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TURÁN-ERŐD TYPE CONVERSE MARKOV INEQUALITIES FOR CONVEX DOMAINS ON THE PLANE

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Abstract

For a convex domain $K \subset \mathbb{C}$ the well-known general Bernstein-Markov inequality holds: a polynomial p of degree n must have $\|p'\| \le c(K)n^2\|p\|$. However, for polynomials in general, $\|p'\|$ can be arbitrarily small, compared to $\|p\|$.

Turán investigated the situation under the condition that p have all its zeroes in the convex body K. With this assumption he proved $\|p'\| \ge (n/2)\|p\|$ for the unit disk D and $\|p'\| \ge c\sqrt{n}\|p\|$ for the unit interval $I := [-1, 1]$. Levenberg and Poletsky provided general lower estimates of order $\sqrt{n}$, and there were certain classes of domains with order n lower estimates.

We show that for *all* compact and convex *domains* K and polynomials p with all their zeroes in K $\|p'\| \ge c(K)n\|p\|$ holds true, while $\|p'\| \le C(K)n\|p\|$ occurs for arbitrary compact connected sets $K \subset \mathbb{C}$. Moreover, the dependence on width and diameter of the set K is found up to a constant factor. Note that if K is *not* a domain ($\mathrm{int}K = \emptyset$), then the order is only $\sqrt{n}$.

Erőd observed that in case the boundary of the domain is smooth and the curvature exceeds a constant $\kappa > 0$, then we can get an order n lower estimation with the curvature occurring in the implied constant. Elaborating on this idea several extensions of the result are given. Again, geometry is in focus, including a new, strong "discrete" version of the classical Blaschke Rolling Ball Theorem.

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Key Words and Phrases: Bernstein-Markov Inequality, Turán's lower estimate of derivative norm, logarithmic derivative, Chebyshev constant, convex sets and domains, width of a set, circular domains, convex curves, smooth convex bodies, curvature, osculating circle, Blaschke's rolling ball theorem, subdifferential or Lipschitz-type lower estimate of increase

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1. Introduction

On the complex plane $\mathbb{C}$ polynomials of degree n admit a Markov inequality[1] $\|p'\|_K \leq c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \geq c_K\, n^A \,\|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials must belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n^{(0)}(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n^{(0)}(K)} M(p) \qquad \text{with} \qquad M := M(p) := \frac{\|p'\|}{\|p\|}\ . \tag{1.1}$$

Theorem 1.1 (**Turán,** [21, p. 90]). *If $p \in \mathcal{P}_n(D)$, where D is the unit disk, then we have*

$$\left\|p'\right\|_D \geq \frac{n}{2} \|p\|_D\ . \tag{1.2}$$

Theorem 1.2 (**Turán,** [21, p. 91]). *If $p \in \mathcal{P}_n(I)$, where $I := [-1, 1]$, then we have*

$$\left\|p'\right\|_I \geq \frac{\sqrt{n}}{6} \|p\|_I\ . \tag{1.3}$$

Theorem **1**.1 is best possible, as the example of $p(z) = 1 + z^n$ shows. This also highlights the fact that, in general, the order of the inverse Markov factor cannot be higher than n. On the other hand, a number of positive results, started with J. Erőd's work, exhibited convex domains having order n inverse Markov factors (like the disk). We come back to this after a moment.

Regarding Theorem **1**.2, Turán pointed out by the example of $(1-x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [8], but the value of the constant is computed for all fixed n precisely in [6]. In fact, about two-third of the paper [6] is occupied by the rather lengthy and difficult calculation of these constants, which partly explains why later authors started to consider this achievement the only content of the paper. Nevertheless, the work of Erőd was much richer, with many important ideas occurring in the various approaches what he had presented.

[1] Namely, to each point z of K there exists another $w \in K$ with $|w - z| \geq \operatorname{diam}(K)/2$, and thus application of Markov's inequality on the segment $[z, w] \subset K$ yields $|p'(z)| \leq (4/\operatorname{diam}(K))n^2 \|p\|_K$.

In particular, Erőd considered ellipse domains, which form a parametric family E_b naturally connecting the two sets I and D. Note that for the same sets E_b the best form of the Bernstein-Markov inequality was already investigated by Sewell, see [19].

Theorem 1.3 (**Erőd,** [6, p. 70]). *Let $0 < b < 1$ and let E_b denote the ellipse domain with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Then*

$$\|p'\| \geq \frac{b}{2} n \|p\| \tag{1.4}$$

for all polynomials p of degree n and having all zeroes in E_b.

Erőd himself provided two proofs, the first being a quite elegant one using elementary complex functions, while the second one fitting more in the frame of classical analytic geometry. In 2004 this theorem was rediscovered by J. Szabados, providing a testimony of the natural occurrence of the sets E_b in this context[2].

In fact, the key to Theorem **1**.1 was the following observation, implicitly already in [21] and [6] and formulated explicitly in [8].

Lemma 1.1 (**Turán, Levenberg-Poletsky**). *Assume that $z \in \partial K$ and that there exists a disc D_R of radius R so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$|p'(z)| \geq \frac{n}{2R} |p(z)| \ . \tag{1.5}$$

So Levenberg and Poletsky [8] found it worthwhile to formally introduce the next definition.

Definition 1.1. A compact set $K \subset \mathbb{C}$ is called *R-circular*, if for any point $z \in \partial K$ there exists a disc D_R of radius R with $z \in \partial D_R$ and $K \subset D_R$.

With this they formulated various consequences. For our present purposes let us chose the following form, c.f. [8, Theorem 2.2].

Theorem 1.4 (**Erőd; Levenberg-Poletsky**). *If K is an R-circular set and $p \in \mathcal{P}_n^{(0)}(K)$, then*

$$\|p'\| \geq \frac{n}{2R} \|p\| \ . \tag{1.6}$$

[2]After learning about the overlap with Erőd's work, the result was not published.

