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UNIQUENESS OF MULTIVARIATE CHEBYSHEV-TYPE EXTREMAL POLYNOMIALS FOR CONVEX BODIES

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To Gàbor Hàlasz on the occasion of his sixtieth birthday

A. Kroó and D. Schmidt have solved the multivariate Chebyshev extremal problem of finding the least possible sup norm of polynomials on convex bodies under certain constraints, generalizing the classical one-dimensional Chebyshev extremal problem. A. Kroó also began a study of uniqueness of the minimal polynomials and settled the question in dimension 2. We continue this work and give a geometric condition on the convex body which is necessary and sufficient for the Chebyshev polynomials to be the unique extremal polynomials in the corresponding optimisation problem.

In particular, we will obtain that for the unit ball $B^d(\mathbf{0}, 1)$ we do not have uniqueness, while for the unit cube $[-1, 1]^d$ the extremal polynomials are essentially unique.

1. Introduction

Let $K \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a convex body (i.e., a convex, closed set with non-empty interior) and denote by $\|\cdot\| = \|\cdot\|_{C(K)}$ the usual supremum norm

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on K. We will denote by $\pi_n^{m,k}$ the set of algebraic polynomials from $\mathbb{R}^m$ to $\mathbb{R}^k$ (i.e., having m variables and k coordinate functions) with total degree of each of the coordinate polynomials not exceeding n. We also write $\pi_n^m := \pi_n^{m,1}$ and $\pi_n := \pi_n^{1,1}$. The Euclidean (ℓ_2) length of a vector in $\mathbb{R}^d$ will be written using the usual absolute value sign $|\cdot|$, and $\langle\cdot,\cdot\rangle$ stands for the usual scalar product.

Chebyshev-type extremal problems consist of minimizing $\|p_n\|_{C(K)}$ under constraints on $p_n \in \pi_n^d$ normalizing either the "leading coefficients/terms", or the value attained at some point outside K. The two type of problems are connected, since the leading coefficient restriction can be regarded as a normalization of the polynomials at the infinity. For a given $p_n \in \pi_n^d$, let us denote by $p_n^\star$ the "n-degree homogeneous part" or "leading terms", i.e.,

$$p_n(\mathbf{x}) = \sum_{\substack{\mathbf{k}\in\mathbb{N}^d \\ |\mathbf{k}|\le n}} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}, \quad p_n^\star(\mathbf{x}) := \sum_{\substack{\mathbf{k}\in\mathbb{N}^d \\ |\mathbf{k}|= n}} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}. \tag{1.1}$$

Then a normalization of the leading terms $p_n^\star$ of p_n can be defined w.r.t. any direction (unit vector) $\mathbf{v} \in \mathbb{R}^d$, $|\mathbf{v}| = 1$, as a restriction $p_n^\star(\mathbf{v}) = 1$. Now for $\lambda > 0$ large enough, $\lambda\mathbf{v} \notin K$, and normalizing $p_n(\lambda\mathbf{v})$ with respect to λ^n connects the two kind of problems in the limiting case $\lambda \to \infty$ as

$$\lim_{\lambda\to+\infty} \frac{p_n(\lambda\mathbf{v})}{\lambda^n} = p_n^\star(\mathbf{v}). \tag{1.2}$$

This could also be an alternative to define $p_n^\star$ and $p_n^\star(\mathbf{v})$.

With the above notations, A. Kroó and D. Schmidt [5] and A.Kroó [2] investigated

$$C_n(K,\mathbf{v}) := \min\{\|p_n\| : p_n \in \pi_n^d, \quad p_n^\star(\mathbf{v}) = 1\}, \tag{1.3}$$

and

$$R_n(K,\mathbf{x}^\star) := \min\{\|p_n\| : p_n \in \pi_n^d, \quad p_n(\mathbf{x}^\star) = 1\}. \tag{1.4}$$

Clearly in case $d = 1$ these extremal problems reduce to the well-known extremal problems of Chebyshev.

We call (1.3) the "leading term Chebyshev problem", and (1.4) the "function value Chebyshev problem". (Kroó mentions that extremal problems of (1.4) are sometimes called Richardson polynomials in the literature.)

The "maximal chord length (or transversal) of K in direction $\mathbf{v}$" is defined* for $\mathbf{v} \in \mathbb{R}^d$, $|\mathbf{v}| = 1$ as

$$\tau(K,\mathbf{v}) := \max\{|\mathbf{a}-\mathbf{b}| : \mathbf{a},\mathbf{b} \in K, \quad \mathbf{a}-\mathbf{b} = \lambda\mathbf{v},\ \lambda \in \mathbb{R}\}. \tag{1.5}$$

*Many papers including [2], [3], [4], [5], [6] term (1.5) the "width of K in direction $\mathbf{v}$". However, the established geometric notion is the above given one, cf. e.g. [8, pp.367–370].

Clearly, if $\mathbf{a}-\mathbf{b}=\lambda\mathbf{v}$ and $|\lambda|=\tau(K,\mathbf{v})$, then $\mathbf{a},\mathbf{b}\in\partial K$ (∂K being the boundary of K). In addition, K has supporting hyperplanes at $\mathbf{a}$ and at $\mathbf{b}$. However, the following more precise statement is also true.

Lemma A. *Let $\mathbf{a},\mathbf{b}\in\partial K$ with $|\mathbf{a}-\mathbf{b}|=\tau(K,\mathbf{v})$. Then K has parallel supporting hyperplanes at $\mathbf{a}$ and $\mathbf{b}$.*

This was proved in [4] as Proposition 3 and it played a key role in [2].

For Problem (1.4) a similar geometric result was proved in [5] (cf. Corollary 1 on p. 421).

Lemma B. *For every $\mathbf{x}^\star\in\mathbb{R}^d\setminus K$ there exists a line ℓ passing through $\mathbf{x}^\star$such that $K\cap\ell=[\mathbf{a},\mathbf{b}]$ with some $\mathbf{a},\mathbf{b}\in\partial K$ and K possesses parallel supporting hyperplanes at $\mathbf{a}$ and $\mathbf{b}$. Moreover, the above occures if and only if $\mu=|\mathbf{x}^\star-\frac{\mathbf{a}+\mathbf{b}}{2}|/|\mathbf{a}-\mathbf{b}|$ is minimal.*

For the pair $\mathbf{a},\mathbf{b}\in\partial K$, or the direction $\mathbf{v}=\frac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$, any unit vector $\mathbf{v}^\star\in\mathbb{R}^d$, $|\mathbf{v}^\star|=1$, normal to the parallel supporting hyperplanes will be called a *conjugate*. Note that for both problems there are simple examples with the point pair $\{\mathbf{a},\mathbf{b}\}$ being unique or being not unique and the conjugate directions $\{\mathbf{v}^\star,-\mathbf{v}^\star\}$ being unique or not unique, independently of each other. However, no ambiguity arises from the use of $\{\mathbf{a},\mathbf{b}\}$, as we can consider any possible choice of them in all our statements below. Namely, whenever we have two pairs of extremal points, say $\mathbf{a},\mathbf{b}$ and $\mathbf{a}',\mathbf{b}'$, then the corresponding set of conjugate vectors and parallel supporting hyperplanes are exactly the same to both of them. Indeed, if $H(\mathbf{a})=\{\mathbf{v}^\star\}^\perp+\mathbf{a}$, $H(\mathbf{b})=\{\mathbf{v}^\star\}^\perp+\mathbf{b}$ form a supporting strip to K, then by $\mathbf{a}'-\mathbf{b}'=\mathbf{a}-\mathbf{b}=\tau\mathbf{v}$ we get $\mathbf{a}'\in H(\mathbf{a})$, $\mathbf{b}'\in H(\mathbf{b})$ and so the same supporting strip forms supporting hyperplanes at $\mathbf{a}'$ and $\mathbf{b}'$, too. The argument is similar, in Case B, with reference to the rule of the parallel secants. Thus the set of conjugate vectors is independent of the particular choice of the extremal point pair $\mathbf{a},\mathbf{b}$.

To formulate Kroó's solutions to the above Chebyshev problems, we have to use the classical Chebyshev polynomials

$$T_n(t):=\cos(n\arccos t)=2^{n-1}t^n+\ldots=2^{n-1}\prod_{j=1}^{n}(t-z_j^{(n)}) \qquad \left(z_j:=z_j^{(n)}:=\cos\Big(\frac{(2j-1)\pi}{2n}\Big)\right). \tag{1.6}$$

Theorem A [2]. *For any convex body $K\subset\mathbb{R}^d$ and direction $\mathbf{v}\in\mathbb{R}^d$, $|\mathbf{v}|=1$, we have*

$$C_n(K,\mathbf{v})=2^{-2n+1}\tau^n(K,\mathbf{v}). \tag{1.7}$$

Moreover, for any $\mathbf{a}, \mathbf{b} \in \partial K$ *with* $\mathbf{a} - \mathbf{b} = \tau(K, \mathbf{v}) \cdot \mathbf{v}$, *and for any conjugate vector* $\mathbf{v}^\star$, *the polynomial*

$$g_n(\mathbf{x}) := 2^{-2n+1}\tau^n(K, \mathbf{v})T_n\left(\frac{\langle \mathbf{v}^\star, 2\mathbf{x} - \mathbf{a} - \mathbf{b}\rangle}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b}\rangle}\right) \tag{1.8}$$

is extremal.

Theorem B [5]. *For any convex body* $K \subset \mathbb{R}^d$ *and any* $\mathbf{x}^\star \in \mathbb{R}^d \setminus K$, *the extremal quantity*

$$R_n(K, \mathbf{x}^\star) = T_n^{-1}\left(\frac{|2\mathbf{x}^\star - \mathbf{a} - \mathbf{b}|}{|\mathbf{a} - \mathbf{b}|}\right), \tag{1.9}$$

where $\mathbf{a}, \mathbf{b} \in \partial K$ *and* $\mathbf{x}^\star$ *lie on some straight line* ℓ *for which* $2\mu := \frac{|2\mathbf{x}^\star - \mathbf{a} - \mathbf{b}|}{|\mathbf{a} - \mathbf{b}|}$, *is minimal. Moreover, for any conjugate vector* $\mathbf{v}^\star$ *an extremal polynomial is furnished by*

$$r_n(\mathbf{x}) := R_n(K, \mathbf{x}^\star) \cdot T_n\left(\frac{\langle \mathbf{v}^\star, 2\mathbf{x} - \mathbf{a} - \mathbf{b}\rangle}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b}\rangle}\right). \tag{1.10}$$

A. Kroó started investigating the question of uniqueness, too. The first observation is that (1.8) or (1.10) is certainly not unique, if $\mathbf{v}^\star$ is not. However, Kroó observed that uniqueness of the conjugate direction is not sufficient for uniqueness of the extremal polynomials. He could fully characterize uniqueness in dimension $d = 2^\dagger$ using the following notion.

Definition. Let $L_\mathbf{a}$ be a straight line supporting the convex body $K \subset \mathbb{R}^2$ at $\mathbf{a} \in \partial K$. Then the point $\mathbf{a}$ is called *flat*, if no disc touching $L_\mathbf{a}$ at $\mathbf{a}$ can contain K.

Theorem C. *Let* $K \subset \mathbb{R}^2$ *be a convex body, and* $n \geq 2$. *Then* (1.8) *is the unique solution of the leading terms Chebyshev extremal problem* (1.3) *if and only if the pair of parallel lines supporting* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *is unique, and at least one of the points* $\mathbf{a}, \mathbf{b}$ *is flat with respect to the corresponding supporting line.*

Theorem D. *For a convex body* $K \subset \mathbb{R}^2$ *and* $\mathbf{x}^\star \in \mathbb{R}^2 \setminus K$ *the extremal polynomial* (1.10) *is the unique solution to the function value extremal problem* (1.4) *if and only if the pair of parallel lines supporting* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *is unique, and at least one of the points* $\mathbf{a}, \mathbf{b}$ *is flat with respect to the corresponding supporting line.*

†Kroó writes in his paper [2]: "Since the question of uniqueness is substantially more delicate and technical, we shall provide a complete solution to this problem only for bivariate polynomials".

2. Definitions and results

We consider a pair of boundary points $\{\mathbf{a}, \mathbf{b}\} \subset \partial K$ (furnished by Lemmas A or B, respectively) with some extremal properties and having parallel supporting hyperplanes with normal vector $\mathbf{v}^\star$, forming a supporting strip of K. Let us denote

$$H := \{\mathbf{v}^\star\}^\perp, \quad H_{\mathbf{a}} := H + \mathbf{a}, \quad H_{\mathbf{b}} := H + \mathbf{b}, \tag{2.1}$$

where, as usual, $S^\perp$ stands for the linear subspace of all vectors orthogonal to (all vectors of) $S \subset \mathbb{R}^d$. Since $\operatorname{int} K \neq 0$, we have $H_{\mathbf{a}} \neq H_{\mathbf{b}}$ and thus $\langle \mathbf{a} - \mathbf{v}, \mathbf{v}^\star \rangle \neq 0$. Without loss of generality let us fix the order of the two points so that $\mathbf{a} - \mathbf{b} = |\mathbf{a} - \mathbf{b}| \cdot \mathbf{v}$ in the leading coefficient Chebyshev problem, and $\langle \mathbf{x}^\star - \mathbf{b}, \mathbf{x}^\star - \mathbf{a} \rangle > 0$ (i.e., $\langle \mathbf{a} - \mathbf{b}, \mathbf{x}^\star - \mathbf{b} \rangle > 0$) in the function value problem. In the latter case we put $\mathbf{v} := (\mathbf{a} - \mathbf{b})/|\mathbf{a} - \mathbf{b}|$, in full accordance to the other case.

Thus we always have $\langle \mathbf{v}, \mathbf{v}^\star \rangle > 0$. We also put

$$G := \left(K - \frac{\mathbf{a} + \mathbf{b}}{2} \right) \cap H \subset H. \tag{2.2}$$

In order to generalize to higher dimensions, we consider quadratic forms $q(\mathbf{y})$ on $\mathbb{R}^{d-1}$ (i.e., $q \in \pi_2^{d-1}$, $q^* = q \not\equiv 0$) which can play the role of the simple quadratic function $c \cdot y^2$ occuring implicitely in Kroó's definition of flat points.

In what follows, we may assume $H = \mathbb{R}^{d-1}$. Thus, for a vector $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x} = (\mathbf{y}, 0)$ with $\mathbf{y} \in \mathbb{R}^{d-1}$, we also have $\mathbf{y} \in H$ and apply quadratic forms q on $\mathbb{R}^{d-1}$ to $\mathbf{x} \in \mathbb{R}^d$ by considering $q(\mathbf{y})$.

Now let us call a quadratic form q of $d-1$ variables *admissible*, if $q(\mathbf{y}) \geq 0$ for all $\mathbf{y} \in G$. In case $d = 2$, solved by Kroó, the only quadratic forms are of the form $q(y) = cy^2$, $c \in \mathbb{R} \setminus \{0\}$, and since K is a convex body, hence $G \neq \emptyset$, q is admissible if and only if $c > 0$. For higher dimensions the quadratic form q can be rather complicated even if the sets K and G are relatively simple. However, if $[\mathbf{a}, \mathbf{b}] \cap \operatorname{int} K \neq \emptyset$, we easily see that $q \geq 0$, i.e., q is positive semidefinite. In particular, this is the typical case when K is a *symmetric* convex body‡ in $\mathbb{R}^d$, or when K has a smooth boundary of order C^1, say. On the other hand, for $d = 3$, choosing $K := [0,1]^3$, $\mathbf{v} = (0,0,1)$ and $\mathbf{a} = (0,0,0)$, $\mathbf{b} = (0,0,1)$, $\mathbf{v}^\star = \mathbf{v}$, we easily see that $q(x,y) := xy$ is admissible, being non-negative in the positive quadrant of $\mathbb{R}^2$, but is indefinite as a quadratic form of $\mathbb{R}^2$.

In the following we are going to use the notations

$$\mathbf{y} := \mathbf{y}(\mathbf{x}) := \mathbf{x} - \mathbf{a} - \frac{\langle \mathbf{x} - \mathbf{a}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \mathbf{v} = \mathbf{x} - \mathbf{b} - \frac{\langle \mathbf{x} - \mathbf{b}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \mathbf{v}. \tag{2.3}$$

‡Not always. For further discussion see §7, Claim 13, and the Proof of Corollary 3.

