entails

$$(-1)^j \hat{F}(t_j, s) \leq 0 \qquad (0 < s < \delta,\ 1 \leq j \leq n-1) \tag{8}$$

and the Weak Sign Changes Lemma (applied to any fixed $s_0$ and the polynomial $\eta_{s_0}(t) := \hat{F}(t, s_0) \in \pi_{n-3}^1$) yields $\hat{F}(t, s_0) \equiv 0$ ($\forall\, t \in \mathbb{R}$). As it holds for an open interval $(0, \delta)$ of $s$, this also holds for all $s \in \mathbb{R}$ hence $\hat{F} \equiv 0$ ($t, s \in \mathbb{R}$) and $F \equiv 0$ proving Lemma 2. ■

We proceed further showing a combination of Theorem 1 and Lemma 2.

**Lemma 3.** *If for some* $\mathbf{b} \in H$, $(\partial^2 h/\partial b^2)(\mathbf{a}) = 0$, *then we also have*

$$\text{grad}\, h(t\mathbf{a} + s\mathbf{b}) \equiv \mathbf{0} \qquad (t, s \in \mathbb{R}). \tag{9}$$

**Proof.** Without loss of generality, we can restrict ourselves to $\mathbf{b} = \mathbf{b}^*$, $|\mathbf{b}^*| = 1$. In view of Lemma 2, we immediately get $(\partial h/\partial \mathbf{a})(t\mathbf{a} + s\mathbf{b}) \equiv 0$, hence it suffices to show

$$G_{\mathbf{v}}(t,s) := \frac{\partial h}{\partial \mathbf{v}}(t\mathbf{a} + s\mathbf{b}^*) \equiv 0 \qquad (s, t \in \mathbb{R}) \tag{10}$$

for all $\mathbf{v} \perp \mathbf{a}$, $|\mathbf{v}| = 1$. Now applying (17) to the vector $s\mathbf{b}^* + r\mathbf{v} \in H$ with $|s|, |r| < \delta/2$ we obtain

$$(-1)^j h(t_j\mathbf{a} + s\mathbf{b}^* + r\mathbf{v}) \le 0 \qquad \left(j = 1, \ldots, n-1,\ |s|,\ |r| < \frac{\delta}{2}\right), \tag{11}$$

while Lemma 2 also entails

$$h(t_j\mathbf{a} + s\mathbf{b}^*) = 0 \qquad (j = 1, \ldots, n-1,\ s \in \mathbb{R})\,. \tag{12}$$

Comparing (31) and (32) we immediately get (30) for $t = t_j$ $(j = 1, \ldots, n-1)$ and first for $|s| < \delta/2$, but then by analytic continuation also for all $s \in \mathbb{R}$. Thus for any fixed $s_0 \in \mathbb{R}$, $G_{\mathbf{v}}(t, s_0) \in \pi_n^{1,1}[t]$ has at least $n-1$ zeros, while its degree is at most $n-1$ being a derivative of $h$. Moreover, for $s_0 = 0$ we know $G_{\mathbf{v}}(t, 0) \equiv 0$ $(t \in \mathbb{R})$ by Lemma 1 in view of the condition $(\partial^2 h/\partial b^{*2})(\mathbf{a}) = 0$. Hence we can also write

$$G_{\mathbf{v}}(t,s) = G_{\mathbf{v}}(t,s) - G_{\mathbf{v}}(t,0) = \Delta_{s\mathbf{b}^*}\frac{\partial h}{\partial \mathbf{v}}(t\mathbf{a}) = \Delta_s G_{\mathbf{v}}(t,0)$$

and it shows that $G_{\mathbf{v}}(t, s_0) \in \pi_n^{1,1}[t]$ has degree at most $n-2$ in $t$. But we also have $n-1$ zeros of $G_{\mathbf{v}}(t, s_0)$, hence it is identically zero for all $t \in \mathbb{R}$, and $s_0 \in \mathbb{R}$ being arbitrary, we arrive to (30). This proves Lemma 3. ∎

In the following we introduce some geometric properties of the convex symmetric body $K$ which is closely connected to the uniqueness of Markov-extremal polynomials.