Note that here it is not assumed that K be convex; a circular arc, or a union of disjoint circular arcs with proper points of join, satisfy the criteria. However, other curves, like e.g. the interval itself, do not admit such inequalities; as said above, the order of magnitude can be as low as $\sqrt{n}$ in general.

Erőd did not formulate the result that way; however, he was clearly aware of that. This can be concluded from his various argumentations, in particular for the next result.

Theorem 1.5 (Erőd, [6, p. 77]**).** *If K is a C^2-smooth convex domain with the curvature of the boundary curve staying above a fixed positive constant $\kappa > 0$, and if $p \in \mathcal{P}_n^{(0)}(K)$, then we have*

$$\|p'\| \geq c(K)n\|p\|. \tag{1.7}$$

From Erőd's argument one can not easily conclude that the constant is $c(K) = \kappa/2$; on the other hand, his statement is more general than that. Although the proof is slightly incomplete, let us briefly describe the idea[3].

Proof. The norm of p is attained at some point of the boundary, so it suffices to prove that $|p'(z)|/|p(z)| \geq cn$ for all $z \in \partial K$. But the usual form of the logarithmic derivative and the information that all the n zeroes $z_1, \ldots, z_n$ of p are located in K allows us to draw this conclusion once we have for a fixed direction $\varphi := \varphi(z)$ the estimate

$$\Re\left(e^{i\varphi}\frac{1}{z - z_k}\right) \geq c > 0 \qquad (k = 1, \ldots, n). \tag{1.8}$$

Choosing φ the (outer) normal direction of the convex curve ∂K at $z \in \partial K$, and taking into consideration that z_k are placed in $K \setminus \{z\}$ arbitrarily, we end up with the requirement that

$$\Re\left(e^{i\varphi}\frac{1}{z - w}\right) = \frac{\cos\alpha}{|z - w|} \geq c \qquad (w \in K \setminus \{z\}, \;\; \alpha := \varphi - \arg(z - w)) \,. \tag{1.9}$$

Now if K is strictly convex, then for $z \neq w$ we do not have $\cos\alpha = 0$, a necessary condition for keeping the ratio off zero. It remains to see if $|z - w|/\cos\alpha$ stays bounded when $z \in \partial K$ and $w \in K \setminus \{z\}$, or, as is easy to see, if only $w \in \partial K \setminus \{z\}$. Observe that $F(z, w) := |z - w|/\cos\alpha$ is a two-variate function on ∂K^2, which is not defined for the diagonal $w = z$, but under certain conditions can be extended continuously. Namely, for given z the limit, when $w \to z$, is the well-known geometric quantity $2\rho(z)$, where $\rho(z)$ is the radius of the osculating circle (i.e., the reciprocal of the curvature $\kappa(z)$). (Note here a gap in the argument for not taking into consideration also $(z', w') \to (z, z)$, which can be removed by

[3] For more about the life and work of János Erőd, see [15] and [16].

showing uniformity of the limit.) Hence, for smooth ∂K with strictly positive curvature bounded away from 0, we can define $F(z,z) := 2/\kappa(z) = 2\rho(z)$. This makes F a continuous function all over ∂K^2, hence it stays bounded, and we are done. $\square$

We will return to this theorem and provide a somewhat different, complete proof giving also the value $c(K) = \kappa/2$ of the constant later in §**6**. For an analysis of the slightly incomplete, nevertheless essentially correct and really innovative proof of Erőd see [15].

From this argument it can be seen that whenever we have the property (1.9) for all given boundary points $z \in \partial K$, then we also conclude the statement. This explains why Erőd could allow even vertices, relaxing the conditions of the above statement to hold only piecewise on smooth Jordan arcs, joining at vertices. However, to have a fixed bound, either the number of vertices has to be bounded, or some additional condition must be imposed on them. Erőd did not elaborate further on this direction.

Convex domains (or sets) *not* satisfying the R-circularity criteria with any fixed positive value of R are termed to be *flat*. Clearly, the interval is flat, like any polygon or any convex domain which is not strictly convex. From this definition it is not easy to tell if a domain is flat, or if it is circular, and if so, then with what (best) radius R. We will deal with the issue in this work, aiming at finding a large class of domains having cn order of the inverse Markov factor with some information on the arising constant as well.

On the other hand a lower estimate of the inverse Markov factor of the same order as for the interval was obtained in full generality in 2002, see [8, Theorem 3.2].

Theorem 1.6 (**Levenberg-Poletsky**). *If $K \subset \mathbb{C}$ is a compact, convex set, $d := \operatorname{diam} K$ is the diameter of K and $p \in \mathcal{P}_n^{(0)}(K)$, then we have*

$$\left\|p'\right\| \geq \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\| . \tag{1.10}$$

Clearly, we can have no better order, for the case of the interval the $\sqrt{n}$ order is sharp. Nevertheless, already Erőd [6, p. 74] addressed the question: "For what kind of domains does the method of Turán apply?" Clearly, by "applies" he meant that it provides cn order of oscillation for the derivative.

The most general domains with $M(K) \gg n$, found by Erőd, were described on p. 77 of [6]. Although the description is a bit vague, and the proof shows slightly less, we can safely claim that he has proved the following result.

Theorem 1.7 (**Erőd**). *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles $< \pi$, with all the arcs being C^2-smooth and being either straight lines of length $\ell < \Delta(K)/4$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from 0 by a fixed constant. Then there is a constant $c(K)$, such that $M_n(K) \geq c(K)n$ for all $n \in \mathbb{N}$.*

To deal with the flat case of straight line boundary arcs, Erőd involved another approach, cf. [6, p. 76], appearing later to be essential for obtaining a general answer. Namely, he quoted Faber [7] for the following fundamental result going back to Chebyshev.