Now it is easy to see that $\langle \mathbf{y}, \mathbf{v}^\star \rangle = 0$ and the two forms given in (2.3) are equal since $\mathbf{a} - \mathbf{b} = \frac{\langle \mathbf{a}-\mathbf{b}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \cdot \mathbf{v}$. For the very same reason we have for any $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{c} \in \ell$ the equivalent formulation

$$(2.4) \quad \mathbf{y} = \mathbf{x} - \mathbf{c} - \frac{\langle \mathbf{x} - \mathbf{c}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \mathbf{v} = \mathbf{x} - \mathbf{z}, \quad \mathbf{z} := \mathbf{z}(\mathbf{x}) := \mathbf{c} + \frac{\langle \mathbf{x} - \mathbf{c}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \mathbf{v},$$

where geometrically $\mathbf{z}$ is the point of intersection of the straight line ℓ through $\mathbf{a}$ and $\mathbf{b}$ and the hyperplane $H_{\mathbf{x}} = H + \mathbf{x}$ perpendicular to $\mathbf{v}^\star$ and containing the point $\mathbf{x}$.

Definition 1. We say that K is *separated* from its supporting hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ by the admissible quadratic form q if with the notation (2.3)–(2.4) above we have

$$(2.5) \qquad \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle \geq q(\mathbf{y}) \quad \text{and} \quad \langle \mathbf{x} - \mathbf{b}, \mathbf{v}^\star \rangle \geq q(\mathbf{y}) \quad (\forall \mathbf{x} \in K).$$

Note that $\langle \mathbf{a}-\mathbf{x}, \mathbf{v}^\star \rangle = \operatorname{dist}(\mathbf{x}, H_{\mathbf{a}})$ and $\langle \mathbf{x}-\mathbf{b}, \mathbf{v}^\star \rangle = \operatorname{dist}(\mathbf{x}, H_{\mathbf{b}})$. To interpret the above definition let us observe that in view of the supporting hyperplanes at $\mathbf{a}$ and $\mathbf{b}$ we always have (2.5) with $q \equiv 0$, and that (2.5) is equivalent to

$$(2.6) \qquad \langle \mathbf{b}, \mathbf{v}^\star \rangle + q(\mathbf{y}) \leq \langle \mathbf{x}, \mathbf{v}^\star \rangle \leq \langle \mathbf{a}, \mathbf{v}^\star \rangle - q(\mathbf{y}) \qquad (\forall \mathbf{x} \in K).$$

To explain (2.6), let us point out that we shall prove in Proposition 5 that $q(\mathbf{y}) \geq 0$ ($\mathbf{y} = \mathbf{y}(\mathbf{x}),\ \mathbf{x} \in K$) for all admissible q.

Definition 2. The support of K at $\mathbf{a}$ and $\mathbf{b}$ by $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$, respectively, is *quadratically flat*, if for all admissible quadratic forms $q \not\equiv 0$ we can find an $\mathbf{x} \in K$ such that with $\mathbf{y} = \mathbf{y}(\mathbf{x})$ defined in (2.3), either

$$(2.7) \qquad \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle < q(\mathbf{y})$$

or

$$(2.8) \qquad \langle \mathbf{b} - \mathbf{x}, \mathbf{v}^\star \rangle < q(\mathbf{y})$$

holds true. In other words, this means that there is no admissible quadratic form separating K from $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$.

Let us give also a special, simpler variant of the above definitions for later use.

Definition 3. K is separated from its supporting hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ *parabolically* in the direction $\mathbf{u} \in \mathbb{R}^d$, $|\mathbf{u}| = 1$ and $\mathbf{u} \in H$, if there exists a positive constant $c > 0$ such that whenever $\mathbf{x} \in K$ and $\mathbf{y} = \mathbf{y}(\mathbf{x})$ is given by (2.5) or (2.3), we have

$$(2.9) \qquad \langle \mathbf{b}, \mathbf{v}^\star \rangle + c\langle \mathbf{y}, \mathbf{u} \rangle^2 \leq \langle \mathbf{x}, \mathbf{v}^\star \rangle \leq \langle \mathbf{a}, \mathbf{v}^\star \rangle - c\langle \mathbf{y}, \mathbf{u} \rangle^2.$$

Definition 4. The support of K at $\mathbf{a}$ and $\mathbf{b}$ by $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$, respectively, is *parabolically flat*, if for all directions $\mathbf{u} \in \mathbb{R}^d$, $|\mathbf{u}| = 1$ and $\mathbf{u} \in H$, and for all $c > 0$ there exists $\mathbf{x} \in K$ such that with $\mathbf{y}$ as in (2.3) we have either

$$\langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle < c \langle \mathbf{y}, \mathbf{u} \rangle^2 \tag{2.10}$$

or

$$\langle \mathbf{x} - \mathbf{b}, \mathbf{v}^\star \rangle < c \langle \mathbf{y}, \mathbf{u} \rangle^2. \tag{2.11}$$

Note that Definitions 3 and 4 restrict the quadratic forms from all admissible forms to parabolic forms $c \cdot y^2$, $y = \langle \mathbf{y}, \mathbf{u} \rangle$, quite similarly to the bivariate case considered by Kroó. This notion of parabolic flatness was used in [6] to investigate unicity of Markov-extremal polynomials on convex *symmetric* bodies. However, here we have to generalize the notion of flatness further, which reflects the fact that the convex body K is not necessarily symmetric.

With the above definitions we can formulate our results.

Theorem 1. *In the leading term Chebyshev problem* (1.3)*, the canonical extremal polynomials* (1.8) *are unique if and only if the conjugate vector* $\mathbf{v}^\star$ *is unique (with the normalization* $\langle \mathbf{v}^\star, \mathbf{v} \rangle > 0$*) and the support of* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *by* $H_{\mathbf{a}}$ *and* $H_{\mathbf{b}}$*, respectively, is quadratically flat.*

Corollary 1. *Suppose that* $\ell \cap [\mathbf{a}, \mathbf{b}] \neq \emptyset$*. Then the leading term Chebyshev problem* (1.3) *has only the unique canonical extremal polynomials* (1.8) *if and only if the conjugate vector* $\mathbf{v}^\star$ *is unique and the support of* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *by* $H_{\mathbf{a}}$ *and* $H_{\mathbf{b}}$ *respectively, is parabolically flat.*

Theorem 2. *In the function value Chebyshev problem* (1.4) *the canonical extremal polynomials* (1.10) *are unique if and only if the conjugate vector* $\mathbf{v}^\star$ *is unique (with the normalization* $\langle \mathbf{v}^\star, \mathbf{v} \rangle > 0$*) and the support of* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *by* $H_{\mathbf{a}}$ *and* $H_{\mathbf{b}}$*, respectively, is quadratically flat.*

Corollary 2. *Suppose that* $\ell \cap [\mathbf{a}, \mathbf{b}] \neq \emptyset$*. Then the function value Chebyshev problem* (1.4) *has only the unique canonical extremal polynomials* (1.10) *if and only if the conjugate vector* $\mathbf{v}^\star$ *is unique and the support of* K *at* $\mathbf{a}$ *and* $\mathbf{b}$ *by* $H_{\mathbf{a}}$ *and* $H_{\mathbf{b}}$ *respectively, is parabolically flat.*

Now let us show that Corollary 1 follows from Theorem 1. If the extremal problem has unique solution, then $\mathbf{v}^\star$ is unique and the support is quadratically, hence even parabolically flat. Conversely, suppose that $\mathbf{v}^\star$ is unique and the support is parabolically flat. In view of Theorem 1 we have to show only that the support is quadratically flat, too. As remarked already before Definition 1, any admissible q has to be positive semidefinite if $[\mathbf{a}, \mathbf{b}] \cap \operatorname{int} K \neq \emptyset$.

However, picking any non-zero term of the canonical square sum representation of the positive semidefinite quadratic form q, we obtain a parabolic form $p(\mathbf{y}) = c\langle \mathbf{u}, \mathbf{y}\rangle^2$ with $c > 0$ and satisfying $0 \le p(\mathbf{y}) \le q(\mathbf{y})$ $(\mathbf{y} \in \mathbb{R}^{d-1})$. Now if the support is parabolically flat, it satisfies (2.10) or (2.11), i.e., (2.7) or (2.8) with q replaced by p. Since $p \le q$, this implies (2.7) or (2.8) also for q. Hence the support is quadratically flat and the corollary is proved. A similar reasoning applies to the proof of Corollary 2, using Theorem 2.

Note also that we have actually shown that "parabolic separation" is equivalent to "positive semidefinite separation".

On the other hand, using spheres instead of our parabolic surfaces would not provide a good characterization of uniqueness. That can be easily seen e.g. from working out the example $K := \{(x, y, z) \in \mathbb{R}^3 : |y| \le 1 \text{ and } x^2 + z^2 \le 1\}$, $\mathbf{v} = \mathbf{e}_3 = (0, 0, 1)$. The cylindrical body K, lying on its side, is not "separated spherically" from the horizontal supporting hyperplanes at $\mathbf{a} = (0, 0, 1)$ and $\mathbf{b} = (0, 0, -1)$, but the definition directly provides a parabolic separation. In view of the above results we can conclude that the extremal polynomials in the leading term Chebyshev problem in direction $\mathbf{v}$ may not be unique, whence "spherical flatness" could not be used to characterize unicity. When e.g. $\mathbf{x}^\star = (0, 0, z_0)$ $|z_0| > 1$, the same situation occurs for the function value Chebyshev problem for K at $\mathbf{x}^\star$.

Let us show finally that in case $[\mathbf{a}, \mathbf{b}] \subset \partial K$ also non-parabolic (i.e., indefinite) quadratic forms have to be involved. Our example will be a convex set $K \subset \mathbb{R}^3$, parabolically flat, but not flat quadratically. We see then from the necessity part of Theorem 1 (cf. §4) that in this case the extremal polynomials (1.8) are not the only ones in the leading term Chebyshev problem (1.3).

Example. Our convex body $K \subset \mathbb{R}^3$ will be situated in $[0, 1] \times [0, 1] \times [-1, 1]$ with $\mathbf{v} = \mathbf{e}_3 = (0, 0, 1)$ and $\mathbf{a} = (0, 0, 1)$, $\mathbf{b} = (0, 0, -1)$ belonging to K. Even from these few conditions it follows that $\tau(K, \mathbf{e}_3) = 2$, $C_n(K, \mathbf{e}_3) = 2^{1-n}$ and $G_n(x, y, z) := 2^{1-n}T_n(z)$ is an extremal polynomial.

We construct K to be symmetric with respect to the line $L = \{y = x, z = 0\}$, i.e., $(x, y, z) \in K$ if and only if $(y, x, -z) \in K$. We also suppose $K \cap \{(x, y, z) : z = 0\} = D := \{(x, y, 0) : 0 \le x, y,\ x + y \le 1\}$. We define first $K^+ := K \cap \{z \ge 0\}$. Since $K^+ \cap \{z = 0\} = D$ and K^+ is convex, the reflection of K^+ to the line L furnishes another convex body K^-, with $K^- \cap K^+ = D$. Then we define K as the union of K^+ and K^- which ensures that K is symmetric to L, located in $[0, 1] \times [0, 1] \times [-1, 1]$, containing $\mathbf{a}$ and $\mathbf{b}$, and having non-empty interior. (We also need to check that K is convex as well, but this will be clear from the construction.)

Now consider the two curves

$$\gamma := \{(x,y,z) : y = 0,\ 0 \le x \le 1,\ 1 - z = x^3\} \tag{2.12}$$

and

$$\Gamma := \{(x,y,z) : x = 0,\ 0 \le y \le 1,\ 1 - z = y^{3/2}\}. \tag{2.13}$$

Define K^+ as the convex hull

$$K^+ := \mathrm{con}\,\{\gamma, \Gamma, D\} \tag{2.14}$$

which is clearly in $[0,1]^3$, contains $\mathbf{a}$, and is convex by definition.

It is also clear that $K^+ \cap \{z = 0\} = D$, and for any $x_0, y_0 \in [0,1]$ the line $\{x = x_0,\ y = y_0\}$ contains a point from K^+ if and only if $(x_0, y_0, 0) \in D$. Hence in this case $\{x = x_0,\ y = y_0\} \cap K^+ = \{(x_0, y_0, z) : 0 \le z \le s(x_0, y_0)\}$ with some surface $s(x,y)$ defined on the triangle $\{(x,y),\ 0 \le x,\ y \le 1,\ x+y \le 1\}$. To describe $s(x,y)$ we refer to (2.14) and write $K^+ = \mathrm{con}\,(\mathrm{con}\,(\gamma, \Gamma), D)$ to obtain

$$\begin{aligned} s(x_0, y_0) & \\ &= \max\{z : \exists 0 \le \alpha, \beta,\ \alpha + \beta = 1,\ 0 \le x, y \le 1 \\ &\qquad \text{s.t. } \alpha(x, 0, 1 - x^3) + \beta(0, y, 1 - y^{3/2}) = (x_0, y_0, z)\} \\ &= 1 - \min\{\alpha x^3 + \beta y^{3/2} : \exists 0 \le \alpha, \beta,\ \alpha + \beta = 1,\ 0 \le x, y \le 1 \\ &\qquad \text{s.t. } \alpha x = x_0,\ \beta y = y_0\}. \end{aligned} \tag{2.15}$$

Now, using the arithmetic-geometric mean inequality, we have

$$\alpha x^3 + \beta y^{3/2} \ge \sqrt[3]{3\alpha x^3 \cdot \left(\frac{3}{2}\beta y^{3/2}\right)^2} = \frac{3\alpha x \beta y}{2^{2/3}\alpha^{2/3}\beta^{1/3}}.$$

It is easy to see that for $0 \le \alpha$, β, $\alpha + \beta = 1$, we also have $\alpha^{2/3}\beta^{1/3} \le 2^{2/3}/3$, hence we are led to

$$\alpha x^3 + \beta y^{3/2} \ge 9/2^{4/3}\alpha x \cdot \beta y \ge \alpha x \cdot \beta y.$$

Applying the above in (2.15) we obtain

$$s(x_0, y_0) \le 1 - x_0 \cdot y_0 \tag{2.16}$$

or, extending (2.16) to the whole of K,

$$x \cdot y \le 1 - |z| \qquad \forall (x,y,z) \in K. \tag{2.17}$$

Note that (2.17) means that K is quadratically separated from $H_{\mathbf{a}} = \{(x, y, 1) : x, y \in \mathbb{R}\}$ and $H_{\mathbf{b}} = \{(x, y, -1) : x, y \in \mathbb{R}\}$ at $\mathbf{a}$ and $\mathbf{b}$, respectively, by the quadratic form $q(x, y) = x \cdot y$ which is admissible, being non-negative in the non-negative quadrant $\{(x, y) : x \geq 0,\ y \geq 0\}$. Also it is easy to see (and will be seen later in the general situation) that the extremal polynomials of the leading term Chebyshev problem for K in direction $\mathbf{v}$ are not unique, e.g.

$$F_n(x, y, z) := G_n(x, y, z) + \sin^2\left(\frac{\pi}{2n}\right) \cdot xy \cdot \frac{G_n(x, y, z)}{z^2 - \cos^2\left(\frac{\pi}{2n}\right)} \tag{2.18}$$

is also an extremal polynomial of degree n.

Finally we check if K is subject to any condition of parabolic separation, i.e., if we can have in place of (2.17) also an inequality

$$(ax + by)^2 \leq 1 - |z| \qquad \forall (x, y, z) \in K \tag{2.19}$$

with $q(x, y) = \langle (a, b), (x, y) \rangle^2$. However, for $0 < \xi < 1$, the choice of the points $(\xi, 0, 1 - \xi^3) \in \gamma \subset K$ yields $a^2\xi^2 \leq \xi^3$, hence $a = 0$, and the points $(0, -\eta, 1 - \eta^3) \in \gamma^-$ (the reflection of γ with respect to L) show that $b^2\eta^2 \leq \eta^3$ $(0 < \eta < 1)$, hence also $b = 0$. Thus (2.19) holds only for $a = b = 0$, and K is actually parabolically flat, as sated.

Remark. We shall see later in Theorems 4 and 5 that the conditions of "smoothness" (i.e., uniqueness of the conjugate vector $bfvs$) and "flatness" can be considered separately as well. Smoothness results in some further connections, but not necessarily identical equality of any extremal polynomials and the canonical examples.

3. Some auxiliary facts about univariate polynomials

In the course of proof the following well-known lemma will be used at several occasions.

Proposition 1 (Weak sign changes property). *Let $p \in \pi_n$ be a polynomial of degree at most n and $t_0 < t_1 < \ldots < t_{n+1}$ be $n + 2$ nodes on the real line. If p has weak sign changes at these nodes, i.e.,*

$$(-1)^j p(t_j) \leq 0 \qquad (j = 0, 1, \ldots, n + 1) \tag{3.1}$$

holds for all $n + 2$ nodes, then p must vanish identically.