Let $K$ be a convex, centrally symmetric body in $\mathbb{R}^d$, let $\mathbf{a}$ (and so also $-\mathbf{a}$) be any point of $K^0 = B(\mathbf{0}, r(K)) \cap \partial K$, and let $H \cong \mathbb{R}^{d-1}$ a tangential hyperplane at $\mathbf{a}$, i.e., suppose that $K$ lies between the affine hyperplanes $\mathbf{a} + H$, $-\mathbf{a} + H$, or, equivalently,

$$|\langle \mathbf{a}^*, \mathbf{x}\rangle| \le \langle \mathbf{a}^*, \mathbf{a}\rangle \qquad (\forall\, \mathbf{x} \in K) \tag{13}$$

for a normal vector $\mathbf{a}^*$ of $H$ (with $\langle \mathbf{a}^*, \mathbf{a}\rangle > 0$).

**Definition 1.** We say that $K$ is *parabolically separated from $H$ in direction* $\mathbf{b} \in H$ if there exists a constant $c > 0$ such that

$$\langle \mathbf{b}, \mathbf{x}\rangle^2 \le c \min\{\langle \mathbf{a}^*, \mathbf{a} - \mathbf{x}\rangle,\ \langle \mathbf{a}^*, \mathbf{a} + \mathbf{x}\rangle\} \qquad (\forall \mathbf{x} \in K) \tag{14}$$

Geometrically, the left-hand side of (34) is just the square of the distance in the direction of $\mathbf{b} \in H$ of $\mathbf{x}$ and the line $\ell_{\mathbf{a}}$, while the right-hand side is just the distance of $\mathbf{x} \in K$ from the closest of the tangential affine hyperplanes $\{\mathbf{a} + H$, and $-\mathbf{a} + H\}$. Observe that since $\mathbf{a}$ is minimal, i.e., $\mathbf{a} \in K^0$, the vectors $\mathbf{a}$ and $\mathbf{a}^*$ are parallell and thus $\mathbf{b} \in H$ is perpendicular to $\mathbf{a}$ and $\mathbf{a}^*$ as well. Thus, in particular, we can write $\langle \mathbf{b}, \mathbf{x} - \mathbf{a}\rangle$ in place of $\langle \mathbf{b}, \mathbf{x}\rangle$ for $\mathbf{a} \in K^0$.

The above definition of "parabolic separation" thus means that not only the hyperplanes, but also the parabolic surfaces

$$(15)\qquad \begin{aligned} P_+ &:= \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{y} - \mathbf{a}, \mathbf{a}^* \rangle + \delta \langle \mathbf{y} - \mathbf{a}, \mathbf{b} \rangle^2 = 0\}, \\ P_- &:= \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{y} + \mathbf{a}, \mathbf{a}^* \rangle - \delta \langle \mathbf{y} - \mathbf{a}, \mathbf{b} \rangle^2 = 0\}, \end{aligned}$$

contain amongst them the body $K$ (with some $\delta = 1/c > 0$).

We also define close supporting.

**Definition 2.** For a convex, centrally symmetric body $K \subset \mathbb{R}^d$, $\mathbf{a}$ (and also $-\mathbf{a}$) in $K^0$ and $H \perp \mathbf{a}^*$ with (33) (i.e., $H$ being a supporting hyperplane with outer normal vector $\mathbf{a}^*$), we say that $H$ supports $K$ at $\mathbf{a}$ "parabolically close," or that "$\mathbf{a}$ is a parabolically flat point of support with respect to $H$," if there is no direction $\mathbf{b} \in H$ with parabolic separation in the sense of Definition 1, i.e., if for all $\mathbf{b} \in H$, $\mathbf{b} \neq \mathbf{0}$ and for all $\varepsilon > 0$ there exists $\mathbf{x} \in K$ satisfying

$$(16)\qquad \langle \mathbf{a}^*, \mathbf{a} - \mathbf{x} \rangle < \varepsilon \cdot \langle \mathbf{b}, \mathbf{x} \rangle^2.$$

(Note that the other option $\langle \mathbf{a}^*, \mathbf{a} + \mathbf{x} \rangle < \varepsilon \langle \mathbf{b}, \mathbf{x} \rangle^2$ comes together with (36) as $K$ is symmetric with respect to the origin.)