Lemma 1.2 (**Chebyshev**). *Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$ and let $J \subset R \subset \mathbb{C}$ be any set containing J. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in R} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \left(\frac{|J|}{4} \right)^k . \tag{1.11}$$

Proof. This is essentially the classical result of Chebyshev for a real interval, cf. [2, 9], and it holds for much more general situations (perhaps with the loss of the factor 2) from the notion of Chebyshev constants and capacity, cf. Theorem 5.5.4. (a) in [11]. □

The relevance of Chebyshev's Lemma is that it provides a quantitative way to handle contribution of zero factors at some properly selected set J. One uses this for comparison: if $|p(\zeta)|$ is maximal at $\zeta \in \partial K$, then the maximum on some J can not be larger. Roughly speaking, combining this with geometry we arrive at an effective estimate of the contribution, hence even on the location of the zeroes.

In his recent work [5], Erdélyi considered various special domains. Apart from further results for polynomials of some special form (e.g. even or real polynomials), he obtained the following.

Theorem 1.8 (**Erdélyi**). *Let Q denote the square domain with diagonal $[-1, 1]$. Then for all polynomials $p \in \mathcal{P}_n(Q)$ we have*

$$\|p'\| \geq C_0 n \|p\| \tag{1.12}$$

with a certain absolute constant C_0.

Note that the regular n-gon K_n is already covered by Erőd's Theorem **1**.7 if $n \geq 26$, but not the square Q, since the side length h is larger than the quarter of the transfinite diameter Δ: actually, $\Delta(Q) \approx 0.59017\ldots h$, while

$$\Delta(K_n) = \frac{\Gamma(1/n)}{\sqrt{\pi}2^{1+2/n}\Gamma(1/2+1/n)} h > 4h \quad \text{iff} \qquad n \geq 26,$$

see [11, p. 135]. Erdélyi's proof is similar to Erőd's argument[4]: sacrificing generality gives the possibility for a better calculation for the particular choice of Q.

Returning to the question of the order in general, let us recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ *having nonempty interior*. Clearly, assuming boundedness is natural, since all polynomials of positive degree have $\|p\|_K = \infty$ when the set K is unbounded. Also, all convex sets with nonempty interior are *fat*, meaning that $\mathrm{cl}(K) = \mathrm{cl}(\mathrm{int}K)$. Hence taking the closure does not change the sup norm of polynomials under study. The only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his $c\sqrt{n}$ lower estimate is of the right order. Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order.

To study (1.1) some geometric parameters of the convex domain K are involved naturally. We write $d := d(K) := \mathrm{diam}\,(K)$ for the *diameter* of K, and $w := w(K) := \mathrm{width}\,(K)$ for the *minimal width* of K. That is,

$$w(K) := \min_{\gamma\in[-\pi,\pi]} \left(\max_{z\in K} \Re(ze^{-i\gamma}) - \min_{z\in K} \Re(ze^{-i\gamma}) \right). \tag{1.13}$$

Note that a (closed) convex domain is a (closed), bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leq d(K) < \infty$. Our main result is the following.

Theorem 1.9. *Let $K \subset \mathbb{C}$ be any convex domain having minimal width $w(K)$ and diameter $d(K)$. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$\frac{\|p'\|}{\|p\|} \geq C(K)n \qquad \text{with} \qquad C(K) = 0.0003\frac{w(K)}{d^2(K)}\ . \tag{1.14}$$

Then again, as regards the order of magnitude, (and in fact apart from an absolute constant factor), this result is sharp for all convex domains $K \subset \mathbb{C}$.

Theorem 1.10. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$*

[4]Erdélyi was apparently not aware of the full content of [6] when presenting his rather similar argument.

there exists a polynomial $p \in \mathcal{P}_n^{(0)}(K)$ *of degree exactly* n *satisfying*

$$\|p'\| \leq C'(K)\, n\, \|p\| \qquad \text{with} \qquad C'(K) := 600\, \frac{w(K)}{d^2(K)}\ . \tag{1.15}$$

Remark 1.1. Note that here we do not assume that K be convex, but only that it is a connected, closed (compact) subset of $\mathbb{C}$. (Clearly the condition of boundedness is not restrictive, $\|p\|$ being infinite otherwise.)

In the proof of Theorem **1**.9, due to generality, the precision of constants could not be ascertained e.g. for the special ellipse domains considered in [6]. Thus it seems that the general results are not capable to fully cover e.g. Theorem **1**.3.

However, even that is possible for a quite general class of convex domains with order n inverse Markov factors and a different estimate of the arising constants. This will be achieved working more in the direction of Erőd's first observation, i.e. utilizing information on curvature.

Since these results need some technical explanations, formulation of these will be postponed until §**6**. But let us mention the key ingredient, which clearly connects curvature and the notion of circular domains. In the smooth case, it is well-known as Blaschke's Rolling Ball Theorem, cf. [1, p. 116].

Lemma 1.3 (**Blaschke**). *Assume that the convex domain* K *has* C^2 *boundary* $\Gamma = \partial K$ *and that there exists a positive constant* $\kappa > 0$ *such that the curvature* $\kappa(\zeta) \geq \kappa$ *at all boundary points* $\zeta \in \Gamma$. *Then to each boundary points* $\zeta \in \Gamma$ *there exists a disk* D_R *of radius* $R = 1/\kappa$, *such that* $\zeta \in \partial D_R$, *and* $K \subset D_R$.

Again, geometry plays the crucial role in the investigations of variants when smoothness and conditions on curvature are relaxed. We will strongly extend the classical results of Erőd, showing that conditions on the curvature suffices to hold only almost everywhere (in the sense of arc length measure) on the boundary.