This lemma seems to be known for a long time and it probably goes back to Chebyshev's time. An interpolatory proof can be found e.g. in [2], cf. Proposition 2 there.

We also use the following easy construction of polynomials.

Proposition 2. *With the notations* (1.6) *let us define the polynomials*

$$V_{n-2}(t) := 2^{n-1}\prod_{j=2}^{n-1}(t-z_j) = \frac{T_n(t)}{t^2-z_1^2} \in \pi_{n-2}. \tag{3.2}$$

Then we have

$$\|V_{n-2}\|_{C[-1,1]} = \frac{1}{\sin^2\left(\frac{\pi}{2n}\right)}, \tag{3.3}$$

$$\operatorname{sgn}(T_n(t)) = \operatorname{sgn}(|t|-z_1)\cdot \operatorname{sgn}(V_{n-2}(t)), \tag{3.4}$$

and

$$1-T_n(t) \geq \frac{1+\cos\left(\frac{\pi}{2n}\right)}{\sin^2\left(\frac{\pi}{2n}\right)}(1-t) \qquad (z_1 \leq t \leq 1). \tag{3.5}$$

The proof requires only direct calculations. However, since we shall need only the obvious sign rule (3.4) and the fact that (3.3), (3.5) hold with some finite constants, we omit the details here. A precise calculation of the above can be found also in [6].

Propositions 1 and 2 were used by Kroó in [2], [3] at several instances in the solution of the bivariate case. We involve two more auxiliary statements.

Proposition 3. *For any $m \in \mathbb{N}$ there exists a positive constant $c(m) > 0$ such that for every polynomial $p(t) \in \pi_m$ we have*

$$\max_{[c(m),1]} |p(t)| \geq \frac{|p(0)|}{2}.$$

The statement is a direct consequence of the well-known Remez inequality, cf. e.g. [1], Theorem 5.1.1.

The following stronger variant of Proposition 1 will be used also.

Proposition 4. *Let $m \in \mathbb{N}$ and S be an arbitrary subset of $(-1,1)$ having $|S| := m+1$ (distinct) elements. Then there exists a finite positive constant $\alpha = \alpha(S)$ with the property that all polynomials $p(t) \in \pi_m$, changing sign in the weak sense $m+1$ times at the nodes of S, necessarily must satisfy*

$$|p(-1)| \leq \alpha(S)\cdot|p(1)|.$$

Proof. Let us denote by Ω the set of all polynomials $p \in \pi_m$, changing signs weakly at the points of S in the meaning of (3.1) and satisfying $p(-1) = 1$. On the finite dimensional vector space π_m we define the norm $||p|| := \max_S |p|$. Note that this seminorm is really a norm, since if p vanishes on S, then it has $m+1$ zeros and must vanish identically. Now clearly Ω is a convex and closed set, and thus the convex set $T := \{p(1) : p \in \Omega\}$ is also closed. Also T does not contain 0. Indeed, if the equality $p(1) = 0$ holds for some $p \in \Omega$, then the nodes of S would provide $m+2$ points of weak sign changes of $p \in \pi_m$, hence $p \equiv 0$ by Proposition 1, a contradiction to the assumption $p \in \Omega$. In all, we find that T is separated from 0, which implies the assertion.

4. Proof of the necessity part

Exploiting the close similarity between the two problems, we give a common construction to prove necessity in both cases. Starting from normalization on $\mathbf{v}$ or at $\mathbf{x}^\star$, we get in the statements of Theorems 1 and 2 a situation where (in view of the choices of sign, preceding Definition 1) $\mathbf{a}-\mathbf{b} = |\mathbf{a}-\mathbf{b}| \cdot \mathbf{v}$ and $\langle \mathbf{v}, \mathbf{v}^\star \rangle > 0$ with any conjugate vector $\mathbf{v}^\star$. Clearly both extremal functions (1.8) and (1.10) are constant multiples of the same Chebyshev polynomial for which, by the chain rule,

$$\partial T_n \left(\frac{\langle \mathbf{v}^\star, 2\mathbf{x} - \mathbf{a} - \mathbf{b} \rangle}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b} \rangle} \right) = T_n' \left(\frac{\langle \mathbf{v}^\star, 2\mathbf{x} - \mathbf{a} - \mathbf{b} \rangle}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b} \rangle} \right) \cdot \frac{2}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b} \rangle} \cdot \mathbf{v}^\star, \tag{4.1}$$

a vector which is parallel to $\mathbf{v}^\star$. Thus (4.1) shows immediately that for different conjugate vectors the canonical examples (1.8), (1.10) are also different. Whence (under the sign condition $\langle \mathbf{v}, \mathbf{v}^\star \rangle > 0$) the conjugate vector $\mathbf{v}^\star$ has to be unique in order to get uniqueness of the extremal polynomials. Let us then proceed supposing the uniqueness of $\mathbf{v}^\star$ and consider any admissible quadratic form q quadratically separating K from its supporting hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ in the sense of Definition 1. Now, with $\delta > 0$ small enough, we define

$$S_n(\mathbf{x}) := T_n\left(t(\mathbf{x})\right) + \delta \cdot q(\mathbf{y}(\mathbf{x})) \cdot V_{n-2}(t(\mathbf{x})), \tag{4.2}$$

where

$$t := t(\mathbf{x}) := \frac{\langle \mathbf{v}^\star, 2\mathbf{x} - \mathbf{a} - \mathbf{b} \rangle}{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b} \rangle}. \tag{4.3}$$

Observe that by (2.4) q is a second degree polynomial in $\mathbf{x}$ while $V_{n-2}(t(\mathbf{x})) \in \pi^d_{n-2}$, whence $S_n \in \pi^d_n$. Also it is easy to see that the second expression on the right-hand side of (4.2) is not identically zero, hence S_n differs essentially from its first term. However, we are going to prove

$$\|S_n\|_{C(K)} = 1. \tag{4.4}$$

Once we have (4.4), the construction culmimates in new examples $\tilde{r}_n := R_n(K, \mathbf{x}^\star) \cdot S_n$ and $\tilde{g}_n := C_n(K, \mathbf{v}) \cdot S_n$ of extremal polynomials, essentially different from (1.10) and (1.8), respectively, since for $\mathbf{x} \in \ell$ (i.e., both for $\mathbf{x}^\star \in \ell$ and for $\lambda\mathbf{v} + \frac{\mathbf{a}+\mathbf{b}}{2} \in \ell$ $(\lambda > 0)$) $\mathbf{y}(\mathbf{x}) = 0$ and $q(\mathbf{y}(\mathbf{x})) = 0$ ensures $S_n(\mathbf{x}) = T_n(t(\mathbf{x}))$, leading to

$$\tilde{r}_n(\mathbf{x}^\star) = R_n(K, \mathbf{x}^\star) \cdot S_n(\mathbf{x}^\star) = 1$$

and also by (1.2), even to

$$\tilde{g}^*_n(\mathbf{v}) = C_n(K, \mathbf{v}) \cdot S^*_n(\mathbf{v}).$$

Now to prove (4.4) we can use (2.6) and also the relation

$$q(\mathbf{y}(\mathbf{x})) \geq 0 \qquad (\mathbf{x} \in K) \tag{4.5}$$

which is equivalent to the admissibility of q (cf. Proposition 5). Since $-1 \leq t(\mathbf{x}) \leq 1$ $(\mathbf{x} \in K)$, and in view of the obvious symmetry of the cases $t \leq 0$ and $t \geq 0$, (reflecting the fact that T_n and V_{n-2} are both even or odd together with n), we can restrict our attention to the two basic cases $|t| \leq z_1$ and $z_1 < t \leq 1$, respectively. Now in the first case (3.4) and (4.5) show that the two summands at the right-hand side of (4.2) can not have the same non-zero sign, thus

$$|S_n(\mathbf{x})| \leq \max\left\{|T_n(t(\mathbf{x}))|,\ \delta \cdot q(\mathbf{y}(\mathbf{x})) \cdot |V_{n-2}(t(\mathbf{x}))|\right\} \leq 1, \tag{4.6}$$

in view of the definition of δ in (4.2) and (3.3) from Proposition 2. On the other hand, for $z_1 < t \leq 1$ we use the first inequality of (2.5) together with (3.3) and (3.5) to obtain

$$\begin{aligned} |S_n(\mathbf{x})| &\leq T_n(t) + \delta \cdot \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle \cdot V_{n-2}(t) \\ &\leq 1 - c_1(1-t) + \delta \cdot \frac{\langle \mathbf{v}^\star, \mathbf{a} - \mathbf{b} \rangle}{2}(1-t)c_2 \\ &\leq 1 - (1-t)\left\{c_1 - \frac{\delta}{2} c_2 \tau(K, \mathbf{v}^\star)\right\} \leq 1. \end{aligned} \tag{4.7}$$

Collecting (4.6) and (4.7) we obtain $\|S_n\| \leq 1$, whence also (4.4).

We would like to note that the above proof closely follows [2], where for the case $d = 2$ the same argument was given. As we have seen, Kroó's proof extends to $d > 2$ without any difficulty. The proof of the sufficiency part when $d > 2$ will contain essential new elements.

5. Notions from convex geometry

Let $S \subset \mathbb{R}^d$ be any set of vectors; similarly to the polar set S^* of S (corresponding to the special case $c = 1$), the sets

$$\mathcal{P}_c(S) := \{\mathbf{u} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{x} \rangle \le c \quad (\forall \mathbf{x} \in S)\}$$

are usually defined in convex analysis, cf. e.g. [7]. The special case $c = 0$ can be regarded as an "infinitezimal polar set", reflecting to properties of S only at $\mathbf{0}$ when $\mathbf{0} \in S$, as will be the case in our applications. That is, throughout the paper we use the sets

$$\mathcal{P}(S) := \mathcal{P}_0(S) := \{\mathbf{u} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{x} \rangle \le 0 \quad (\forall \mathbf{x} \in S)\}. \tag{5.1}$$

Corresponding to the special property $\lambda \cdot \mathbf{0} = \mathbf{0} \quad (\forall \lambda \in \mathbb{R})$ of $c = 0$, the set $\mathcal{P}(S) := \mathcal{P}_0(S)$ is always homogeneous, and does not reflect global properties of S at all (unlike $\mathcal{P}_1(S)$).

The following properties of $\mathcal{P}$ are easy consequences from the definition.

(i) $\mathcal{P}(S)$ is homogeneous, i.e., for $\lambda > 0$ and $\mathbf{u} \in \mathcal{P}(S)$, also $\lambda\mathbf{u} \in \mathcal{P}(S)$;
(ii) $\mathcal{P}(S)$ is convex;
(iii) $\mathcal{P}(S)$ is topologically closed;

that is, $\mathcal{P}(S)$ is a closed convex cone in $\mathbb{R}^d$ and, moreover,

(iv) $\mathcal{P}(\mathrm{hom}(S)) = \mathcal{P}(S)$, where $\mathrm{hom}(S) := \{\lambda\mathbf{x} : \lambda \ge 0,\ \mathbf{x} \in S\}$ is the homogeneous set generated by S;
(v) $\mathcal{P}(\mathrm{con}\,(S)) = \mathcal{P}(S)$, where $\mathrm{con}\,(S)$ is the convex hull of S;
(vi) $\mathcal{P}(\overline{S}) = \mathcal{P}(S)$, where $\overline{S}$ is the (topological) closure of S,

that is, $\mathcal{P}$ is preserved after homogeneous, convex, or topological closure of S. It is clear that $\mathcal{P}(S) \cap S$ is either $\{\mathbf{0}\}$ or $\emptyset$, according to $\mathbf{0} \in S$ or $\mathbf{0} \notin S$, respectively. However, a repeated application of $\mathcal{P}$ satisfies

(vii) $S \subseteq \mathcal{P}^2(S)$;
(viii) $\mathcal{P}^2(S) = \cap\{C \subseteq \mathbb{R}^d : C \supset S \text{ and } C \text{ is a convex, closed cone}\}$;
(ix) $\mathcal{P}^2(S) = \overline{\mathrm{hom}(\mathrm{con}\,(S))}$.

From this it is also evident that we have

(x) $\mathcal{P}^3(S) = \mathcal{P}(S)$.

Finally, the following set algebraic properties of the operation $\mathcal{P}$ are easyly verified.

(xi) For $E, F \subseteq \mathbb{R}^d$ $\mathcal{P}(E \cup F) = \mathcal{P}(E) \cap \mathcal{P}(F)$,
(xii) $\mathcal{P}(E \cap F) \supseteq \mathrm{con}\,(\mathcal{P}(E) \cup \mathcal{P}(F))$,
(xiii) $\mathcal{P}(\mathcal{P}(E) \cap \mathcal{P}(F)) = \mathrm{con}\,(\mathcal{P}^2(E) \cup \mathcal{P}^2(F)) = \mathcal{P}^2(E \cup F\}$.

Let $K \subset \mathbb{R}^d$ be a set, and $\mathbf{a} \in \partial K$ be a boundary point. A supporting hyperplane $H_{\mathbf{a}}$ can be drawn to K at $\mathbf{a}$ if and only if there exists a non-zero vector $\mathbf{n}$, the outer normal vector of $H_{\mathbf{a}}$, such that $\langle \mathbf{n}, \mathbf{x} \rangle \le \langle \mathbf{n}, \mathbf{a} \rangle$ $(\forall \mathbf{x} \in K)$.

In other words, the set of outer normal vectors to the supporting hyperplanes of K at $\mathbf{a}$ is $\mathcal{P}(K-\mathbf{a}) \setminus \{\mathbf{0}\}$.

Now if K is a convex body, $\mathbf{a}, \mathbf{b} \in \partial K$ with $\mathbf{a}-\mathbf{b} = |\mathbf{a}-\mathbf{b}| \cdot \mathbf{v}$ (i.e., $\mathbf{v} = \frac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$), and we are looking for parallel supporting hyperplanes at $\mathbf{a}$ and $\mathbf{b}$, with normal vectors $\mathbf{v}^\star$ chosen so that $\langle \mathbf{v}^\star, \mathbf{v} \rangle > 0$ (i.e., $\mathbf{v}^\star$ is an outer normal vector at $\mathbf{a}$ and an inner normal vector at $\mathbf{b}$), then the following statement is true.

Claim 1. *The set of all conjugate vectors $\mathbf{v}^\star$ (satisfying $\langle \mathbf{v}^\star, \mathbf{a}-\mathbf{b} \rangle > 0$) are the unitary vectors of the closed convex cone $N := \mathcal{P}(K-\mathbf{a}) \cap (-\mathcal{P}(K-\mathbf{b}))$.*

The following two statements are also elementary (and equivalent, according to (v), (x), (xi) and (xiii)).

Claim 2. *If $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$, i.e., $\mathbf{c} = \mu\mathbf{a} + (1-\mu)\mathbf{b}$ with $0 < \mu < 1$, then we have*

$$\mathcal{P}(K-\mathbf{c}) = \mathcal{P}(K-\mathbf{a}) \cap \mathcal{P}(K-\mathbf{b}).$$

Let us denote $M := \operatorname{con}\{\mathcal{P}^2(K-\mathbf{a}), \mathcal{P}^2(K-\mathbf{b})\}$.

Claim 3. *For any $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$ we have $\mathcal{P}^2(K-\mathbf{c}) = M$.*

In what follows, we set $V := \mathcal{P}^2(K-\mathbf{c}) \cap H = M \cap H$.

Claim 4. *For both G and V the relative interior is the relative portion of the interior of their defining set, i.e.,*

$$\operatorname{r\,int} G = \operatorname{int}\left(K - \frac{\mathbf{a}+\mathbf{b}}{2}\right) \cap H$$

and

$$\operatorname{r\,int} V = \operatorname{int}(\mathcal{P}^2(K-\mathbf{c})) \cap H = \operatorname{int} M \cap H.$$

Proof. We prove only the first statement, the second being similar.

The $\supseteq$ part is trivial, so let us prove the inclusion $\subseteq$.