As an example, we mention that a ball $B(\mathbf{0}, r)$ has parabolic separation at all boundary points in all tangential directions, while for a cube only edge and vertex points can be separated, and points $\mathbf{a}$ on side surfaces (inside the squares) are all flat points of support. (Note that even vertex or edge points are flat with respect to some special tangential hyperplanes.)

It turns out that "parabolic flatness" is closely connected to Markov-unicity. We also note that for $\mathbf{a} \in K^0$, $H$ is unique, hence for $\mathbf{a} \in K^0$ in the definition of flatness we may drop the direct reference to the tangential hyperplane $H$ and can say loosely the shorter term "$\mathbf{a}$ is (parabolically) flat."

## 4.

**Theorem 3.** *Suppose $K$ is a symmetric convex body in $\mathbb{R}^d$ and $\mathbf{a} \in K^0$. If $K$ is parabolically separated at $\mathbf{a}$ from the (unique) tangential hyperplane $H + \mathbf{a}$ in some direction $\mathbf{b}^* \in H$, $|\mathbf{b}^*| = 1$, then there exists a polynomial $p_n \in \pi_n^{d,1}$, distinct from $g$ (defined in (2)), such that*

$$\begin{aligned} &\text{(i)} & \|p_n\|_{C(K)} &= 1, \\ &\text{(ii)} & p_n|_{\ell_{\mathbf{a}}} &\equiv g|_{\ell_{\mathbf{a}}}, \\ &\text{(iii)} & \operatorname{grad} p_n|_{\ell_{\mathbf{a}}} &\equiv \operatorname{grad} g|_{\ell_{\mathbf{a}}}. \end{aligned}$$

Note that (2), (4), and (iii) clearly entails that $\operatorname{grad} p_n(\mathbf{a}) = (n^2/r^2(K))\mathbf{a}$, $|\operatorname{grad} p_n(\mathbf{a})| = n^2/r(K) = M_n(K)$, hence $p_n$ is Markov-extremal and there is no uniqueness in this case.

**Proof.** Denote

$$V_{n-2}(t) := \frac{T_n(t)}{(t-z_1^2)} = \frac{T_n(t)}{(t-z_1)(t-z_n)} = 2^{1-n} \cdot \prod_{j=2}^{n-1}(t-z_j). \tag{37}$$

We shall need the following properties of $V_{n-2} \in \pi_{n-2}^{1,1}$. First, we compute

$$\begin{aligned} \|V_{n-2}\|_{C[-1,1]} &= \max\{\|V_{n-2}\|_{C[-z_2,z_2]}, \|V_{n-2}\|_{C[z_2,1]}\} \\ &\le \max\left\{\frac{\|T_n\|_{C[-z_2,z_2]}}{z_1^2-z_2^2}, 2^{1-n}\prod_{j=2}^{n-1}(1-z_j)\right\} \\ &= \max\left\{\frac{1}{z_1^2-z_2^2}, \frac{T_n(1)}{1-z_1^2}\right\} = \frac{1}{1-\cos^2\pi/2n}, \end{aligned}$$

hence

$$\|V_{n-2}\|_{C[-1,1]} = \frac{1}{\sin^2\pi/2n} = V_{n-2}(1). \tag{38}$$

Second, we have, for any $0 \le \Delta \le 1$, the inequality

$$T_n(t) + \Delta \cdot (1-t)V_{n-2}(t) \le 1 \qquad (z_1 \le t \le 1). \tag{39}$$