Theorem 1.11. *Assume that the convex domain* K *has boundary* $\Gamma = \partial K$ *and that the a.e. existing curvature of* Γ *exceeds* κ *almost everywhere, or, equivalently, assume the subdifferential condition* (3.6) *(or any of the equivalent formulations in* (3.1)-(3.6)*) with* $\lambda = \kappa$. *Then for all* $p \in \mathcal{P}_n^{(0)}(K)$ *we have*

$$\|p'\| \geq \frac{\kappa}{2} n \|p\|\ . \tag{1.16}$$

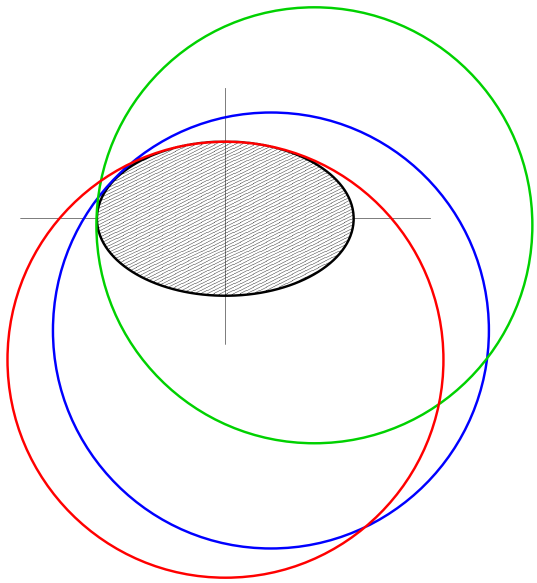

Figure 1. The Ellipse E_b is b-circular according to the Rolling Ball Theorem of Blaschke. Therefore, $\|p'\|/\|p\| \geq bn/2$.

This also hinges upon geometry, and we will have two proofs. One is essentially an application of a recent, quite far-reaching extension of the Blaschke Theorem by Strantzen. The other involves even more geometry: it hinges upon a new, discrete version of the Blaschke Rolling Ball Theorem, (which easily implies also Strantzen's Theorem), but which is suitable, at least in principle, to provide also some degree-dependent estimate of $M_n(K)$ by means of the *minimal oscillation* or *change* of the outer unit normal vector(s) along the boundary curve.

For applications to various domains, where yields of the different estimates can also be compared, see the later sections. Before that, in the next section we discuss the most general result, Theorem **1**.9, and its sharpness, as expressed by Theorem **1**.10.

In §**3** we start with describing the underlying geometry, and in §**5** we will describe variants and extensions on the theme of the Blaschke Rolling Ball Theorem. Finally, in §**6** we will formulate the resulting theorems and analyze the yields of them on various parametric classes of domains.

2. A discussion of the proof of the main theorem and its sharpness

Here we only comment on the ideas of the proofs, which otherwise are several pages long fine estimates and calculations. However, the point is not in the calculations themselves, but in the geometrical ideas behind. Those are what we try to explain a bit here. With the ideas clarified, it is still a matter of rather delicate, technical work, but still possible – without the proper insight it is rather improbable that one would just blindly compute them. In this regard we also thank a great deal to Prof. Gábor Halász, who provided us one of the key geometrical ingredients by suggesting a really insightful modification of our original argument. On the other hand, the proof is elementary in the sense that no special theoretical knowledge is required from the reader to fully follow the proofs, check the calculations in [13] or [17].

Let us recall that after the general result of Theorem **1**.6 of Levenberg and Poletsky it was widely felt that no better, than the $\sqrt{n}$ order, can be obtained for arbitrary convex bodies. Research was thus directed to *special* sets, still admitting better order Turán-Markov constants. It was a surprise when our preprint [14] surpassed $\sqrt{n}$ proving $n^{2/3}$ in general. In fact, there were serious people stating that this is the right order and that they have computed the then seemingly extremal, difficult to handle triangle having Turán-Markov constant of that order. As we could not reconstruct, could not conclude the allegedly working counterexample, we discussed the situation with Gábor Halász, who first also tried to fix the calculations, but then came up with the observation that our method, considering exactly normal lines to the selected maximal point $\zeta \in \partial K$ with $|p(\zeta)| = \|p\|_K$ is not optimal for the triangle.

He observed, that with ζ situated close (but not at) a vertex, the normal line provides a loss in the estimates, as the distance from zeroes lying possibly on (or close to) the (longer) part of the side of the triangle where ζ sits, grows, in view of the Pithagorean Theorem, only proportionally to δ^2, if δ is the distance, measured inward along the normal line, from ζ. For small δ and h (length of intersection of the normal line and K) this is a serious loss, compared to linear increase $c\delta$ if we can consider a *slanted* line, not normal, but tilted towards the short end of the side, where ζ sits (i.e towards the close vertex of the triangle). Actually, this observation gave almost immediately an order n Turán-Markov constant for the triangle, and finally proved to be equally powerful for the general case, too.

Let us thus go over the idea of the full proof now. Throughout we will assume, as we may, that K is also closed, hence a compact convex set with nonempty interior.

We start with picking up a boundary point $\zeta \in \partial K$ of maximality of $|p|$, and consider a supporting line at ζ to K. Our original argument of [14] then used a normal direction and compared values of p at ζ and on the intersection of K and this normal line.

Essential use is made of the fact that in case the length h of this intersection is small (relative to w), then, due to convexity, the normal line cuts K to halves unevenly: one part has to be small (of the order of h). That is, the situation in

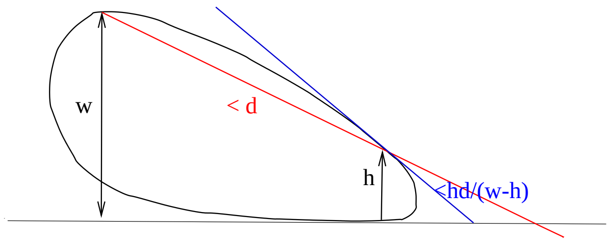

FIGURE 2. If the normal line is short, it cuts K to halves unevenly.

the general case is rather similar to the triangle – if the intersection of K with the normal is small, then one half of K, as divided by the normal, is altogether small. That was explicitly formulated in [14], and is used implicitly also in [17] through various calculations with the angles.