Let $\mathbf{y} \in \operatorname{r\,int}((K-\mathbf{c}) \cap H)$ (where $\mathbf{c} := \frac{\mathbf{a}+\mathbf{b}}{2}$ here) and let $r > 0$ be such that

$$B^{d-1}(\mathbf{y}, r) \subseteq (K-\mathbf{c}) \cap H. \tag{5.2}$$

Let us suppose indirectly that $\mathbf{y} \notin \operatorname{int}(K-\mathbf{c})$. Then it is clear that $\mathbf{y} \in (K-\mathbf{c}) \setminus \operatorname{int}(K-\mathbf{c}) \subseteq \partial(K-\mathbf{c})$. Since the set $K-\mathbf{c}$ is convex, we can draw a supporting hyperplane at $\mathbf{y}$ to it, that is, there exists a normal vector $\mathbf{0} \neq \mathbf{n} \in \mathbb{R}^d$ such that

$$\langle \mathbf{n}, \mathbf{x} \rangle \geq \langle \mathbf{n}, \mathbf{y} \rangle \qquad (\forall \mathbf{x} \in K-\mathbf{c}). \tag{5.3}$$

Comparing (5.2) and (5.3) we conclude that for all vectors $\mathbf{u} \in H$, $|\mathbf{u}| \leq r$, we have $\langle \mathbf{n}, \mathbf{y} + \mathbf{u} \rangle \geq \langle \mathbf{n}, \mathbf{y} \rangle$, i.e., $\langle \mathbf{n}, \mathbf{u} \rangle \geq 0$ $(\forall \mathbf{u} \in B^{d-1}(\mathbf{0}, r))$, which yields $\langle \mathbf{n}, \mathbf{u} \rangle = 0$ $(\forall \mathbf{u} \in H)$, i.e., $\mathbf{n} \perp V$, $\mathbf{n} \| \mathbf{v}^\star$, $\mathbf{n} = \lambda \mathbf{v}^\star$. In particular, also $\langle \mathbf{n}, \mathbf{y} \rangle = 0$. Since $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$, $\pm \delta \mathbf{v} \in K - \mathbf{c}$ for some $\delta > 0$, and hence we get from (5.3) also that $\langle \mathbf{n}, \mathbf{v} \rangle = 0$. However $\mathbf{n} = \lambda \mathbf{v}^\star$ and $\langle \mathbf{v}, \mathbf{v}^\star \rangle > 0$, hence $\lambda = 0$ and $\mathbf{n} = \mathbf{0}$, a contradiction.

Claim 5. *With the above definitions of G in* (2.2)*, M in Claim 3 and V immediately preceding Claim 4, we have $V = \mathcal{P}^2(G)$.*

Proof. Since V is the intersection of M and H, and both $M = \mathcal{P}^2(K - \mathbf{c})$ and H contains $\mathcal{P}^2(G)$, the $\supseteq$ part is obvious. Now we prove the $\subseteq$ direction.

Let us recall first that any convex body is *fat*, that is, the closure of its interior equals to the closure of the convex body itself. Moreover, the interior of a convex body equals to the interior of the closure of the convex body. Indeed, let us take any point not belonging to the interior. Then through this point a supporting hyperplane can be drawn to the interior. However, that would entail that also the closure of the interior, (containing the whole convex set itself), and so even the closure of the convex body, would lie in one of the half-spaces cut by the hyperplane. Thus our point on the hyperplane can not be an interior point of the closure, either.

By (ix) this reasoning applies to $\hom(K - \mathbf{c})$ and M, hence $\operatorname{int} M = \operatorname{int} \hom(K - \mathbf{c}) \subseteq \hom(K - \mathbf{c})$ and so we get

$$\overline{\operatorname{int} M \cap H} \subseteq \overline{\hom(K - \mathbf{c}) \cap H} = \overline{\hom(K - \mathbf{c} \cap H)} = \mathcal{P}^2(G),$$

using again (ix).

But by Claim 4 we have $\operatorname{int} M \cap H = \operatorname{r\,int} V$, and from the relative fatness of $V \supseteq G$ in H we obtain $\overline{\operatorname{r\,int} V} = \overline{V} = V$. Thus we are led to $V = \overline{\operatorname{int} M \cap H} \subseteq \mathcal{P}^2(G)$ and the proof is completed.

Proposition 5. *The quadratic form $q(\mathbf{y})$ is admissible if one of the following equivalent conditions holds:*

(*i*) $q(\mathbf{y}) \geq 0$ $(\forall \mathbf{y} \in G)$ *for G in* (2.2)*;*

(*ii*) $q(\mathbf{y}(\mathbf{x})) \geq 0$ $(\forall \mathbf{x} \in K)$ *for $\mathbf{y}(\mathbf{x})$ defined in* (2.3) *or* (2.4)*;*

(*iii*) $q(\mathbf{y}) \geq 0$ $(\forall \mathbf{x} \in V)$ *for V defined preceding Claim 4.*

Proof. Since by definition $V \supseteq G$, it is clear that (iii) implies (i). Conversely, as q is homogeneous and continuous, together with G it has to be non-negative also on $\overline{\hom(G)} = \mathcal{P}^2(G)$, by (ix). Referring to Claim 5 here we obtain (iii).

Now let us prove that (i) implies (ii). For $\mathbf{x} \in K$, (2.2)–(2.3) yield $\mathbf{z} = \mathbf{z}(\mathbf{x}) = \alpha \mathbf{a} + (1 - \alpha)\mathbf{b}$ with some $0 \leq \alpha \leq 1$. Without loss of generality we may suppose e.g. $\frac{1}{2} \leq \alpha \leq 1$, i.e., $\mathbf{c} - \mathbf{b} = \frac{1}{2\alpha}(\mathbf{z} - \mathbf{b})$ with $1/2 \leq \frac{1}{2\alpha} \leq 1$.

By convexity of K we obtain that $\mathbf{x}^\star = \mathbf{b} + \frac{1}{2\alpha}(\mathbf{x}-\mathbf{b}) \in K$, $H \ni \mathbf{y}^\star = \mathbf{y}(\mathbf{x}^\star)$, $\mathbf{z}^\star = \mathbf{z}(\mathbf{x}^\star) = \mathbf{b} - \frac{1}{2\alpha}\mathbf{b} - \frac{1}{2\alpha}\mathbf{z}(\mathbf{x}) = \mathbf{c}$. Hence $\mathbf{y}^\star \in K - \mathbf{c} \cap H = G$. Thus, by condition (i), $\mathbf{y}^\star \in G$ implies $q(\mathbf{y}^\star) \geq 0$. Since q is homogeneous, we get also $q(\mathbf{y}) = 4\alpha^2 q(\mathbf{y}^\star) \geq 0$, whence the statement.

Finally we prove that (ii) implies (i). For any $\mathbf{y} \in G$, $\mathbf{x} := \mathbf{y} + \frac{\mathbf{a}+\mathbf{b}}{2} \in K$ and $\mathbf{y} = \mathbf{x} - \frac{\mathbf{a}+\mathbf{b}}{2} \in G \subset H$ entails $\mathbf{z} = \frac{\mathbf{a}+\mathbf{b}}{2}$, $\mathbf{y}(\mathbf{x}) = \mathbf{y}$. Hence using (ii) we obtain $q(\mathbf{y}) = q(\mathbf{y}(\mathbf{x})) \geq 0$.

At the end of this section, let us describe the linear algebra connecting the geometry and analysis of the problem. Let f be a smooth function $f : \mathbb{R}^d \to \mathbb{R}$. Then the second derivative $\partial^2 f$ can be represented as a matrix

$$\partial^2 f = \begin{pmatrix} & \vdots & \\ \cdots & \dfrac{\partial^2 f}{\partial x_i \partial x_j} & \cdots \\ & \vdots & \end{pmatrix}_{i,j=1,\dots,d}, \tag{5.4}$$

which is symmetric in view of the interchangeability of the partial derivatives. For a fixed point $\mathbf{z} \in \mathbb{R}^d$ the symmetric matrix $B = \partial^2 f(\mathbf{z})$ is a bilinear mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$, mapping the vector pair $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ to $B(\mathbf{u}, \mathbf{v}) = \langle B\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, B\mathbf{v} \rangle$. (Note that here the symmetry of B is used as $B^\top = B$.) Now the "diagonal mapping" of this bilinear map is a quadratic form $Q(\mathbf{x}) := B(\mathbf{x}, \mathbf{x})$. Conversely, a quadratic form $Q(\mathbf{x})$ determines uniquely the corresponding symmetric bilinear mapping through the equation $B(\mathbf{x}, \mathbf{y}) := \frac{1}{4}\{Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}-\mathbf{y})\}$. Standard calculus with the coordinates then yields

$$\frac{\partial^2 f}{\partial \mathbf{b}^2}(\mathbf{z}) = Q(\mathbf{b}) \qquad (\mathbf{b} \in \mathbb{R}^d) \tag{5.5}$$

describing the connection between the quadratic form and successive directional derivation. Restricting f to the line $\ell_{\mathbf{b}}(\mathbf{z})$ through $\mathbf{z}$ and in direction $\mathbf{b}$, an application of the usual Taylor formula with the Lagrange remainder term of second order will give

$$\begin{aligned} f(\mathbf{z}+\mathbf{b}) &= f(\mathbf{z}) + \frac{\partial f}{\partial \mathbf{b}^\star}(\mathbf{z}) \cdot \beta + \frac{1}{2}\,\frac{\partial^2 f}{\partial^2 \mathbf{b}^\star}(\mathbf{z} + \xi\beta\mathbf{b}^\star) \cdot \beta^2 \\ &= f(\mathbf{z}) + \frac{\partial f}{\partial \mathbf{b}}(\mathbf{z}) + \frac{1}{2}\,\frac{\partial^2 f}{2\partial \mathbf{b}^2}(\mathbf{z} + \xi\mathbf{b}) \end{aligned} \tag{5.6}$$

for any $\mathbf{b} = \beta\mathbf{b}^\star$ ($|\mathbf{b}^\star| = 1\ \beta > 0$) and with some suitable $0 \leq \xi \leq 1$ (depending on all quantities involved).

6. First connection between arbitrary and canonical extremal polynomials

From this point on we start investigating the sufficiency part of Theorems 1 and 2. As a first step, in Theorem 3 below, we shall formulate some connection between an arbitrary extremal polynomial and an arbitrary canonical one. This connection holds in full generality, without any reference to geometric conditions.

The proof of Theorem 3 will be a direct extension of the corresponding argument of Kroó [2] in the bivariate case, since his proof goes through in the higher dimensional setting without difficulty.

In full accordance to the choice of orientation fixed at the beginning of §2, the line ℓ can be parametrized as

$$\mathbf{c}(t) := \frac{\mathbf{a}+\mathbf{b}}{2} + t\frac{\mathbf{a}-\mathbf{b}}{2} \qquad (t \in \mathbb{R}). \tag{6.1}$$

Clearly, with the notations (2.3), (2.4) and (4.3), we have $\mathbf{y}(\mathbf{c}(t)) \equiv \mathbf{0}$ and $\mathbf{z}(\mathbf{c}(t)) \equiv \mathbf{c}(t)$, while $t(\mathbf{c}(t)) \equiv t$. Putting

$$t_j := \cos\left(\frac{j\pi}{n}\right) \qquad (j = 0, 1, \ldots, n) \tag{6.2}$$

we can define

$$\mathbf{c}_j := \mathbf{c}(t_j) \in [\mathbf{a}, \mathbf{b}] \qquad (j = 0, 1, \ldots, n), \quad \mathbf{c}_0 = \mathbf{a}, \quad \mathbf{c}_n = \mathbf{b}. \tag{6.3}$$

Now, in view of the way we fixed the orientation of $\mathbf{a}, \mathbf{b}$, we have $\langle \mathbf{v}^\star, \mathbf{v} \rangle > 0$ for any conjugate vector $\mathbf{v}^\star \in N$ (cf. Claim 1). Let $\tilde{g}_n$ and $\tilde{r}_n$ be arbitrary extremal polynomials to the problems (1.3) and (1.4), respectively. We renormalize $\tilde{g}_n$, $\tilde{r}_n$, and also g_n and r_n from (1.8) and (1.10), taking

$$s_n(\mathbf{x}) := \frac{1}{C_n(K, \mathbf{v})} g_n(\mathbf{x}) = T_n(t(\mathbf{x})) \quad \text{in problem (1.3)} \tag{6.4}$$

and

$$s_n(\mathbf{x}) := \frac{1}{R_n(K, \mathbf{x}^\star)} r_n(\mathbf{x}) = T_n(t(\mathbf{x})) \quad \text{in problem (1.4)}, \tag{6.5}$$

using notation (4.3) again, and also taking

$$f_n(\mathbf{x}) := \frac{1}{C_n(K, \mathbf{v})} \tilde{g}_n(\mathbf{x}) \quad \text{in problem (1.3)} \tag{6.6}$$

and

$$f_n(\mathbf{x}) := \frac{1}{R_n(K, \mathbf{x}^\star)} \tilde{r}_n(\mathbf{x}) \quad \text{in problem (1.4).} \tag{6.7}$$

Moreover, in both problems (1.3) and (1.4), we put

$$h_n(\mathbf{x}) := f_n(\mathbf{x}) - s_n(\mathbf{x}). \tag{6.8}$$

By definition we then have $s_n, f_n, h_n \in \pi_n^d$, and from the extremality,

$$\|f_n\|_{C(K)} = \|s_n\|_{C(K)} = 1. \tag{6.9}$$

According to the defining normalizations of our extremal polynomials we also have

$$h_n^*(\mathbf{v}) = 0 \quad \text{in problem (1.3),} \tag{6.10}$$

and with $\mathbf{c}(t^*) = \mathbf{x}^\star$, $t^* > 1$

$$h_n(\mathbf{c}(t^*)) = h_n(\mathbf{x}^\star) = 0 \quad \text{in problem (1.4).} \tag{6.11}$$

Theorem 3. *Suppose that $K, \mathbf{a}, \mathbf{b} \in \partial K$, $\mathbf{v}$ (or $\mathbf{x}^\star$) are as above. Consider any extremal polynomial $\tilde{g}_n$ (or $\tilde{r}_n$) and any of the possible canonical extremal polynomials g_n (or r_n) in problem* (1.3) *(or problem* (1.4)*, respectively). With the notations* (6.4)–(6.8) *above we have*

$$h_n|_\ell \equiv 0, \tag{6.12}$$

i.e., $\tilde{g}_n|_\ell \equiv g_n|_\ell$ (or $\tilde{r}_n|_\ell \equiv r_n|_\ell$, respectively).

Proof. Using the definitions (6.4)–(6.8) and (6.2) with the fact that $T_n(t_j) = (-1)^j$ $(j = 0, \ldots, n)$, we obtain in both case (1.3) and (1.4)

$$(-1)^j h_n(\mathbf{c}_j) = (-1)^j f_n(\mathbf{c}_j) - 1 \le 0 \qquad (j = 0, 1, \ldots, n). \tag{6.13}$$

Consider the auxiliary function $H(t) := h_n(\mathbf{c}(t)) \in \pi_n$. In case (1.4) we have $n+2$ weak sign changes of $H(t)$ at the points

$$-1 = t_n < \ldots < t_j < \ldots < t_0 = 1 < t^*$$

in reference to (6.13) and (6.11). Thus Proposition 1 yields $H \equiv 0$ and so (6.12). In case (1.3) we have only $n+1$ weak sign changes furnished by (6.13), but then by (6.10) we obtain

$$\begin{aligned} \lim_{t\to+\infty} \frac{H(t)}{t^n} &= \lim_{t\to+\infty} h_n\left(\frac{\mathbf{a}+\mathbf{b}}{2} + t \cdot \frac{\mathbf{a}-\mathbf{b}}{2}\right) t^{-n} \\ &= h_n^*\left(\frac{\mathbf{a}-\mathbf{b}}{2}\right) = \left(\frac{|\mathbf{a}-\mathbf{b}|}{2}\right)^n h_n^*(\mathbf{v}) = 0 \end{aligned} \tag{6.14}$$

proving that actually $H \in \pi_{n-1}$. Thus (6.12) follows again by an application of Proposition 1.