Indeed, $T_n$ is convex in $[z_1, 1]$, and so

$$T_n(t) \le \frac{T_n(1)-T_n(z_1)}{1-z_1}(t-1) + T_n(1) = \frac{1}{1-\cos\pi/2n}(t-1)+1$$

in this interval, while (38) yields

$$T_n(t) + \Delta(1-t)V_{n-2}(t) \le T_n(t) + |\Delta| \cdot \frac{1}{\sin^2\pi/2n}(1-t),$$

hence we get that the left-hand side of (39) cannot exceed

$$1 + \left(\frac{|\Delta|}{\sin^2\pi/2n} - \frac{1}{1-\cos\pi/2n}\right) \cdot (1-t) \le 1$$

if

$$|\Delta| \le \frac{\sin^2\pi/2n}{1-\cos\pi/2n} = 1 + \cos\frac{\pi}{2n}.$$

Third, we also have

$$|T_n(t) + \Delta(1-|t|)V_{n-2}(t)| \le 1 \qquad \left(-1 \le t \le 1,\ 0 \le \Delta \le \sin^2\frac{\pi}{2n}\right) \tag{40}$$

since for $z_1 \le |t| \le 1$, $|T_n(t) + \Delta(1-|t|)V_{n-2}(t)| \le T_n(|t|) + \Delta(1-|t|)V_{n-2}(|t|)$, and (39) applies, while for $|t| \le z_1$, $T_n$ and $\Delta(1-|t|)V_{n-2}(|t|)$ are of different signs or zero in view of (37), and thus

$$\begin{aligned} |T_n(t) + \Delta(1-|t|)V_{n-2}(t)| &\le \max\{|T_n(t)|,\ \Delta|V_{n-2}(t)|\} \\ &\le \max\left\{\|T_n\|_{C[-1,1]},\ \sin^2\left(\frac{\pi}{2n}\right)\|V_{n-2}\|_{C[-1,1]}\right\} = 1. \end{aligned} \tag{41}$$

Now let us choose a $c > 0$ which satisfies (34). Set

$$q(\mathbf{x}) := \langle \mathbf{b}, \mathbf{x} \rangle^2 \in \pi_2^{d,1} \tag{42}$$

and note that in view of (34) we have

$$0 \le q(\mathbf{x}) \le c \min\{\langle \mathbf{a}, \mathbf{a} - \mathbf{x} \rangle, \langle \mathbf{a}, \mathbf{a} + \mathbf{x} \rangle\}. \tag{43}$$

With $\varepsilon > 0$ to be specified later, put

$$k_n(\mathbf{x}) := \varepsilon \cdot q(\mathbf{x}) \cdot V_{n-2}\left(\frac{\langle \mathbf{a}, \mathbf{x} \rangle}{r^2(K)}\right) \in \pi_n^{d,1}, \tag{44}$$

and denote

$$t := \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{r^2(K)}. \tag{45}$$

Then we have by (42), (43), and (44) that

$$\begin{cases} \operatorname{sgn} k_n(\mathbf{x}) = \operatorname{sgn} V_{n-2}(t) & (\mathbf{x} \in K,\ \mathbf{x} \notin \ell_{\mathbf{a}}), \\ |k_n(\mathbf{x})| \le \varepsilon \cdot cr^2(K)(1 - |t|)|V_{n-2}(t)| & (\mathbf{x} \in K). \end{cases} \tag{46}$$

Finally, we define

$$p_n(\mathbf{x}) := g_{n,\mathbf{a}}(\mathbf{x}) + k_n(\mathbf{x}) \in \pi_n^{d,1}. \tag{47}$$

Plainly the definitions (42), (44), and (47) immediately give (ii) and (iii), while it remains to show (i). This is also clear in view of (2), (43), (44), (46), and (47) whenever (40) applies, i.e., when

$$|p_n(\mathbf{x})| = |T_n(t) + \Delta(\mathbf{x}, t)(1 - |t|)V_{n-2}(t)| \le 1 \qquad (|t| < 1, \mathbf{x} = t\mathbf{a}) \tag{48}$$

holds with

$$\Delta(\mathbf{x}, t) := \frac{\varepsilon \cdot q(\mathbf{x})}{1 - |t|}. \tag{49}$$

Now by (42), $0 \le \Delta(\mathbf{x}, t)$, and (42), (45) entails $\Delta(\mathbf{x}, t) \le \varepsilon \cdot cr^2(K)$. Whenever $\varepsilon$ is chosen not exceeding $(\sin^2 \pi/2n)/cr^2(K)$, we conclude $0 \le \Delta(\mathbf{x}, t) \le \sin^2 \pi/2n$ and (48) follows from (40). Hence $\|p_n\|_{C(K)} = 1$ and the validity of condition (i) of Theorem 3 is proved, too. ■

In the next section we reverse this statement and prove that parabolic flatness ensures unicity of Markov-extremal polynomials.