It is a major geometric feature at our help that when h is small, then one portion of K, cut into half by our slightly tilted line, is also small. This is the key feature which allows us to bend the direction of the normal a bit *towards the smaller portion* of K[5].

As said, in the proof written up in [13], we compare the values of p at ζ and *on a line slightly slanted off from the normal.* Comparing these calculations and the ones in [14] one can observe how this change led to a further, essential improvement of the result through improving the contribution of the factors belonging to zeroes z close to the supporting line. In [14] we could get a square term only (in terms of the distance $t < h$ we move away from the tangent point $\zeta = O$ along the normal), due to orthogonality and the consequent use of the Pythagorean Theorem in calculating the distances. However, here we obtain *linear dependence* in t via the general cosine theorem for the slanted segment J.

As a result of the improved estimates squeezed out this way, we do not need to employ the second usual technique, also going back to Turán, i.e. integration of $(p'/p)'$ over a suitably chosen interval. As pointed out already in [14], this part of the proof yields weaker estimates than cn, so avoiding it is not only a matter of convenience, but is an essential necessity.

The proof of the sharpness result Theorem **1**.10 also relies on the understanding of the geometry of the situation. Let us recall, how it starts.

Take $a, b \in K$ with $|a - b| = d$ and $m \in \mathbb{N}$ with $m > m_0$ to be determined later. Consider the polynomials $q(z) := (z-a)(z-b)$, $p(z) = (z-a)^m(z-b)^m =$

[5]If we try tilting the other way we would fail badly, even if the reader may find it difficult to distill from the proof where, and how. But if there were zeroes close to (or on) the supporting line and far from ζ *in the direction of the tilting*, then these zeroes were farther off from ζ, than from the other end of the intersecting segment. That would spoil the whole argument. However, since K is small in one direction of the supporting line, tilting towards this smaller portion does work.

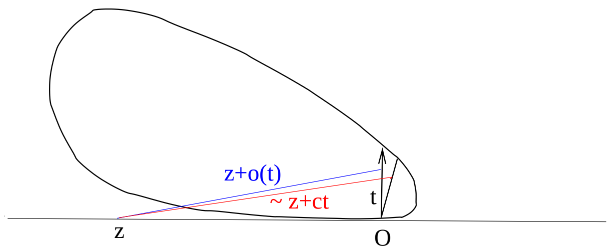

FIGURE 3. Tilting the normal line improves the growth of distance from zeroes z on or close to the supporting line to linear.

$q^m(z)$ and $P(z) = (z-a)^m(z-b)^{m+1} = (z-b)q^m(z)$. Clearly, $p, P \in \mathcal{P}_n^{(0)}(K)$ with $n = \deg p = 2m$ and $n = \deg P = 2m+1$, respectively. We claim that for appropriate choice of m_0 these polynomials satisfy inequality (1.15) for all $n > 2m_0$.

Without loss of generality we may assume $a = -1$, $b = 1$ and thus $d = 2$, as substitution by the linear function $\Phi(z) := \frac{2}{b-a}z - \frac{a+b}{b-a}$ shows. Indeed, if we prove the assertion for $\widetilde{K} := \Phi(K)$ and for $\widetilde{p}(z) = (z+1)^m(z-1)^m$, $\widetilde{P}(z) = (z+1)^m(z-1)^{m+1}$ defined on $\widetilde{K}$, we also obtain estimates for $p = \widetilde{p} \circ \Phi$ and $P = \widetilde{P} \circ \Phi$ on K. The homothetic factor of the inverse substitution Φ^{-1} is $\Lambda := \left|\frac{b-a}{2}\right| = d(K)/2$, and width changes according to $w(\widetilde{K}) = 2w(K)/d(K)$. Note also that under the linear substitution Φ the norms are unchanged but for the derivatives $\|p'\| = \Lambda^{-1}\|\widetilde{p}'\|$ and $\|P'\| = \Lambda^{-1}\|\widetilde{P}'\|$. So now we restrict to $a = -1$, $b = 1$, $d = 2$ and $q(z) := z^2 - 1$ etc.

First we make a few general observations. One obvious fact is that the imaginary axes separates $a = -1$ and $b = 1$, and as K is connected, it also contains some point $c = it$ of K. Therefore, $\|q\| \geq |q(c)| = 1 + t^2 \geq 1$. Also, it is clear that $q'(z) = 2z = (z-1) + (z+1)$: thus, by definition of the diameter

$$\|q'\| \leq \||z-1| + |z+1|\| \leq 4 \,. \tag{2.1}$$

Let us put $w^+ := \sup_{z\in K} \Im z$ and $w^- := -\inf_{z\in K} \Im z$. We can estimate $w' := \max(w^+, w^-)$ from above by a constant times w. That is, we claim that for any point $\omega = \alpha + i\beta \in K$ we necessarily have $|\beta| \leq \sqrt{2}w$ and so the domain K lies in the rectangle $R := \mathrm{con}\{-1 - i\sqrt{2}w, 1 - i\sqrt{2}w, 1 + i\sqrt{2}w, -1 + i\sqrt{2}w\}$.

To see this first note that $\beta \leq \sqrt{3}$, since $d(K) = 2$ by assumption. Recalling (1.13), take $e^{i\gamma}$ be the direction of the minimal width of K: by symmetry, we may take $0 \leq \gamma < \pi$. Then there is a strip of width w and direction $ie^{i\gamma}$ containing K, hence also the segments $[-1, 1]$ and $[\alpha, \alpha + i\beta]$. It follows that $2|\cos\gamma| \leq w$ and $\beta \sin\gamma \leq w$. The second inequality immediately leads to $\beta \leq \sqrt{2}w$ if $\gamma \in [\pi/4, 3\pi/4]$. So let now $\gamma \in [0, \pi/4) \cup [3\pi/4, \pi)$, i.e. $|\cos\gamma| \geq 1/\sqrt{2}$.