7. Gradients of extremal polynomials and conjugate directions

We now proceed to find a closer relationship between extremal polynomials "at large" and the canonical ones. Keeping the previous notations and normalizations, our notation will be extended to the following formulas:

$$
\begin{array}{rcl}
 & & \partial s_n(t) = T_n'(t(x))\dfrac{1}{\langle \mathbf{v}^\star, \mathbf{v}\rangle}\cdot \mathbf{v}^\star \in \pi_{n-1}^{d,d}, \\
\sigma(t) & := & \partial s_n(\mathbf{c}(t)) = a_{n-1}\mathbf{v}^\star t^{n-1} + \ldots + a_1\mathbf{v}^\star t + a_0\mathbf{v}^\star \in \pi_{n-1}^{1,d}; \\
 & & \partial f_n(\mathbf{x}) \in \pi_{n-1}^{d,d}, \\
\mathbf{\Phi}(t) & := & \partial f_n(\mathbf{c}(t)) = \mathbf{d}t^{n-1} + \mathbf{e}t^{n-2} + \mathbf{\Psi}(t) \in \pi_{n-1}^{1,d},\ \mathbf{\Psi} \in \pi_{n-3}^{1,d}; \\
 & & \partial h_n(\mathbf{x}) \in \pi_{n-1}^{d,d}, \\
\eta(t) & := & \partial h_n(\mathbf{c}(t)) \\
 & = & (\mathbf{d} - a_{n-1}\mathbf{v}^\star)t^{n-1}(\mathbf{e} - a^{n-2}\mathbf{v}^\star)t^{n-2} + \mathbf{\Theta}(t) \in \pi_{n-1}^{1-d}, \\
 & & \mathbf{\Theta}(t) \in \pi_{n-3}^{1,d}.
\end{array}
\tag{7.1}
$$

In view of Theorem 3, we obviously have

$$
\langle \eta(t), \mathbf{v}\rangle = \langle \partial h_n(\mathbf{c}(t), \mathbf{v}\rangle = \frac{\partial h_n}{\partial \mathbf{v}}(\mathbf{c}(t)) \equiv 0, \qquad \frac{\partial h_n}{\partial \mathbf{v}}\Big|_\ell \equiv 0. \tag{7.2}
$$

However we can say something more in some cases.

Theorem 4. *Suppose that the supporting strip to K at $\mathbf{a}$ and $\mathbf{b}$ is unique, i.e.,* $\dim N = 1$ *and* $N = \{\lambda \mathbf{v}^\star : \lambda > 0\}$ *with a conjugate vector* $\mathbf{v}^\star$. *Then, besides* (6.12) *and* (7.2), *we have*

$$
\partial h_n\Big|_\ell \equiv 0. \tag{7.3}
$$

Theorem 4 asserts that for *any* pair of (normalized) extremal polynomials their gradient vectors coincide on the whole line ℓ since both of them coincide with the gradient of a given canonical extremal polynomial. While Theorem 3 formulates a weaker, but unconditional connection between any f_n and canonical s_n, Theorem 4 formulates a stronger connection under condition of unicity of the supporting strip, i.e., provided $\dim N = 1$. It is clear that Theorem 4 can be reversed, as for $\dim N > 1$ there are different $\mathbf{v}^\star, \mathbf{v}^{\star\star} \in N$ generating different canonical examples with differing gradients at, e.g. the point $\mathbf{a} \in \ell$. Thus we have got a one-to-one correspondence between some geometric characteristics of the situation ($\dim N = 1$) and an analytic description of closeness of extremal polynomials (expressed by (7.3)).

To gain ground first we present a series of simpler assertions, and the proof of the theorem will be postponed until the end of §7.

Claim 6. *For the gradient vectors of f_n at $\mathbf{a}$ and $\mathbf{b}$ we have*

i) $\dfrac{\partial f_n}{\partial \mathbf{u}}(\mathbf{a}) \le 0 \quad \forall \mathbf{u} \in \mathcal{P}^2(K-\mathbf{a})$, *i.e.,* $\partial f_n(\mathbf{a}) \in \mathcal{P}(K-\mathbf{a})$,

and

ii) $(-1)^n \dfrac{\partial f_n}{\partial \mathbf{u}}(\mathbf{b}) \le 0 \quad \forall \mathbf{u} \in \mathcal{P}^2(K-\mathbf{b})$, *i.e.,* $(-1)^n \partial f_n(\mathbf{b}) \in \mathcal{P}(K-\mathbf{b})$.

Proof. In view of property (x) in §5, the formulation expresses equivalent statements both in i) and ii). For i), we have to prove that $\langle \partial f(\mathbf{a}), \mathbf{u}\rangle \le 0$ for $\mathbf{u} \in K-\mathbf{a}$. Put $\mathbf{x} := \mathbf{u}+\mathbf{a} \in K$ and consider $x_\lambda = \mathbf{a} + \lambda\mathbf{u} \in K$ $(0 \le \lambda \le 1)$. Using (6.4)–(6.7) and the fact that $f_n(\mathbf{a}) = s_n(\mathbf{a}) = 1$, by (6.8), (6.12) and (6.4), (6.5), we get from (6.9)

$$f_n(\mathbf{x}_\lambda) \le 1 = f_n(\mathbf{a}),$$

which leads immediately to

$$\frac{\partial f_n(\mathbf{a})}{\partial \mathbf{u}} = \lim_{\lambda \to +0} \frac{f_n(\mathbf{a}+\lambda\mathbf{u}) - f_n(\mathbf{a})}{\lambda} \le 0.$$

Similarly, to prove ii) we have to show that

$$\langle (-1)^n \partial f_n(\mathbf{b}), \mathbf{u}\rangle \le 0 \quad \text{for} \quad \mathbf{u} \in K-\mathbf{b}.$$

Now taking $\mathbf{x} := \mathbf{u}+\mathbf{b} \in K$ and $\mathbf{x}_\lambda := \lambda\mathbf{u}+\mathbf{b} \in K$ $(0 \le \lambda \le 1)$, and recalling (6.4)–(6.7), (6.8), (6.12) and (6.9) again, we are led to

$$(-1)^n f_n(\mathbf{x}_\lambda) \le 1 = (-1)^n T_n(-1) = (-1)^n s_n(\mathbf{b}) = (-1)^n f_n(\mathbf{b}).$$

As above, directional differentiation then gives part ii), too.

Claim 7. *With the notations* (6.1)–(6.3) *and Claim* 3 *in* §5, *we have*

$$(-1)^j \frac{\partial f_n}{\partial \mathbf{u}}(\mathbf{c}_j) \le 0 \quad (j = 1, \dots, n-1,\ \mathbf{u} \in M).$$

Proof. According to Claim 3, for all $j = 1, \dots, n-1$,

$$\mathcal{P}^2(K-\mathbf{c}_j) = M := \operatorname{con}\{\mathcal{P}^2(K-\mathbf{a}), \mathcal{P}^2(K-\mathbf{b})\}.$$

Now, in view of (x) of §5, in order to prove Claim 7 it suffices to show the formula for $\mathbf{u} \in K-\mathbf{c}_j$. Let now $\mathbf{u} \in K-\mathbf{c}_j$ and $1 \le j \le n-1$ be fixed arbitrarily. By the convexity of K we have $\mathbf{x}_\lambda := \mathbf{c}_j + \lambda\mathbf{u} \in K$ together with $\mathbf{c}_j$ and $\mathbf{x} = \mathbf{u} + \mathbf{c}_j$ whenever $0 \le \lambda \le 1$. Hence, in view of (6.1)–(6.7), (4.3) and (6.9), we have

$$(-1)^j f_n(\mathbf{x}_\lambda) \le 1 = (-1)^j T_n(t_j) = (-1)^j s_n(\mathbf{c}_j) = (-1)^j f_n(\mathbf{c}_j),$$

applying also (6.12) (i.e., Theorem 3) in the last step. As in the proof of Claim 6, an application of directional differentiation now yields the statement since

$$\begin{aligned}(-1)^j \frac{\partial f_n}{\partial \mathbf{u}}(\mathbf{c}_j) &= (-1)^j \lim_{\lambda\to+0} \frac{f_n(\mathbf{c}_j+\lambda\mathbf{u})-f_n(\mathbf{c}_j)}{\lambda} \\ &= \lim_{\lambda\to+0} \frac{(-1)^j f_n(\mathbf{x}_\lambda)-(-1)^j f_n(\mathbf{c}_j)}{\lambda} \le 0.\end{aligned}$$

Claim 8. *With the above notations and definitions we have*

$$(-1)^j \frac{\partial h_n}{\partial \mathbf{u}}(\mathbf{c}_j) \le 0 \quad (j=1,\dots,n-1,\ \mathbf{u}\in M).$$

Proof. Recalling notations (2.1) and (2.2), take $\mathbf{u}^\star \in G$ arbitrarily. Since $G \subset H$, $\mathbf{u}^\star \perp \mathbf{v}^\star$ and using (6.4)–(6.8) we obtain

$$\langle (-1)^j \partial h_n(\mathbf{c}_j), \mathbf{u}^\star\rangle = \langle (-1)^j \partial f_n(\mathbf{c}_j), \mathbf{u}^\star\rangle - (-1)^j \langle \partial s_n(\mathbf{c}_j), \mathbf{u}^\star\rangle.$$

However the condition $\mathbf{u}^\star \perp \mathbf{v}^\star$ and (4.1) imply that the second term is zero, while the first one is non-negative, by Claim 7, for

$$G \subset K - \frac{\mathbf{a}+\mathbf{b}}{2} \subset \mathcal{P}^2(K - \frac{\mathbf{a}+\mathbf{b}}{2}) = M.$$

Hence we get the statement at least for $\mathbf{u}^\star \in G$. To verify the general case, in view of Claim 3 and the property (x) from §5 , it suffices to consider $\mathbf{u} \in K - \frac{\mathbf{a}+\mathbf{b}}{2}$. Since $\mathbf{x} := \mathbf{u} + \frac{\mathbf{a}+\mathbf{b}}{2} \in K$, it belongs to the supporting strip to K at $\mathbf{a}$ and $\mathbf{b}$, and so $\mathbf{x} = \mathbf{z}(\mathbf{x}) + \mathbf{y}(\mathbf{x})$ with $\mathbf{z}(\mathbf{x}) \in [\mathbf{a},\mathbf{b}]$ (cf. notations (2.3) and (2.4)). Thus, it is also immediate that for the vector $\mathbf{y}(\mathbf{x}) \in H$ we have $\mathbf{u}^\star := \frac{1}{2}\mathbf{y}(\mathbf{x}) \in G$. Hence we can rewrite $\mathbf{u}$ in the slanted coordinate system generated by H and $\mathbf{v}$ as

$$\mathbf{u} = \mathbf{x} - \frac{\mathbf{a}+\mathbf{b}}{2} = \mathbf{z}(\mathbf{x}) - \frac{\mathbf{a}+\mathbf{b}}{2} + \mathbf{y}(\mathbf{x}) = \frac{\langle \mathbf{u},\mathbf{v}^\star\rangle}{\langle \mathbf{v},\mathbf{v}^\star\rangle}\mathbf{v} + 2\mathbf{u}^\star = \nu\mathbf{v} + 2\mathbf{u}^\star$$

with $\nu := \langle \mathbf{u},\mathbf{v}^\star\rangle / \langle \mathbf{v},\mathbf{v}^\star\rangle$. But now we can use (7.2) (i.e., Theorem 3) to obtain

$$\langle \partial h_n(\mathbf{c}_j), \mathbf{u}\rangle = \nu\langle \partial h_n(\mathbf{c}_j), \mathbf{v}\rangle + 2\langle \partial h_n(\mathbf{c}_j), \mathbf{u}^\star\rangle = 2\langle \partial h_n(\mathbf{c}_j), \mathbf{u}^\star\rangle$$

and so the first part can be applied to prove the general statement.

Claim 9. *For the leading coefficient* $\mathbf{d}$ *of* $\boldsymbol{\Phi}(t) = \partial f_n(\mathbf{c}(t))$ *in* (7.1) *we have* $\mathbf{d} \in N$. *Moreover,* $\mathbf{d} \neq \mathbf{0}$.

Proof. First let us take $\mathbf{u} \in K - \mathbf{a}$. From item i) of Claim 6 and Claim 7 we already know that

$$(-1)^j \frac{\partial f}{\partial \mathbf{u}}(\mathbf{c}_j) \leq 0 \quad (j = 0, 1, \ldots, n-1). \tag{7.4}$$

Let us suppose that $\langle \mathbf{d}, \mathbf{u} \rangle > 0$. Consider the polynomial

$$p(t) := \frac{\partial f}{\partial \mathbf{u}}(\mathbf{c}(t)) = \langle \mathbf{\Phi}(t), \mathbf{u} \rangle \in \pi_{n-1}.$$

Since the leading coefficient of $p(t)$ is $\langle \mathbf{d}, \mathbf{u} \rangle > 0$, we have $p(t) > 0$ $(\forall t > \widetilde{t})$ for some $\widetilde{t}$. Taking any $t_\infty > \max\{\widetilde{t}, t_0\}$ we obtain for $\mathbf{c}_\infty := \mathbf{c}(t_\infty)$ the inequality

$$-\frac{\partial f}{\partial \mathbf{u}}(\mathbf{c}_\infty) = -p(t_\infty) < 0. \tag{7.5}$$

Now (7.4) and (7.5) provide $n+1$ sign changes of the polynomial $p \in \pi_{n-1}$ at the points $t_{n-1} < \cdots < t_0 = 1 < t_\infty$. Thus, by Proposition 1, we get $p \equiv 0$, contradicting the assumption that $p(t_\infty) > 0$. This contradiction excludes $\langle \mathbf{d}, \mathbf{u} \rangle > 0$. Hence $\langle \mathbf{d}, \mathbf{u} \rangle \leq 0$ and consequently $\mathbf{d} \in \mathcal{P}(K - \mathbf{a})$ since $\mathbf{u} \in K - \mathbf{a}$ was chosen arbitrarily.

Next we take $\mathbf{u} \in K - \mathbf{b}$ and suppose that $\langle \mathbf{u}, \mathbf{d} \rangle < 0$. As above, we infer the existence of a point $t_{-\infty} < -1 = t_n < \ldots < t_1$ such that $\operatorname{sgn}(p(t_{-\infty})) = (-1)^{n-1} \cdot \operatorname{sgn}(\lim_{t \to -\infty} \frac{p(t)}{t^{n-1}}) = (-1)^n$, i.e.,

$$(-1)^{n+1} \frac{\partial f_n}{\partial \mathbf{u}}(\mathbf{c}_{-\infty}) = (-1)^{n+1} p(t_{-\infty}) < \infty, \tag{7.6}$$

while in view of item ii) of Claim 6 and Claim 7 we now have

$$(-1)^j \frac{\partial f_n}{\partial \mathbf{u}}(\mathbf{c}_j) \leq 0 \qquad (j = 1, \ldots, n-1, n). \tag{7.7}$$

Collecting (7.6) and (7.7) we get $n+1$ sign changes of $p \in \pi_{n-1}$ at the points $t_{-\infty} < t_n < \ldots < t_1$, hence $p \equiv 0$, contrary to (7.6). This contradiction excludes $\langle \mathbf{u}, \mathbf{d} \rangle < 0$, hence $\langle \mathbf{u}, \mathbf{d} \rangle \geq 0$ and, since $\mathbf{u}$ is arbitrary, $\mathbf{d} \in -\mathcal{P}(K - \mathbf{b})$.

In all cases, $\mathbf{d} \in \mathcal{P}(K - \mathbf{a}) \cap (-\mathcal{P}(K - \mathbf{b})) = N$, as stated.

Now if $\mathbf{d}$ is a zero vector, then $\mathbf{\Phi}(t) \in \pi_{n-2}^{1,d}$ and thus $p(t) := \langle \partial \mathbf{\Phi}(\mathbf{c}(t)), \mathbf{u} \rangle \in \pi_{n-2}$. Hence e.g. for $\mathbf{u} \in K - \mathbf{a}$ (7.4) would give $n \geq \deg p + 2$ weak sign changes and p would be identically zero by Proposition 1. In other words, for any given $t \in \mathbb{R}$ the vector $\partial \mathbf{\Phi}(\mathbf{c}(t))$ would be orthogonal to $K - \mathbf{a}$, having non-empty interior, which implies $\partial \mathbf{\Phi}(\mathbf{c}(t)) \equiv \mathbf{0}$. However, $\partial f \mid_\ell \equiv \mathbf{0}$ and (7.2) would lead to $\frac{\partial s_n}{\partial \mathbf{v}}(\mathbf{c}(t)) \equiv 0$, i.e., $s_n \mid_\ell \equiv$ const., which is clearly impossible in view of (6.4)–(6.5) and the definition (1.6) of the Chebyshev polynomials. Hence $\mathbf{d} = \mathbf{0}$ leads to a contradiction and so $\mathbf{d}$ can not be the zero vector.

In the following Claims 10 and 11 we use the condition that the leading coefficient $\mathbf{d}$ of $\boldsymbol{\Phi}(t) = \partial f_n(\mathbf{c}(t))$ in (7.1) is parallel to $\mathbf{v}^\star$. In view of Claim 1 and Claim 9, we at least have $\mathbf{v}^\star \in N$, $\mathbf{d} \in N$, and the condition can be viewed as a proper choice of the canonical extremal polynomial (1.8), (6.4) or (1.10), (6.5) to the given general extremal polynomial f_n in (6.6) or (6.7). This point of view will be utilized later in Claim 12.