**Theorem 4.** *Suppose $K$ is a symmetric convex body in $\mathbb{R}^d$ and $p_n \in \pi_n^{d,1}$ is a given Markov-extremal polynomial normalized according to*

$$\|p_n\|_{C(K)} = 1. \tag{i}$$

*Let $\mathbf{a} \in K^0$ be such that*

$$p_n|_{\ell_{\mathbf{a}}} \equiv g|_{\ell_{\mathbf{a}}} \tag{ii}$$

*also holds true. If* $\mathbf{a}$ *is parabolically flat, then we also have*

$$p_n \equiv g \tag{50}$$

*on the whole of* $\mathbb{R}^d$.

(Note that by the theorem of Kroó some $\mathbf{a} \in K^0$ satisfying (ii) must always exist to any Markov-extremal $p_n$.)

**Proof.** Let us consider now the symmetric matrix or bilinear form $A = \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ and the corresponding quadratic form defined by

$$M(\mathbf{b}) := \frac{\partial^2 h}{\partial \mathbf{b}^2}(\mathbf{a}) \qquad (\mathbf{b} \in H \simeq \mathbb{R}^{d-1}), \tag{51}$$

where here, as usual, we write $h = p_n - g$. Recall that $A$ (i.e., $M$) determines a set $\mathbf{u}_j \in H$ $(j = 1, \dots, d-1)$ of orthonormal eigenvectors with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{d-1}$ satisfying $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$.

Now Lemma 1 means that the quadratic form $M$ has only nonnegative values, i.e., $M$ is *positive semidefinite*, which has a number of useful structural consequences.

First, $0 \le \lambda_j$ entails that there exists some $k \in \mathbb{N}$, $0 \le k \le d-1$, with $0 = \lambda_1 = \cdots = \lambda_k < \lambda_{k+1} \le \cdots \le \lambda_{d-1}$, and thus

$$H = N \oplus U, \quad \text{i.e.,} \quad \mathbf{b} = \mathbf{n} + \mathbf{u} \qquad (\forall\, \mathbf{b} \in H) \quad (\mathbf{n} \in N,\ \mathbf{u} \in U), \tag{52}$$

where $\oplus$ is to denote the direct sum of subspaces and

$$N := \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}], \qquad U := \operatorname{span}\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_{d-1}\}. \tag{53}$$

(We should remark here that the bilinear form $A$ and the spaces $U$ and $N$ depend on the point $\mathbf{a}$, so that we might have put $A_{\mathbf{a}}$, $U_{\mathbf{a}}$, and $N_{\mathbf{a}}$ as well. That more precise notation will be used at the end when turning to the case of the other point $-\mathbf{a}$.)

Second, we easily obtain that $M(\mathbf{b}) = M(\mathbf{n}+\mathbf{u}) = \langle \mathbf{n}+\mathbf{u}, A(\mathbf{n}+\mathbf{u})\rangle = \langle \mathbf{n}+\mathbf{u}, A\mathbf{u}\rangle = \langle \mathbf{u}, A\mathbf{u}\rangle = M(\mathbf{u})$ and

$$N = \{\mathbf{b} \in H : M(\mathbf{b}) = 0\} = \{\mathbf{b} \in H : A\mathbf{b} = 0\} = \operatorname{Ker} A. \tag{54}$$

Moreover, if $k < d-1$, then

$$\inf\left\{\frac{M(\mathbf{u})}{\langle \mathbf{u}, \mathbf{u}\rangle} : \mathbf{u} \in U\right\} = \min\{\lambda_j : j = k+1, \dots, d-1\} = \lambda_{k+1} > 0 \tag{55}$$

and also

$$M(\mathbf{b}) \ge \lambda_{k+1} |\mathbf{u}|^2 \qquad (\mathbf{b} \in H,\ \mathbf{b} = \mathbf{n} + \mathbf{u},\ \mathbf{n} \in N,\ \mathbf{u} \in U). \tag{56}$$

In order to prove Theorem 3, it suffices to show that we have

$$H = N, \qquad U = \{\mathbf{0}\}, \tag{57}$$

as this is the very case when Lemma 2 can be applied to all $\mathbf{b} \in H$ and (24) ensures (50).