Applying also $\beta \le \sqrt{3}$ now we deduce $\beta \le \sqrt{3} \le \sqrt{3/2}\,2|\cos\gamma| \le \sqrt{3/2}w$, whence the asserted $w^{\pm} \le \sqrt{2}w$ is proved.

The rest is a (tedious, delicate) computation of norms of the polynomials $\widetilde{p}, \widetilde{P}$ and their derivatives in the rectangle R. We spare the reader from details referring to [13] or [17].

3. Some geometrical notions

Let $\mathbb{R}^d$ be the usual Euclidean space of dimension d, equipped with the Euclidean distance $|\cdot|$. Our starting point is the following classical result of Blaschke [1, p. 116].

Theorem 3.1 (Blaschke). *Assume that the convex domain $K \subset \mathbb{R}^2$ has C^2 boundary $\Gamma = \partial K$ and that with the positive constant $\kappa_0 > 0$ the curvature satisfies $\kappa(\mathbf{z}) \le \kappa_0$ at all boundary points $\mathbf{z} \in \Gamma$. Then to each boundary points $\mathbf{z} \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{z} \in \partial D_R$, and $D_R \subset K$.*

Note that the result, although seemingly local, does not allow for extensions to non-convex curves Γ. One can draw pictures of leg-bone like shapes of arbitrarily small upper bound of (positive) curvature, while at some points of touching containing arbitrarily small disks only. The reason is that the curve, after starting off from a certain boundary point $\mathbf{x}$, and then leaning back a bit, can eventually return arbitrarily close to the point from where it started: hence a prescribed size of disk cannot be inscribed.

On the other hand the Blaschke Theorem extends to any dimension $d \in \mathbb{N}$.

Also, the result has a similar, dual version, too, see [1, p. 116]. This was formulated already in Lemma **1**.3 above.

Now we start with introducing a few notions and recalling auxiliary facts. In §**4** we formulate and prove the two basic results – the discrete forms of the Blaschke Theorems. Then we show how our discrete approach yields a new, straightforward proof for a more involved sharpening of Theorem **3**.1, originally due to Strantzen.

Recall that the term planar *convex body* stands for a compact, convex subset of $\mathbb{C} \cong \mathbb{R}^2$ having nonempty interior. For a (planar) convex body K any interior point z defines a parametrization $\gamma(\varphi)$ – the usual polar coordinate representation of the boundary ∂K, – taking the unique point $\{z + te^{i\varphi} : t \in (0,\infty)\} \cap \partial K$ for the definition of $\gamma(\varphi)$. This defines the closed Jordan curve $\Gamma = \partial K$ and its parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$. By convexity, from any boundary point $\zeta = \gamma(\theta) \in \partial K$, locally the chords to boundary points with parameter $< \theta$ or with $> \theta$ have arguments below and above the argument of the direction of any supporting line at ζ. Thus the tangent direction or argument function $\alpha_-(\theta)$

can be defined as e.g. the supremum of arguments of chords from the left; similarly, $\alpha_+(\theta) := \inf\{\arg(z-\zeta) \ : \ z=\gamma(\varphi), \ \varphi > \theta\}$, and any line $\zeta + e^{i\beta}\mathbb{R}$ with $\alpha_-(\theta) \le \beta \le \alpha_+(\theta)$ is a supporting line to K at $\zeta = \gamma(\theta) \in \partial K$. In particular the curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with the unique value of $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$. It is clear that interpreting $\alpha_\pm$ as functions on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function. In other words, we are allowed to change parameterizations to arc length, say, when in case of $|\Gamma| = \ell$ ($|\Gamma|$ meaning the length of $\Gamma := \partial K$) the functions $\alpha_\pm$ map $[0, \ell]$ to $[0, 2\pi]$.

Observe that $\alpha_\pm$ are nondecreasing functions with total variation $\mathrm{Var}\,[\alpha_\pm] = 2\pi$, and that they have a common value precisely at continuity points, which occur exactly at points where the supporting line to K is unique. At points of discontinuity $\alpha_\pm$ is the left-, resp. right continuous extension of the same function. For convenience, and for better matching with [3], we may even define the function $\alpha := (\alpha_+ + \alpha_-)/2$ all over the parameter interval.

For obvious geometric reasons we call the jump function $\beta := \alpha_+ - \alpha_-$ the *supplementary angle* function. In fact, β and the usual Lebesgue decomposition of the nondecreasing function α_+ to $\alpha_+ = \sigma + \alpha_* + \alpha_0$, consisting of the pure jump function σ, the nondecreasing singular component α_*, and the absolute continuous part α_0, are closely related. By monotonicity there are at most countable many points where $\beta(x) > 0$, and in view of bounded variation we even have $\sum_x \beta(x) \le 2\pi$, hence the definition $\mu := \sum_x \beta(x)\delta_x$ defines a bounded, non-negative Borel measure on $[0, 2\pi)$. Now it is clear that $\sigma(x) = \mu([0,x])$, while $\alpha_*' = 0$ a.e., and α_0 is absolutely continuous. In particular, α or α_+ is differentiable at x provided that $\beta(x) = 0$ and x is not in the exceptional set of non-differentiable points with respect to α_* or α_0. That is, we have differentiability almost everywhere, and

$$\begin{aligned}
&\int_x^y \alpha' = \alpha_0(y) - \alpha_0(x) = \lim_{z\to x-0} \alpha_0(y) - \alpha_0(z) \\
&= \lim_{z\to x-0} \{[\alpha_+(y) - \sigma(y) - \alpha_*(y))] - [\alpha_+(z) - \sigma(z) - \alpha_*(z)]\} \\
&= \alpha_+(y) - \beta(y) - \mu([x,y)) - \lim_{z\to x-0} \alpha_+(z) - \lim_{z\to x-0} [\alpha_*(y) - \alpha_*(z)] \le \alpha_-(y) - \alpha_+(x).
\end{aligned} \tag{3.1}$$

It follows that

$$\alpha'(t) \ge \lambda \qquad \text{a.e.} \quad t \in [0,a] \tag{3.2}$$

holds true if and only if we have

$$\alpha_\pm(y) - \alpha_\pm(x) \ge \lambda(y-x) \qquad \forall x, y \in [0,a]\ . \tag{3.3}$$

Here we restricted ourselves to the arc length parametrization taken in positive orientation. Recall that one of the most important geometric quantities, curvature, is just $\kappa(s) := \alpha'(s)$, whenever parametrization is by arc length s.