Claim 10. *Suppose that* $\mathbf{d}\|\mathbf{v}^\star$. *Then, with the notations in* (7.1), *we also have* $\mathbf{d} = a_{n-1}\mathbf{v}^\star$ *and*

$$\eta(t) = (\mathbf{e} - a_{n-2}\mathbf{v}^\star)t^{n-2} + \boldsymbol{\Theta}(t) \in \pi_{n-2}^{1,d}.$$

Proof. We have $\langle \mathbf{v}, \mathbf{v}^\star \rangle > 0$. Making use of (7.2), we see that the polynomial

$$\langle \mathbf{v}, \eta(t) \rangle = \langle \mathbf{v}, \mathbf{d} - a_{n-1}\mathbf{v}^\star \rangle t^{n-1} + \langle \mathbf{v}, \mathbf{e} - a_{n-2}\mathbf{v}^\star \rangle t^{n-2} + \langle \mathbf{v}, \boldsymbol{\Theta}(t) \rangle$$

is identically vanishing, hence its leading coefficient must be zero. Since $\mathbf{d}\|\mathbf{v}^\star$, we can write $\mathbf{d} = \alpha \cdot \mathbf{v}^\star$ ($\alpha \in \mathbb{R}$) and thus

$$0 = \langle \mathbf{v}, \alpha\mathbf{v}^\star - a_{n-1}\mathbf{v}^\star \rangle = (\alpha - a_{n-1})\langle \mathbf{v}, \mathbf{v}^\star \rangle,$$

which holds true only if $\alpha = a_{n-1}$, i.e., $\mathbf{d} = a_{n-1}\mathbf{v}^\star$, as stated.

Claim 11. *Suppose that* $\mathbf{d}\|\mathbf{v}^\star$. *Then we also have*

i) $\partial h_n(\mathbf{a}) \in -\mathcal{P}(M)$,
ii) $(-1)^n \partial h_n(\mathbf{b}) \in -\mathcal{P}(M)$,
iii) $\mathbf{e} - a_{n-2}\mathbf{v}^\star \in -\mathcal{P}(M)$.

Proof. By Claim 10, $\eta(t) \in \pi_{n-2}^{1,d}$. Let $\mathbf{u} \in M$ be arbitrary. If either of the inequalities $\langle \mathbf{u}, \partial h_n(\mathbf{a})\rangle \leq 0$ or $\langle \mathbf{u}, (-1)^n \partial h_n(\mathbf{b})\rangle \leq 0$ holds, then we have the n^{th} weak sign change of $p(t) := \langle \mathbf{u}, \eta(t)\rangle$ apart from the $n-1$ weak sign changes already supplied by Claim 8. Hence then $p \equiv 0$ and $\langle \mathbf{u}, \partial h_n(\mathbf{a})\rangle = \langle \mathbf{u}, \partial h_n(\mathbf{b})\rangle = 0$ proving that $\langle \mathbf{u}, \partial h_n(\mathbf{a})\rangle \geq 0$ and $\langle \mathbf{u}, (-1)^n \partial h_n(\mathbf{b})\rangle \geq 0$ in all cases. That is, since $\mathbf{u} \in M$ is arbitrary, we can conclude i) and ii) of Claim 11. Similarly, if for some $\mathbf{u} \in M$ $\langle \mathbf{e} - a_{n-2}\mathbf{v}^\star, \mathbf{u}\rangle$ were negative, i.e., if $p(t)$ had negative leading coefficient, then for some $t_{-\infty} < -1$ and also for some $t_\infty > 1$ we would have $(-1)^{n-1}p(t_{-\infty}) > 0$, $-p(t_\infty) > 0$, providing two additional sign changes in addition to the other $n-1$ already given by Claim 8. Hence Proposition 1 would lead to $p \equiv 0$, a contradiction. Whence $\langle \mathbf{u}, \mathbf{e} - a_{n-2}\mathbf{v}^\star\rangle \geq 0$ ($\forall \mathbf{u} \in M$) and iii) follows.

Now we can demonstrate the following sharpening of Claim 6.

Claim 12. *We have*

i) $\partial f_n(\mathbf{a}) \in N$

ii) $(-1)^{n+1}\partial f_n(\mathbf{b}) \in N.$

Proof. The statement is about f_n and does not refer to $\mathbf{v}^\star$ or s_n. Thus we are free to select a vector $\mathbf{v}^\star \in N$ and the corresponding canonical extremal polynomial as auxiliary objects in the proof. Our choice will be $\mathbf{v}^\star := \mathbf{d}/|\mathbf{d}|$, (or, in case $\mathbf{d} = 0$, any $\mathbf{v}^\star \in N$) so that we can apply also Claim 11 with this choice of $\mathbf{v}^\star$, s_n and $h_n = f_n - s_n$.

Let us put first $\mathbf{u} = \mathbf{x} - \mathbf{b} \in K - \mathbf{b}$. Note that by (4.1) and (6.4)–(6.5) we have

$$\partial s_n(\mathbf{a}) = T_n'(1) \cdot \frac{2}{\langle \mathbf{v}^\star, \mathbf{v}\rangle \cdot \tau(K, \mathbf{v})} \cdot \mathbf{v}^\star = \alpha \cdot \mathbf{v}^\star \quad \text{with} \quad \alpha > 0.$$

We write

$$\langle \partial f_n(\mathbf{a}), \mathbf{u}\rangle = \langle \partial s_n(\mathbf{a}), \mathbf{u}\rangle + \langle \partial h_n(\mathbf{a}), \mathbf{u}\rangle = \alpha \cdot \langle \mathbf{v}^\star, \mathbf{u}\rangle + \langle \partial h_n(\mathbf{a}), \mathbf{u}\rangle. \tag{7.8}$$

As $\mathbf{v}^\star \in N$, Claim 1 and $\mathbf{u} \in K - \mathbf{b}$ ensure $\langle \mathbf{v}^\star, \mathbf{u}\rangle \geq 0$. On the other hand, i) of Claim 11, with Claim 2 and (x) of §5 yield $\partial h_n(\mathbf{a}) \in -\mathcal{P}(M) \subset -\mathcal{P}(K - \mathbf{b})$ which implies $\langle \partial h_n(\mathbf{a}), \mathbf{u}\rangle \geq 0$ as $\mathbf{u} \in K - \mathbf{b}$. Hence we obtain from (7.8) $\langle \partial f_n(\mathbf{a}), \mathbf{u}\rangle \geq 0$ $(\forall \mathbf{u} \in K - \mathbf{b})$, i.e., $\partial f_n(\mathbf{a}) \in -\mathcal{P}(K - \mathbf{b})$. Comparing this with i) in Claim 6 we obtain i).

Similarly, for $\mathbf{u} \in K - \mathbf{a}$, we write

$$\begin{aligned} \langle (-1)^n \partial f_n(\mathbf{b}), \mathbf{u}\rangle &= (-1)^n \{\langle \partial s_n(\mathbf{b}), \mathbf{u}\rangle + \langle \partial h_n(\mathbf{b}), \mathbf{u}\rangle\} \\ &= -\alpha \langle \mathbf{v}^\star, \mathbf{u}\rangle + \langle (-1)^n \partial h_n(\mathbf{b}), \mathbf{u}\rangle \geq 0, \end{aligned}$$

since now $\mathbf{u} \in K - \mathbf{a}$ and $\mathbf{v}^\star \in \mathcal{P}(K - \mathbf{a})$ by Claim 1, while $(-1)^n \partial h_n(\mathbf{b}) \in -\mathcal{P}(M) \subset -\mathcal{P}(K - \mathbf{a})$ in view of ii), Claim 11 and using also Claim 2 and (x) of §5. Again, we get $(-1)^n \partial f_n(\mathbf{b}) \in -\mathcal{P}(K - \mathbf{a})$. Comparing this and ii) of Claim 6, we obtain now $(-1)^n \partial f_n(\mathbf{b}) \in -\mathcal{P}(K - \mathbf{a}) \cap \mathcal{P}(K - \mathbf{b}) = -N$, hence ii).

Note that in the case $\dim N = 1$, i.e., when $N = \{\lambda \mathbf{v}^\star : \lambda > 0\}$, we get $\partial f_n(\mathbf{a}) \| \mathbf{v}^\star$, $\partial f_n(\mathbf{b}) \| \mathbf{v}^\star$. This can be compared with Claim 9.

Proof of Theorem 4. Since $\dim N = 1$ by assumption, we have $\partial f_n(\mathbf{a}) \| \mathbf{v}^\star$ and $\partial f_n(\mathbf{b}) \| \mathbf{v}^\star$, according to Claim 12. In view of (6.4)–(6.5), (6.8) and (4.1) we conclude that $\partial s_n(\mathbf{a}) \| \mathbf{v}^\star$ and $\partial s_n(\mathbf{b}) \| \mathbf{v}^\star$, whence $\partial h_n(\mathbf{a}) \| \mathbf{v}^\star$ and $\partial h_n(\mathbf{b}) \| \mathbf{v}^\star$. However, (7.2) and the relation $\langle \mathbf{v}, \mathbf{v}^\star\rangle > 0$ give that $\partial h_n(\mathbf{a}) = \mathbf{0}$, $\partial h_n(\mathbf{b}) = \mathbf{0}$ since e.g. $0 = \langle \partial h_n(\mathbf{a}), \mathbf{v}\rangle$ and the expression $|\langle \partial h_n(\mathbf{a}), \mathbf{v}\rangle| =$

$\langle |\partial h_n(\mathbf{a})|\mathbf{v}^\star, \mathbf{v}\rangle = |\partial h_n(\mathbf{a})| \cdot \langle \mathbf{v}^\star, \mathbf{v}\rangle$ can vanish only if $|\partial h_n(\mathbf{a})| = 0$. We apply Claim 8 and the equality $\partial h_n(\mathbf{a}) = \mathbf{0} = \partial h_n(\mathbf{b})$ to obtain

$$(-1)^j \frac{\partial h_n}{\partial \mathbf{u}}(\mathbf{c}(t_j)) \le 0 \quad (j = 0, 1, \dots, n-1, n,\ \mathbf{u} \in M). \tag{7.9}$$

The last inequalities induce $n+1$ sign changes for the polynomial $p(t) := \langle \mathbf{u}, \eta(t)\rangle \in \pi_{n-1}$, hence Proposition 1 yields $p \equiv 0$. In other words, $\eta(t)$ is orthogonal to (all vectors of) $M \supset K - \mathbf{a}$, while $\operatorname{int}(K - \mathbf{a}) \neq \emptyset$. Hence $\eta(t) \equiv \mathbf{0}$.

In the concluding subsection of §7 we give another geometric condition, sufficient for the same closeness of an extremal polynomial f_n to some of the canonical examples.

Theorem 5. *Suppose that K has the property that the segment $(\mathbf{a}, \mathbf{b})$ belongs to $\operatorname{int} K$. Consider any extremal polynomial $f_n(\mathbf{x})$ and the leading coefficient $\mathbf{d}$ of $\mathbf{\Phi}(t) := \partial f_n(\mathbf{c}(t))$ (c.f. (7.1)). For the canonical extremal polynomial $s_n(\mathbf{x})$ defined in (1.8), (6.4) or (1.10), (6.5), respectively, with the use of $\mathbf{v}^\star := \mathbf{d}/|\mathbf{d}|$, we always have (7.3). Moreover, the same conclusion holds if we start with $\mathbf{v}^\star := \partial \mathbf{\Phi}_n(\mathbf{a})/|\partial f_n(\mathbf{a})|$ or $\mathbf{v}^\star := (-1)^{n+1}\partial \mathbf{\Phi}_n(\mathbf{b})/|\partial f_n(\mathbf{b})|$.*

Corollary 3. *Suppose that K is a (centrally) symmetric convex body. Consider any extremal polynomial $f_n(\mathbf{x})$. Then there exists $\mathbf{v}^\star \in N$ such that the canonical example $s_n(\mathbf{x})$ defined in (1.8), (6.4) or (1.10), (6.5), respectively, with this conjugate vector, satisfies (7.3). Moreover, $\mathbf{v}^\star$ is the direction of $\mathbf{d}$, or $\partial f_n(\mathbf{a})$, or $(-1)^{n+1}\partial f_n(\mathbf{b})$, or $T_n'(t)\partial f_n(\mathbf{c}(t))$ ($t \in \mathbb{R}$, $t \neq t_1, \dots, t_{n-1}$).*

Proof of Theorem 5. With the conjugate vector $\mathbf{v}^\star \| \mathbf{d}$, Claim 10 yields $\mathbf{d} \neq \mathbf{0}$, hence $\mathbf{v}^\star = \mathbf{d}/|\mathbf{d}|$. In this case we prove again that $\partial h_n(\mathbf{a}) = \mathbf{0}$, $\partial h_n(\mathbf{b}) = \mathbf{0}$, but now we use Claim 11 to this end. Indeed, if e.g. $\frac{\mathbf{a}+\mathbf{b}}{2} \in \operatorname{int} K$, then $K - \frac{\mathbf{a}+\mathbf{b}}{2}$ contains a sufficiently small open ball centered at $\mathbf{0}$ and thus $M = \mathbb{R}^d$, $\mathcal{P}(M) = \{\mathbf{0}\}$. Whence i), ii) of Claim 11 give $\partial h_n(\mathbf{a}) = \mathbf{0}$, $\partial h_n(\mathbf{b}) = 0$, while Claim 8 extends this to get (7.9) again. Thus the proof can be concluded similarly to that in Theorem 4. Once arriving to (7.3) we also see that $\partial f_n(\mathbf{a}) = \partial s_n(\mathbf{a})$ and $\partial f_n(\mathbf{b}) = \partial s_n(\mathbf{b})$, hence by (4.1) we get the statement regarding the connection of $\mathbf{v}^\star$ to $\partial f_n(\mathbf{a})$ or $\partial f_n(\mathbf{b})$.

Proof of Corollary 3. One would try to apply Theorem 5 proving $(\mathbf{a}, \mathbf{b}) \subset \operatorname{int} K$ first. However, even for centrally symmetric convex bodies this is not necessarily true. Take e.g. $d = 2$ and $K = [0,1]^2$ (the unit square). If $\mathbf{v} := (1, 0)$ in Problem (1.3), or if $\mathbf{x}^\star := (2, 0)$ in Problem (1.4), then $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (0, 0)$ is a valid choice and then

$$M = \{\alpha \cdot (\cos\beta, \sin\beta) : \alpha \ge 0,\ 0 \le \beta \le \pi\} \neq \mathbb{R}^2, \quad [\mathbf{a}, \mathbf{b}] \subset \partial K.$$

Since here $N = \{(\gamma, 0) : \gamma \geq 0\}$ is one dimensional, Theorem 4 can be applied in this particular case. However, it is not too complicated to construct further examples where $[\mathbf{a}, \mathbf{b}] \subset \partial K$, and $\dim N > 1$. Take e.g. $d = 3$, $K = [0, 1]^3$ (unit cube), $\mathbf{v} = (1, 1, 0)$ (or $\mathbf{x}^\star = (2, 2, 0)$, for Problem (1.4)) and $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 0, 0)$. Clearly $[\mathbf{a}, \mathbf{b}] \subset \partial K$ and $N = \{\alpha(\cos\beta, \sin\beta, 0) : \alpha \geq 0,\ 0 \leq \beta \leq \frac{\pi}{2}\}$, $\dim N = 2$. Hence for this situation neither Theorem 4, nor Theorem 5 can be applied directly.

Thus we shall look at the geometry of the configuration more closely. The extremal point pairs $\mathbf{a}, \mathbf{b}$ furnished by Lemma A (or Lemma B, respectively), are not necessarily unique. In particular, in the above examples one can choose other points $\mathbf{a}', \mathbf{b}'$ so that $(\mathbf{a}', \mathbf{b}') \subset \operatorname{int} K$. Take e.g. $\mathbf{a}' = (1, \frac{1}{2})$, $\mathbf{b}' = (0, \frac{1}{2})$ (or $\mathbf{a}' = (1, \frac{1}{3})$ and $\mathbf{b}' = (0, \frac{2}{3})$, respectively) for $K = [0, 1]^2$ and $\mathbf{a}' = (1, 1, \frac{1}{2})$, $\mathbf{b}' = (0, 0, \frac{1}{2})$ (or $\mathbf{a}'(1, 1, \frac{1}{3})$, $\mathbf{b}' = (0, 0, \frac{2}{3})$, respectively) for $K = [0, 1]^3$. Observe that here the line ℓ passes through the centre of K, hence it obviously passes through the interior of K. Now to grasp the general lesson drawn from the above examples, we come to Claim 13 below, which clearly suffices to conclude the proof of Corollary 3, using Theorem 5.