Thus it sufficies to show that in case $U \neq \{\mathbf{0}\}$, $\mathbf{a}$ cannot be parabolically flat, i.e., $K$ is parabolically separated at $\mathbf{a}$ in some direction $\mathbf{b} \in H$. We will show below that indeed, $K$ is parabolically separated (from $H$) at $\mathbf{a}$ in *any* direction $\mathbf{u}_0 \in U$, $|\mathbf{u}_0| = 1$. We start with the observation that for all $\mathbf{u} \in U$, (56) entails (with some $\delta_0 > 0$):

$$\frac{\partial^2 h}{\partial \mathbf{u}^2}(\mathbf{a}+\mathbf{v}) \geq \tfrac{1}{2}\lambda_{k+1}|\mathbf{u}|^2 \qquad (\mathbf{u} \in U, \mathbf{v} \in H,\ |\mathbf{v}| \leq \delta_0) \tag{58}$$

in view of the uniform continuity of the second partial (directional) derivative polynomial $\partial^2 h/\partial u^2$ at $\mathbf{a}$.

Now let $\mathbf{x} \in K$ be arbitrary and denote

$$\mathbf{x}' := \mathbf{a} + \rho(\mathbf{x}-\mathbf{a}), \qquad \mathbf{y} := \mathbf{x}' + \rho\frac{\langle \mathbf{a}-\mathbf{x}, \mathbf{a}\rangle}{r(K)^2}\cdot \mathbf{a}, \qquad \rho := \frac{\delta_0}{\operatorname{diam} K}, \tag{59}$$

where $\operatorname{diam} K$ is the diameter of the convex set $K$. Clearly $\mathbf{x}' \in K$, $\mathbf{y} \in H + \mathbf{a}$, and $|\mathbf{y}-\mathbf{a}| \leq \delta_0$, hence for

$$\mathbf{v} := \mathbf{y} - \mathbf{a} \in H, \quad \mathbf{v} = \mathbf{n} + \mathbf{u} \qquad (\mathbf{n} \in N,\ \mathbf{u} \in U) \tag{60}$$

we can apply (58) to get

$$\frac{\partial^2 h}{\partial \mathbf{u}^2}(\mathbf{a}+\mathbf{v}_\xi) \geq \tfrac{1}{2}\lambda_{k+1}|\mathbf{u}|^2 \qquad (0 \leq \xi \leq 1,\ \mathbf{v}_\xi := \mathbf{n} + \xi\mathbf{u}). \tag{61}$$

Next we apply the Taylor expansion formula (21) for $h$ at the point $\mathbf{a}+\mathbf{n}$ and get, using also Lemmas 2 and 3 at $\mathbf{a}+\mathbf{n}$, that

$$\begin{aligned} h(\mathbf{y}) \ &= \ h(\mathbf{a}+\mathbf{n}) + \frac{\partial h(\mathbf{a}+\mathbf{n})}{\partial \mathbf{u}} + \frac{1}{2}\frac{\partial^2 h}{\partial \mathbf{u}^2}(\mathbf{a}+\mathbf{n}+\xi\mathbf{u}) \\ &= \ 0 + 0 + \frac{1}{2}\frac{\partial^2 h}{\partial \mathbf{u}^2}(\mathbf{a}+\mathbf{v}_\xi) \geq \tfrac{1}{4}\lambda_{k+1}|\mathbf{u}|^2. \end{aligned} \tag{62}$$