Thus we can rewrite (3.2) as

$$\kappa(t) \geq \lambda \qquad \text{a.e.} \quad t \in [0,a] \ , \tag{3.4}$$

or, with radius of curvature $\rho(t) := 1/\kappa(t)$ introduced (writing $1/0 = \infty$),

$$\rho(t) \leq \frac{1}{\lambda} \qquad \text{a.e.} \quad t \in [0,a] \ . \tag{3.5}$$

Again, ρ is a parametrization-invariant quantity (describing the radius of the osculating circle). Actually, it is easy to translate all these conditions to arbitrary parametrization of the tangent angle function α. Since also curvature and radius of curvature are parametrization-invariant quantities, all the above hold for any parametrization.

Moreover, with a general parametrization let $|\Gamma(\eta,\zeta)|$ stand for the length of the counterclockwise arc $\Gamma(\eta,\zeta)$ of the rectifiable Jordan curve Γ between the two points $\zeta,\eta \in \Gamma = \partial K$. We can then say that the curve satisfies a Lipschitz-type increase or *subdifferential condition* whenever

$$|\alpha_{\pm}(\eta) - \alpha_{\pm}(\zeta)| \geq \lambda |\Gamma(\eta,\zeta)| \qquad (\forall \zeta, \eta \in \Gamma) \ , \tag{3.6}$$

here meaning by $\alpha_{\pm}(\xi)$, for $\xi \in \Gamma$, not values in $[0,2\pi)$, but a locally monotonously increasing branch of $\alpha_{\pm}$, with jumps in $(0,\pi)$, along the counterclockwise arc $\Gamma(\eta,\zeta)$ of Γ. Clearly, the above considerations show that all the above are equivalent.

In the paper we use the notation α (and also $\alpha_{\pm}$) for the tangent angle, κ for the curvature, and ρ for the radius of curvature. The counterclockwise taken right hand side tangent unit vector(s) will be denoted by $\mathbf{t}$, and the outer unit normal vectors by $\mathbf{n}$. These notations we will use basically in function of the arc length parametrization s, but with a slight abuse of notation also $\alpha_{-}(\varphi)$, $\mathbf{t}(\mathbf{x})$, $\mathbf{n}(\mathbf{x})$ etc. may occur with the obvious meaning.

Note that $\mathbf{t}(\mathbf{x}) = i\mathbf{n}(\mathbf{x}))$ and also $\mathbf{t}(\mathbf{x}) = \dot{\gamma}(s)$ when $\mathbf{x} = \mathbf{x}(s) \in \gamma$ and the parametrization/differentiation, symbolized by the dot, is with respect to arc length; moreover, with $\nu(s) : \arg(\mathbf{n}(\mathbf{x}(s))$ we obviously have $\alpha \equiv \nu + \pi/2 \mod 2\pi$ at least at points of continuity of α and ν. To avoid mod 2π equality, we can shift to the universal covering spaces and maps and consider $\widetilde{\alpha}, \widetilde{\nu}$, i.e. $\widetilde{\mathbf{t}}, \widetilde{\mathbf{n}}$ – e.g. in case of $\widetilde{\mathbf{n}}$ we will somewhat detail this right below. However, note a slight difference in handling α and $\widetilde{\mathbf{n}}$: the first is taken as a singlevalued function, with values $\alpha(s) := \frac{1}{2}\{\alpha_{-}(s) + \alpha_{+}(s)\}$ at points of discontinuity, while $\widetilde{\mathbf{n}}$ is a multivalued function attaining a full closed interval $[\widetilde{\mathbf{n}}_{-}(s), \widetilde{\mathbf{n}}_{+}(s)]$ whenever s is a point of discontinuity. Also recall that curvature, whenever it exists, is $|\ddot{\gamma}(s)| = \alpha'(s) = \widetilde{\mathbf{n}}'(s)$.

In this work we mean by a multi-valued function Φ from X to Y a (non-empty-valued) mapping $\Phi : X \to 2^Y \setminus \{\emptyset\}$, i.e. we assume that the domain of Φ is always the whole of X and that $\emptyset \neq \Phi(x) \subset Y$ for all $x \in X$. Recall the notions of modulus of continuity and minimal oscillation in the full generality of multi-valued functions between metric spaces.

Definition 3.1 (**modulus of continuity and minimal oscillation**). Let (X, d_X) and (Y, d_Y) be metric spaces. We call the *modulus of continuity* of the multivalued function Φ from X to Y the quantity

$$\omega(\Phi,\tau) := \sup\{d_Y(y,y') \ : \ x, x' \in X,\ d_X(x,x') \le \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

Similarly, we call *minimal oscillation* of Φ the quantity

$$\Omega(\Phi,\tau) := \inf\{d_Y(y,y') \ : \ x, x' \in X,\ d_X(x,x') \ge \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