Claim 13. *If K is centrally symmetric, then there are extremal point pairs $\mathbf{a}', \mathbf{b}'$ in Lemma A (and also in Lemma B) so that the line $\ell' := \ell_{\mathbf{a}', \mathbf{b}'}$ through $\mathbf{a}'$ and $\mathbf{b}'$ passes through the center of K.*

Proof. Having any extremal point pair $\mathbf{a}, \mathbf{b}$, consider first $\widetilde{\mathbf{a}} := 2\mathbf{c} - \mathbf{b}$, $\widetilde{\mathbf{b}} := 2\mathbf{c} - \mathbf{a}$, where $\mathbf{c}$ is the centre of K. Clearly $\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}} = \mathbf{a} - \mathbf{b}$ and $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}$ are the reflections of $\mathbf{b}$ and $\mathbf{a}$ with respect to $\mathbf{c}$, hence $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}} \in K$. If $\mathbf{a} = \widetilde{\mathbf{a}}$ and $\mathbf{b} = \widetilde{\mathbf{b}}$, then $\mathbf{c} \in \ell_{\mathbf{a},\mathbf{b}}$ and we are done.

If not, then in case A let us consider $\widehat{\mathbf{a}} := \frac{1}{2}(\mathbf{a} + \widetilde{\mathbf{a}}) \in K$ and $\widehat{\mathbf{b}} := \frac{1}{2}(\mathbf{b} + \widetilde{\mathbf{b}}) \in K$. Now $\widehat{\mathbf{a}} = \mathbf{c} + \frac{\tau}{2}\mathbf{v}$, $\widehat{\mathbf{b}} = \mathbf{c} - \frac{\tau}{2}\mathbf{v}$, hence $\widehat{\mathbf{a}} - \widehat{\mathbf{b}} = \tau\mathbf{v}$ and $\mathbf{c} \in \ell_{\widehat{\mathbf{a}}, \widehat{\mathbf{b}}}$ as stated.

In case B the construction is similar, but having $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ we take now

$$\mathbf{a}^\star := \frac{\mu + \frac{1}{2}}{2\mu}\widetilde{\mathbf{a}} + \frac{\mu - \frac{1}{2}}{2\mu}\mathbf{a} \in K, \quad \mathbf{b}^\star := \frac{\mu - \frac{1}{2}}{2\mu}\widetilde{\mathbf{b}} + \frac{\mu + \frac{1}{2}}{2\mu}\mathbf{b} \in K,$$

where μ is as in Lemma B. It can be checked easily that again $\mathbf{c}, \mathbf{x}^\star \in \ell_{\mathbf{a}^\star, \mathbf{b}^\star}$ while $|\mathbf{x}^\star - \frac{\mathbf{a}^\star + \mathbf{b}^\star}{2}| / |\mathbf{a}^\star - \mathbf{b}^\star| = \mu$. Hence $\mathbf{a}^\star, \mathbf{b}^\star$ are points satisfying the requirements of the statement.

8. Second derivatives of the extremal polynomials

In the remaining part of this paper we proceed towards the proof of the sufficiency part of Theorems 1 and 2. Thus, from now on we take the conditions of these theorems for granted. That is, we suppose $\dim N = 1$ and hence by Claim 9 we have also $\mathbf{d}\|\mathbf{v}^\star$, which implies (6.12) and even (7.3) according to Theorem 4. Moreover, we are allowed to use the condition of quadratic flatness, but we will do so only in §9. However, we keep the numerous notations and definitions introduced earlier, cf. in particular (1.6), (1.8), (1.10), (2.1), (2.2), (2.3), (2.4), (4.1), (4.3), (5.1), N in Claim 1, M in before Claim 3, V before Claim 4, (6.1)–(6.8) and (7.1).

Additionally, we also introduce two quadratic forms (on $\mathbb{R}^{d-1} \cong H$, as in §2), namely

$$\begin{aligned} q_{\mathbf{a}} &:= \partial^2 f_n|_H(\mathbf{a}), \\ \text{i.e.,}\quad q_{\mathbf{a}}(\mathbf{y}) &= \frac{\partial^2 f_n}{\partial \mathbf{y}^2}(\mathbf{a}) := \langle \mathbf{y}, Q_{\mathbf{a}}\mathbf{y}\rangle \quad (\mathbf{y} \in H), \\ Q_{\mathbf{a}} &= \text{Jacobian of } f_n|_H(\mathbf{a}) \in \mathbb{R}^{(d-1)\times(d-1)}, \end{aligned} \tag{8.1}$$

and

$$\begin{aligned} q_{\mathbf{b}} &:= \partial^2 f_n|_H(\mathbf{b}), \\ \text{i.e.,}\quad q_{\mathbf{b}}(\mathbf{y}) &= \frac{\partial^2 f_n}{\partial \mathbf{y}^2}(\mathbf{b}) := \langle \mathbf{y}, Q_{\mathbf{b}}\mathbf{y}\rangle \quad (\mathbf{y} \in H), \\ Q_{\mathbf{b}} &= \text{Jacobian of } f_n|_H(\mathbf{b}) \in \mathbb{R}^{(d-1)\times(d-1)}. \end{aligned} \tag{8.2}$$

Note that for any $\mathbf{y} \in H = \{\mathbf{v}^\star\}^\perp$, $\frac{\partial s_n}{\partial \mathbf{y}} \equiv \mathbf{0}$ on $\mathbb{R}^d$ by (4.1), hence also

$$\frac{\partial^2 f_n}{\partial \mathbf{y}^2} \equiv \frac{\partial^2 h_n}{\partial \mathbf{y}^2} \qquad (\forall \mathbf{y} \in H). \tag{8.3}$$

Thus f_n can be substituted by h_n as well at all occurrences in (8.1), (8.2).

Claim 14. *If $\mathbf{u} \in \operatorname{r int} V = \operatorname{int} M \cap H$, then there exists a number $r_0(\mathbf{u}) > 0$ such that for all $0 \le r \le r_0 := r_0(\mathbf{u})$*

$$(-1)^j h_n(\mathbf{c}_j + r\mathbf{u}) \le 0 \quad (j = 1, \ldots, n-1). \tag{8.4}$$

Proof. By Claim 4 we have $\operatorname{r int} V = \operatorname{int} M \cap H$. Let us fix j, $1 \le j \le n-1$, and consider the representation $\mathcal{P}^2(K - \mathbf{c}_j) = M$ which is valid, by Claim 3, for $\mathbf{c}_j \in (\mathbf{a}, \mathbf{b})$. M is a closed convex cone (cf. §5, (ix)) which contains the body $K - \mathbf{c}_j$. Thus $\operatorname{int} M \ne \emptyset$ and M is *fat*, i.e., $\overline{\operatorname{int} M} = \overline{M} = M$. Moreover, (ix) of §5 gives the representation $\mathcal{P}^2(K - \mathbf{c}_j) = \overline{\hom(\operatorname{con}(K - \mathbf{c}_j))}$, hence also $\operatorname{int} M \subset \hom(K - \mathbf{c}_j)$, in view of the fact that $\operatorname{con}(K - \mathbf{c}_j) = K - \mathbf{c}_j$, which is a consequence of the convexity of $K - \mathbf{c}_j$. Thus we are led to $\mathbf{u} \in \hom(K - \mathbf{c}_j)$, i.e., $r_j\mathbf{u} \in K - \mathbf{c}_j$ for some $r_j := r_j(\mathbf{u}) > 0$. Taking now the minimum, $r_0(\mathbf{u}) := \min\{r_j(\mathbf{u}) : j = 1, \ldots, n-1\}$, we obtain $[\mathbf{c}_j + r_0\mathbf{u}, \mathbf{c}_j] = \operatorname{con}\{r_0\mathbf{u} + \mathbf{c}_j, \mathbf{c}_j\} \subset \operatorname{con}\{r_j\mathbf{u} + \mathbf{c}_j, \mathbf{c}_j\} \subset \operatorname{con} K = K$. Note that above $\mathbf{u}$ had to be an interior point, and this is really essential, since it is easy to construct sets in $\mathbb{R}^2$ so that $[\mathbf{c}_j, \mathbf{c}_j + r_0\mathbf{u}]$ will not belong to K for any positive choice of r_0.

Now it remains to prove that $(-1)^j h_n(\mathbf{x}) \le 0$ for an arbitrary $\mathbf{x} \in (\mathbf{c}_j + H) \cap K$ and for each j, $0 \le j \le n$. Now we argue similarly to the proof of Claim 7 using $\mathbf{x} \in K$, (6.6)–(6.7), (6.1)–(6.3), $t(\mathbf{c}_j) = t_j$ with $T_n(t_j) = (-1)^j$ and $t(\mathbf{x}) = t(\mathbf{c}_j)$ (as $\mathbf{x} - \mathbf{c}_j \perp \mathbf{v}^\star$) to obtain

$$(-1)^j f_n(\mathbf{x}) \le 1 = (-1)^j T_n(t_j) = (-1)^j T_n(t(\mathbf{x})) = (-1)^j s_n(\mathbf{x}).$$

By (6.8), the last inequality is just $(-1)^j h_n(\mathbf{x}) \le 0$ which was to be shown.

Claim 15. *For any* $\mathbf{u} \in V = M \cap H$ *we have*

$$(-1)^j \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}_j) \le 0 \qquad (j = 1, \dots, n-1). \tag{8.5}$$

Proof. For $\mathbf{u} \in \operatorname{r\,int} V$ we can use Claim 14. Then the Taylor formula (5.6) yields

$$0 \ge (-1)^j h_n(\mathbf{c}_j + r\mathbf{u}) = (-1)^j \{h_n(\mathbf{c}_j) + \frac{\partial h_n}{\partial \mathbf{u}}(\mathbf{c}_j) \cdot r + \frac{1}{2}\,\frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}_j + \xi r\mathbf{u}) \cdot r^2\},$$

for any $0 < r \le r_0(\mathbf{u})$. Applying Theorems 3 and 4, we infer, after dividing by $\frac{1}{2}r^2 > 0$, the inequalities

$$(-1)^j \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}_j + \xi r\mathbf{u}) \le 0$$

$$(\xi = \xi_j(r, \mathbf{u}) \in (0,1), \quad 0 < r \le r_0,\ j = 1, \dots, n-1).$$

A passage to limit as $r \to +0$ gives

$$(-1)^j q_j(\mathbf{u}) := (-1)^j \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}_j) \le 0 \qquad (j = 1, \dots, n-1) \tag{8.6}$$

for arbitrarily fixed $\mathbf{u} \in \operatorname{r\,int} V$. On the other hand, for fixed j the expression $q_j(\mathbf{u})$ is a quadratic form on H, hence continuous with respect to the variable $\mathbf{u} \in H$. By continuity, (8.6) extends to $V = \overline{\operatorname{r\,int} V}$ and thus proving (8.5). This concludes the proof.

Claim 16. *For any* $\mathbf{u} \in V$ *we have*

$$\begin{aligned} q_{\mathbf{a}}(\mathbf{u}) &= \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{a}) = \frac{\partial^2 f_n}{\partial \mathbf{u}^2}(\mathbf{a}) \ge 0; \\ (-1)^n q_{\mathbf{b}}(\mathbf{u}) &= (-1)^n \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{b}) = (-1)^n \frac{\partial^2 f_n}{\partial \mathbf{u}^2}(\mathbf{b}) \ge 0. \end{aligned} \tag{8.7}$$

Moreover, if for some particular $\mathbf{u} \in V$ *we have equality in either of the above inequalities, then we have*

$$\frac{\partial^2 h_n}{\partial \mathbf{u}^2}\Big|_\ell \equiv 0. \tag{8.8}$$

Proof. In case $q_{\mathbf{a}}(\mathbf{u}) > 0$ and $q_{\mathbf{b}}(\mathbf{u}) > 0$ we have nothing to prove. Suppose now that e.g. $q_{\mathbf{a}}(\mathbf{u}) \le 0$. With $q_0 := q_{\mathbf{a}}$ this implies the extension of (8.5) even to $j = 0$, hence the polynomial $p(t) := \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}(t)) \in \pi_{n-2}$ has n weak sign changes. However, by Proposition 1, then $p \equiv 0$, leading to (8.8) and, in particular, to $q_{\mathbf{a}}(\mathbf{u}) = 0$. The proof is similar if $q_{\mathbf{b}}(\mathbf{u}) = 0$ is supposed. Hence (8.7) holds in all cases and the statements concerning equality are proved, too.

Claim 17. *Suppose that for some* $\mathbf{u} \in \operatorname{r\,int} V$ *either* $\frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{a}) = 0$ *or* $\frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{b}) = 0$ *occurs. Then besides* (6.12) *we have*

$$h_n\big|_L \equiv 0 \qquad (L := L(\mathbf{u}, \mathbf{v}) := \{\mathbf{c}(t) + s\mathbf{u} :\ t, s \in \mathbb{R}\}). \tag{8.9}$$

Proof. We have to prove that the polynomial

$$P(t,s) := h_n(\mathbf{c}(t) + s\mathbf{u}) \in \pi_n^2$$

vanishes identically under the conditions of Claim 17. Write

$$P(t,s) = \sum_{j=0}^{n} p_j(t) s^j, \qquad p_j \in \pi_{n-j},$$

and note that

$$p_0(t) = h_n(\mathbf{c}(t)) \equiv 0$$

by (6.12),

$$p_1(t) = \frac{\partial h_n}{\partial \mathbf{u}}(\mathbf{c}(t)) \equiv 0$$

by Theorem 4, and also even

$$p_2(t) = \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}(t)) \equiv 0$$

in view of (8.8) from Claim 16 because of the equality $\frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{a}) \cdot \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{b}) = 0$. Thus P can be rewritten as

$$P(t,s) = \sum_{j=3}^{n} p_j(t) s^j = s^3 p(t,s), \qquad p \in \pi_{n-3}^2.$$

Now we refer to Claim 14, using essentially the fact that $\mathbf{u}$ is in the relative interior of V, and get (8.4) for all $0 \leq s \leq r_0 = r_0(\mathbf{u})$. Consequently, if $0 < s \leq r_0$,

$$(-1)^j p(t_j, s) \leq 0 \qquad (j = 1, \ldots, n-1,\ 0 < s < r_0).$$

However, for any given particular value of $s \in (0, r_0)$ this provides $n-1$ weak sign changes of $p(t,s) =: P_s(t) \in \pi_{n-3}$ leading immediately by Proposition 1 to the identity $P_s \equiv 0$. This being so for all $s \in (0, r_0)$ implies that $P(t,s) = 0$ on a non-empty open set $\mathbb{R} \times (0, r_0) \subset \mathbb{R}^2$. Since P is a polynomial, this is enough to prove the assertion.

Claim 18. *If the product of $q_{\mathbf{a}}$ and $q_{\mathbf{b}}$, defined in* (8.1)–(8.2)*, vanishes on some relatively open subset of H, then we have*

$$h_n \equiv 0, \quad \text{i.e.,} \quad f_n \equiv s_n$$

on the whole space $\mathbb{R}^d$.

Proof. The polynomial $q_{\mathbf{a}}(\mathbf{y}) \cdot q_{\mathbf{b}}(\mathbf{y}) \in \pi_4^{d-1}$ can vanish on an open set only if it is identically zero. Hence, for all $\mathbf{u} \in \operatorname{r\,int} V \neq \emptyset$, either $q_{\mathbf{a}}(\mathbf{u}) = 0$ or $q_{\mathbf{b}}(\mathbf{u}) = 0$. Applying Claim 17 we are led to $h_n(\mathbf{c}(t) + s\mathbf{u}) = 0$ $(\forall t, s \in \mathbb{R})$. That is, the polynomial $h_n(\mathbf{x}) \in \pi_n^d$ vanishes on the non-empty open set $\ell + \operatorname{r\,int} V \subset \mathbb{R}^d$. Hence h_n is identically zero and Claim 18 is proved.