On the other hand, for any fixed unit vector $\mathbf{u}_0 \in U$:

$$\begin{aligned} \langle \mathbf{u}_0, \mathbf{x}\rangle^2 \ &= \ \langle \mathbf{u}_0, \mathbf{x}-\mathbf{a}\rangle^2 = \frac{1}{\rho^2}\langle \mathbf{u}_0, \mathbf{x}'-\mathbf{a}\rangle^2 = \frac{1}{\rho^2}\langle \mathbf{u}_0, \mathbf{y}-\mathbf{a}\rangle^2 \\ &= \ \frac{1}{\rho^2}\langle \mathbf{u}_0, \mathbf{v}\rangle^2 = \frac{1}{\rho^2}\langle \mathbf{u}_0, \mathbf{n}+\mathbf{u}\rangle^2 = \frac{1}{\rho^2}\langle \mathbf{u}_0, \mathbf{u}\rangle^2 \leq \frac{1}{\rho^2}|\mathbf{u}|^2, \end{aligned} \tag{63}$$

hence for the fixed unit vector $\mathbf{u}_0 \in U$ we get, by (62):

$$h(\mathbf{y}) \geq \frac{\lambda_{k+1}}{4}\rho^2\langle \mathbf{u}_0, \mathbf{x}\rangle^2. \tag{64}$$

Next we estimate $h(\mathbf{y})$ from above. Clearly

$$h(\mathbf{y}) = p(\mathbf{y}) - g(\mathbf{y}) \leq |p(\mathbf{y}) - p(\mathbf{x}')| + p(\mathbf{x}') - g(\mathbf{y}) \leq |p(\mathbf{y}) - p(\mathbf{x}')|$$

since $g(\mathbf{y}) = g(\mathbf{a}+H) = 1$ and $\mathbf{x}' \in K$ entails $p(\mathbf{x}') \leq 1$. Certainly the segment $[\mathbf{x}', \mathbf{y}]$ lies in the cylinder $\{\mathbf{z} \in \mathbb{R}^d : |\langle \mathbf{z}, \mathbf{a}\rangle| \leq r^2(K),\ |\operatorname{dist}(\mathbf{z}, \ell_{\mathbf{a}})| \leq \operatorname{diam} K\} = \hat{K}$, which is a compact subset of $\mathbb{R}^d$ and thus $\|\mathrm{grad}\, p\|_{C(\hat{K})} =: L < \infty$. Thus we get with this constant $L < \infty$ that

$$h(\mathbf{y}) \leq L \cdot |\mathbf{y} - \mathbf{x}'| = L \frac{\langle \mathbf{a} - \mathbf{x}, \mathbf{a} \rangle}{r(K)} \rho. \tag{65}$$

Comparing (64) and (65) we obtain the inequality

$$\langle \mathbf{a} - \mathbf{x}, \mathbf{a} \rangle \geq \frac{\lambda_{k+1} \cdot r(K) \cdot \rho}{4 \cdot L} \langle \mathbf{u}_0, \mathbf{x} \rangle^2 \tag{66}$$

which is just the same as

$$\langle \mathbf{u}_0, \mathbf{x} \rangle^2 \leq c_1 \langle \mathbf{a} - \mathbf{x}, \mathbf{a} \rangle \qquad (\mathbf{u}_0 \in U,\ |\mathbf{u}_0| = 1). \tag{67}$$

A similar argument can be used for $-\mathbf{a}$ to show

$$\langle \mathbf{u}_0, \mathbf{x} \rangle^2 \leq c_2 \langle \mathbf{x} + \mathbf{a}, \mathbf{a} \rangle \qquad (\mathbf{u}_0 \in \tilde{U},\ |\mathbf{u}_0| = 1). \tag{68}$$

Now the only thing to be checked is that $U = U(\mathbf{a})$, the subspace spanned by the positive eigenvectors of $A_{\mathbf{a}}$, is the same as $\tilde{U} = U(-\mathbf{a})$, the subspace spanned by the positive eigenvectors of $A_{-\mathbf{a}}$. Now $U := N^{\perp}$ and $\tilde{U} := \tilde{N}^{\perp}$ ($\tilde{N} :=$ Ker $A_{-\mathbf{a}}$), hence it suffices to show $N = \tilde{N}$. However, this is an easy consequence of our Lemma 1 (applied to both $\mathbf{a}$ and to $-\mathbf{a}$). Thus (67) and (68) cover the same set of vectors $\mathbf{u}_0 \in U$, and with $c = \min\{c_1, c_2\}$ we arrive at (34) as stated. The proof is complete. ∎

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