Claim 19. *There exists a positive constant $\rho := \rho(n) > 0$ with the following property: Whenever $\mathbf{x} \in K$ and $\mathbf{y} = \mathbf{y}(\mathbf{x})$ is defined as in* (2.3) *and* (2.4)*, then for all $0 \leq r \leq \rho$ we have*

$$(-1)^j h_n(\mathbf{c}_j + r\mathbf{y}) \leq 0 \qquad (j = 1, \ldots, n-1).$$

Proof. Let us take any particular $j \in \{1, \ldots, n-1\}$. If $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$, then $\mathbf{y}(x) = 0$ and by (6.9) we have nothing to prove. If $\mathbf{x} \in K \setminus [\mathbf{a}, \mathbf{b}]$, then $\mathbf{y}(x) \neq 0$ and $\mathbf{a}, \mathbf{b}$ and $\mathbf{x}$ span a non-degenerate triangle Δ belonging to K. With $\mathbf{z}(\mathbf{x}) = \mathbf{z} \in [\mathbf{a}, \mathbf{b}]$ as in (2.4), suppose now, e.g., that $\mathbf{c}_j \in [\mathbf{b}, \mathbf{z}]$ (the other case $\mathbf{c}_j \in [\mathbf{z}, \mathbf{a}]$ being quite similar). Then with a homothetic transformation, having $\mathbf{b}$ as its fixed point and $\lambda_j := \frac{\langle \mathbf{c}_j - \mathbf{b}, \mathbf{v} \rangle}{\langle \mathbf{z} - \mathbf{b}, \mathbf{v} \rangle}$ as its ratio, we get the point $\mathbf{x}_j = \mathbf{b} + \lambda_j(\mathbf{x} - \mathbf{b}) = \mathbf{b} + \lambda_j(\mathbf{z} - \mathbf{b}) + \lambda_j \mathbf{y}$ as the image of $\mathbf{x}$. Note that λ_j is well defined since $\mathbf{c}_j \in [\mathbf{b}, \mathbf{z}]$ ensures

$$\langle \mathbf{z} - \mathbf{b}, \mathbf{v} \rangle \geq \langle \mathbf{c}_j - \mathbf{b}, \mathbf{v} \rangle \geq \langle \mathbf{c}_{n-1} - \mathbf{b}, \mathbf{v} \rangle = \frac{1}{2}(t_{n-1} + 1)\langle \mathbf{v}, \mathbf{v} \rangle = \sin^2\left(\frac{\pi}{2n}\right)\langle \mathbf{v}, \mathbf{v} \rangle > 0,$$

and that $0 < \lambda_j \leq 1$ because $\langle \mathbf{z} - \mathbf{b}, \mathbf{v} \rangle \geq \langle \mathbf{c}_j - \mathbf{b}, \mathbf{v} \rangle$. Hence $\mathbf{x}_j \in [\mathbf{b}, \mathbf{x}]$ and thus $\mathbf{x}_j \in K$. Note that $\mathbf{b} + \lambda_j(\mathbf{z} - \mathbf{b}) = \mathbf{c}_j$ and $\lambda_j \mathbf{y} \in H$.

Now choosing $\rho := \sin^2(\frac{\pi}{2n})$ and computing

$$\langle \mathbf{c}_j - \mathbf{b}, \mathbf{v} \rangle = \frac{1}{2}(t_j + 1)\langle \mathbf{v}, \mathbf{v} \rangle = \sin^2(\frac{(n-j)\pi}{2n})\langle \mathbf{v}, \mathbf{v} \rangle$$

we see also that $\rho \le \lambda_j \le 1$. From this we get $[\mathbf{c}_j, \mathbf{c}_j + \rho \mathbf{y}] \subset [\mathbf{c}_j, \mathbf{x}_j] \subset K$ by convexity. However, then for any $0 \le r \le \rho$ also $\mathbf{x}_j(r) = \mathbf{c}_j + r\mathbf{y} \in K$. From here the proof reduces to the proof of $(-1)^j h_n(\mathbf{x}_j(r)) \le 0$ provided $\mathbf{x}_j(r) \in K \cap (\mathbf{c}_j + H)$. That was already done in the second part of the proof of Claim 14. Since j was arbitrarily chosen, Claim 19 is proved with the constant $\rho = \rho(n) = \sin^2(\frac{\pi}{2n})$.

Claim 20. *Let $\mathbf{x} \in K$ and $\mathbf{y} = \mathbf{y}(\mathbf{x}) \in H$ be as in* (2.3)*,* (2,6)*. With the same constant $\rho = \rho(n) > 0$ as in Claim* 19 *we have*

$$h_n(\mathbf{a} + r\mathbf{y}) \ge 0, \qquad (-1)^n h_n(\mathbf{b} + r\mathbf{y}) \ge 0.$$

Proof. Using Claim 19 it is easy to see that for any fixed $r \in [0, \rho]$ the polynomial $H_r(t) := h_n(\mathbf{c}(t) + r\mathbf{y})$ changes sign weakly at n points if either $H_r(1) = h_n(\mathbf{a} + r\mathbf{y}) \le 0$ or $H_r(-1) = h_n(\mathbf{b} + r\mathbf{y}) \le 0$. On the other hand $H_r \in \pi_n$, but its degree can be reduced similarly to the calculations in Claim 17. Indeed, we have

$$H(t,r) := H_r(t) = \sum_{k=0}^{n} p_k(t) r^k,$$

and $p_0 = h_n|_\ell \equiv 0$, $p_1 = \frac{\partial h_n}{\mathbf{y}}|_\ell \equiv 0$ according to (6.12) and (7.3). This leads to

$$H(t,r) = r^2 \sum_{k=2}^{n} p_k(t) r^{k-2} = r^2 F(r,t), \quad F(r,t) \in \pi^2_{n-2}$$

and also $F_r(t) \in \pi_{n-2}$ with $F_r(t) := F(r,t) = r^{-2}H(t,r) = r^{-2}H_r(t)$. Hence F_r has n sign changes, too, and Proposition 1 applies showing that $F_r \equiv 0$, i.e., also $h_n(\mathbf{a} + r\mathbf{y}) = 0$ and $h_n(\mathbf{b} + r\mathbf{y}) = 0$.

Claim 21. *There exists a finite positive constant $\gamma = \gamma(n)$, depending only on n (but not on any other occurring objects), so that for any $\mathbf{u} \in V$ we have*

$$\frac{1}{\gamma}\, q_{\mathbf{a}}(\mathbf{u}) \le (-1)^n q_{\mathbf{b}}(\mathbf{u}) \le \gamma\, q_{\mathbf{a}}(\mathbf{u}).$$

Proof. In view of Claim 16, $q_{\mathbf{a}}(u) = |q_{\mathbf{a}}(u)|$ and $(-1)^n q_{\mathbf{b}}(u) = |q_{\mathbf{b}}(u)|$. Similarly to the proof of Claim 16 we consider $p(t) := \frac{\partial^2 h_n}{\partial \mathbf{u}^2}(\mathbf{c}(t)) \in \pi_{n-2}$ which changes sign (at least in the weak sense) on the point set $S = \{t_j : j = 1, \ldots, n-1\}$. Hence Proposition 4 can be applied and since $p(-1) = q_{\mathbf{b}}(\mathbf{u})$,

$p(1) = q_{\mathbf{a}}(\mathbf{u})$, we conclude that at least $|q_{\mathbf{b}}(\mathbf{u})| \leq \alpha(S)q_{\mathbf{a}}(\mathbf{u})$. Taking now $\gamma := \alpha(s) < \infty$, we arrive at the second inequality of Claim 21. To get the other one, in view of the obvious symmetry of S with respect to 0, we may consider $p^*(t) := (-1)^n p(-t)$ with node set $S^* = S$. Hence we can get also the other inequality with the same constant γ.

9. Quadratic separation and proof of the sufficiency part

The key to the proof of the sufficiency part of Theorems 1 and 2 is the following explicite estimate, formulated as a lemma. Note that (9.1) and (9.2) are very close to quadratic separation, but they express only some kind of "one-sided" separation, different at $\mathbf{a}$ and $\mathbf{b}$. Also admissibility is not discussed here yet.

Lemma. *There exists a positive constant $\delta = \delta(\mathbf{a}, \mathbf{b}, K, f) > 0$ such that we have*

$$\langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle \geq \delta \cdot q_{\mathbf{a}}(\mathbf{y}(\mathbf{x})) \qquad (\forall \mathbf{x} \in K), \tag{9.1}$$

and similarly also

$$\langle \mathbf{x} - \mathbf{b}, \mathbf{v}^\star \rangle \geq \delta (-1)^n q_{\mathbf{b}}(\mathbf{y}(\mathbf{x})) \qquad (\forall \mathbf{x} \in K). \tag{9.2}$$

Proof. We prove only (9.1) since the variant (9.2) for $\mathbf{b}$ can be derived mutatis mutandis. Note that if $q_{\mathbf{a}}(\mathbf{y}(\mathbf{x})) = 0$ for a certain $\mathbf{x} \in K$, then (9.1) simplifies to the inequality expressing the fact that $H_{\mathbf{a}}$ supports K at $\mathbf{a}$. Thus, if $q_{\mathbf{a}}(\mathbf{y}(\mathbf{x}))$ happens to be zero, then we have nothing more to do. However in the forthcoming argument we do not distinguish the cases whether $q_{\mathbf{a}}$ vanishes or not.

We define the set

$$F := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{b}, \mathbf{v}^\star \rangle \leq \langle \mathbf{x}, \mathbf{v}^\star \rangle \leq \langle \mathbf{a}, \mathbf{v}^\star \rangle \quad \text{and} \quad (\ell + \mathbf{y}(\mathbf{x})) \cap K \neq \emptyset\}. \tag{9.3}$$

The first condition means that $\mathbf{x} \in F$ is in the supporting strip between $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$, the second expresses that the projection (in the direction $\mathbf{v}$) of F to $H_{\mathbf{a}}$ equals to the projection of K. Clearly F is closed and bounded, thus $\partial f \in \pi_{n-1}^{d,d}$ is bounded on F and we can write

$$D := \max_F \left| \frac{\partial f}{\partial \mathbf{v}} \right| < \infty. \tag{9.4}$$

At the outset we fix an element $\mathbf{x} \in K$ arbitrarily. Let now $0 \leq \lambda \leq 1$ be arbitrary and consider the points $\mathbf{x}_\lambda := \lambda(\mathbf{x} - \mathbf{a}) + \mathbf{a} \in K$, $\mathbf{y}_\lambda := \mathbf{y}(\mathbf{x}_\lambda) =$

$\lambda\mathbf{y}(\mathbf{x}) \in H$. Clearly the line segment $[\mathbf{x}_\lambda, \mathbf{y}_\lambda + \mathbf{a}] = \lambda[\mathbf{x} - \mathbf{a}, \mathbf{y}] + \mathbf{a}$ is parallel to $\mathbf{v}$ and

$$\begin{aligned} \mathbf{y}_\lambda + \mathbf{a} - \mathbf{x}_\lambda &= \lambda(\mathbf{y} - (\mathbf{x} - \mathbf{a})) = \lambda \frac{\langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \mathbf{v}, \\ |\mathbf{y}_\lambda + \mathbf{a} - \mathbf{x}_\lambda| &= \lambda \frac{\langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \end{aligned}$$

while $[\mathbf{x}_\lambda, \mathbf{y}_\lambda + \mathbf{a}] \subset F$. Thus, applying the Lagrange mean value theorem, we get

$$\begin{aligned} f_n(\mathbf{y}_\lambda + \mathbf{a}) &= f_n(\mathbf{x}_\lambda) + \langle \partial f_n((1-\xi)\mathbf{x}_\lambda + \xi(\mathbf{y}_\lambda + \mathbf{a})),\ \mathbf{y}_\lambda + \mathbf{a} - \mathbf{x}_\lambda \rangle \\ &\leq 1 + \frac{\lambda D}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle \end{aligned}$$

which yields, in view of $\mathbf{y}_\lambda + \mathbf{a} \in H_\mathbf{a}$, (6.8) and $s_n(\mathbf{y}_\lambda + \mathbf{a}) = 1$, the estimate

$$h_n(\mathbf{y}_\lambda + \mathbf{a}) \leq \frac{\lambda D}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle. \tag{9.5}$$

Let us consider now the auxiliary polynomial $P(r) := h_n(\mathbf{a} + r\mathbf{y}) \in \pi_n$. Since $P(0) = h_n(\mathbf{a}) = 0$ and $P'(0) = \frac{\partial h_n}{\partial \mathbf{y}}(\mathbf{a}) = 0$ (cf. (6.12) and (7.3)), we can write $P(r) = r^2 P^*(r)$ with $P^* \in \pi_{n-2}$. Note that

$$P^*(0) = \frac{\partial^2 h_n}{\partial \mathbf{y}^2}(\mathbf{a}) = q_\mathbf{a}(\mathbf{y}) \geq 0$$

as it was shown in Claim 16. Moreover, Claim 20 furnishes an interval $[0, \rho]$ with $\rho = \rho(n-2) > 0$ so that P, and hence also P^*, is non-negative for $0 \leq r \leq \rho$. Now an application of Proposition 3 to the polynomial $p(t) := P^*(t\rho) \geq 0$ $(0 \leq t \leq 1)$ ensures the existence of some $r_0 = t_0\rho$ such that $c(n-2) \leq t_0 \leq 1$ and $p(t_0) \geq \frac{1}{2}p(0)$, i.e.,

$$P^*(r_0) \geq \frac{1}{2} P^*(0) = \frac{1}{2} q_\mathbf{a}(\mathbf{y})$$

with some r_0 in $[c(n-2) \cdot \rho(n-2), \rho(n-2)]$. Using the definitions of P^* and P, we infer for the very same $r_0 \in [c(n-2)\rho(n-2), \rho(n-2)]$ that

$$h_n(\mathbf{y}_{r_0} + \mathbf{a}) = P(r_0) = r_0^2 P^*(r_0) \geq \frac{1}{2} q_\mathbf{a}(\mathbf{y}) r_0^2. \tag{9.6}$$

Note that $r_0 \in [0, 1]$. Hence $\lambda = r_0$ can be used in (9.5), and we obtain from (9.5) and (9.6) the estimate

$$\frac{1}{2} q_\mathbf{a}(\mathbf{y}) r_0^2 \leq r_0 \frac{D}{\langle \mathbf{v}, \mathbf{v}^\star \rangle} \langle \mathbf{a} - \mathbf{x}, \mathbf{v}^\star \rangle,$$

or, after cancellation and an application of $r_0 \geq \rho(n-2)\cdot c(n-2)$,

$$\langle \mathbf{a}-\mathbf{x}, \mathbf{v}^\star\rangle \geq \frac{\rho(n-2)c(n-2)\langle \mathbf{v}, \mathbf{v}^\star\rangle}{2D} \cdot q_{\mathbf{a}}(\mathbf{y}(\mathbf{x})). \tag{9.7}$$

Choosing now

$$\delta := \delta(\mathbf{a}, \mathbf{b}, K, f) := \frac{\rho(n-2)c(n-2)\langle \mathbf{v}, \mathbf{v}^\star\rangle}{2D},$$

we really obtain (9.1) from (9.7), which completes the proof of the lemma.

Here we can start the proof of the sufficiency part. Uniqueness of the conjugate vector (i.e., $\dim N = 1$) is supposed, and we also suppose that there exists another extremal function f_n not identically equal to the canonical example s_n. In other words, we start with an h_n not identically vanishing and aim to find a quadratic separation of K at $\mathbf{a}$ and $\mathbf{b}$ from $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$, respectively. Our candidate for quadratic separation is the quadratic form

$$q := \frac{\delta}{\gamma} q_{\mathbf{a}}, \tag{9.8}$$

where δ and γ are as in the Lemma and Claim 21, respectively. Since δ/γ is a positive constant, q will be admissible if and only if $q_{\mathbf{a}}$ is admissible. Now the identity $q_{\mathbf{a}} \equiv 0$ can not occure since then Claim 18 would give $h_n \equiv 0$, contrary to our starting hypothesis. Also $q_{\mathbf{a}}(\mathbf{u}) \geq 0$ ($\mathbf{u} \in V$) has been proved already in Claim 16. Hence $q_{\mathbf{a}}$ (and consequently, q) is admissible (cf. iii) in Claim 5). Now we apply the Lemma, (9.1) and $\gamma \geq 1$ (which is obvious from its definition, and also from the statement of Claim 21), and obtain

$$\langle \mathbf{a}-\mathbf{x}, \mathbf{v}^\star\rangle \geq q(\mathbf{y}(\mathbf{x})) \qquad (\mathbf{x} \in K). \tag{9.9}$$

Similarly we also apply (9.2) from the Lemma to $\mathbf{b}$ and get via an application of Claim 21

$$\langle \mathbf{x}-\mathbf{b}, \mathbf{v}^\star\rangle \geq \delta(-1)^n q_{\mathbf{b}}(\mathbf{y}(\mathbf{x})) \geq q(\mathbf{y}(\mathbf{x})) \qquad (\mathbf{x} \in K). \tag{9.10}$$

Combining (9.9) and (9.10) leads to (2.5), and since q is admissible, this provides a quadratic separation in the sense of Definition 1.

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