If we are given a multi-valued *unit vector function* $\mathbf{v}(\mathbf{x}) : H \to 2^{S^{d-1}} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^d$ and S^{d-1} is the unit ball of $\mathbb{R}^d$, then the derived formulae become:

$$\omega(\tau){:=}\omega(\mathbf{v},\tau){:=}\sup\{\arccos\langle \mathbf{u},\mathbf{w}\rangle :\ \mathbf{x},\mathbf{y} \in H,\ |\mathbf{x}–\mathbf{y}| \le \tau,\ \mathbf{u}{\in} \,|!\mathbf{v}(\mathbf{x}), \mathbf{w}{\in}\mathbf{v}(\mathbf{y})\}, \tag{3.7}$$

and

$$\Omega(\tau){:=}\Omega(\mathbf{v},\tau){:=}\inf\{\arccos\langle \mathbf{u},\mathbf{w}\rangle : \mathbf{x},\mathbf{y} \in H,\ |\mathbf{x}-\mathbf{y}| \ge \tau,\ \mathbf{u}{\in}\mathbf{v}(\mathbf{x}),\ \mathbf{w}{\in}\mathbf{v}(\mathbf{y})\}. \tag{3.8}$$

For a *planar* multi-valued unit vector function $\mathbf{v} : H \to 2^{S^1} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^2 \simeq \mathbb{C}$ and S^1 is the unit circle in $\mathbb{R}^2$, we can parameterize the unit circle S^1 by the corresponding angle φ and thus write $\mathbf{v}(\mathbf{x}) = e^{i\Phi(\mathbf{x})}$ with $\Phi(\mathbf{x}) := \arg(\mathbf{v}(\mathbf{x}))$ being the corresponding angle. We will somewhat elaborate on this observation in the case when our multi-valued vector function is the outward normal vector(s) function $\mathbf{n}(\mathbf{x})$ of a closed convex curve.

Let γ be the boundary curve of a convex body in $\mathbb{R}^2$, which will be considered as oriented counterclockwise, and let the multivalued function $\mathbf{n}(\mathbf{x}) : \gamma \to 2^{S^1} \setminus \{\emptyset\}$ be defined as the set of all outward unit normal vectors of γ at the point $\mathbf{x} \in \gamma$. Observe that the set $\mathbf{n}(\mathbf{x})$ of the set of values of $\mathbf{n}$ at any $\mathbf{x} \in \gamma$ is either a point, or a closed segment of length less than π. Then there exists a unique lifting $\tilde{\mathbf{n}}$ of $\mathbf{n}$ from the universal covering space $\tilde{\gamma}(\simeq \mathbb{R}$, see below) of γ to the universal covering space $\mathbb{R} = \tilde{S^1}$ of S^1, with the respective universal covering maps $\pi_\gamma : \tilde{\gamma} \to \gamma$ and $\pi_{S^1} : \tilde{S^1} \to S^1$, with properties to be described below. Here we do not want to recall the concept of the universal covering spaces from algebraic topology in its generality, but restrict ourselves to give it in the situation described above. As already said, $\tilde{S^1} = \mathbb{R}$ and the corresponding universal covering map is $\pi_{S^1} : x \to (\cos x, \sin x)$ (We consider, as usual, S^1 as $\mathbb{R} \mod 2\pi$.) Similarly, for γ we have $\tilde{\gamma} = \mathbb{R}$, with universal covering map $\pi_\gamma : \mathbb{R} \to \gamma$ given in the following way. Let us fix some arbitrary point $\mathbf{x_0} \in \gamma$, (the following considerations will be independent of $\mathbf{x_0}$, in the natural sense). Let us denote by ℓ the length of γ. Then for $\lambda \in \mathbb{R} = \tilde{\gamma}$ we have that $\pi_\gamma(\lambda) \in \gamma$

is that unique point $\mathbf{x}$ of γ, for which the counterclockwise measured arc $\mathbf{x_0x}$ has a length $\lambda \mod \ell$.

Now we describe the postulates for the multivalued function $\tilde{\mathbf{n}} : \mathbb{R} = \tilde{\gamma} \to \tilde{S^1} = \mathbb{R}$, which determine it uniquely. First of all, we must have the equality $\pi_{S^1} \circ \tilde{\mathbf{n}} = \mathbf{n} \circ \pi_\gamma$, where $\circ$ denotes the composition of two multivalued functions. (In algebraic topology this is called *commutativity of a certain square of mappings.*) Second, the values of $\tilde{\mathbf{n}}$ must be either points or non-degenerate closed intervals (of length less than π; however this last property follows from the other ones). Third, $\tilde{\mathbf{n}}$ must be non-decreasing in the following sense: for $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 < \lambda_2$ we have $r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2) \implies r_1 \le r_2$. Further, $\tilde{\mathbf{n}}$ must be a non-decreasing multivalued function, continuous from the left, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda-\delta,\lambda)} \tilde{\mathbf{n}}(\mu) \subset (\min \tilde{\mathbf{n}}(\lambda) - \varepsilon, \min \tilde{\mathbf{n}}(\lambda))$. Analogously, $\tilde{\mathbf{n}}$ must be a non-decreasing multi-valued function continuous from the right, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda,\lambda+\delta)} \tilde{\mathbf{n}}(\mu) \subset (\max \tilde{\mathbf{n}}(\lambda), \max \tilde{\mathbf{n}}(\lambda) + \varepsilon)$. These are all the postulates for the multi-valued function $\tilde{\mathbf{n}}$. It is clear, that $\tilde{\mathbf{n}}$ exists and is uniquely determined, for fixed $\mathbf{x_0}$ (and, for $\mathbf{x_0}$ arbitrary, only the parametrization of $\mathbb{R} = \tilde{\gamma}$ changes, by a translation.)

The above listed properties imply still one important property of the multi-valued function $\tilde{\mathbf{n}}$: we have for any $\lambda \in \mathbb{R}$ that $\tilde{\mathbf{n}}(\lambda + \ell) = \tilde{\mathbf{n}}(\lambda) + 2\pi$.

Definition 3.2. We define the modulus of continuity of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) modulus of continuity of the multi-valued lift-up function $\tilde{\mathbf{n}} : \mathbb{R} \to \mathbb{R} \setminus \{\emptyset\}$, i.e. as

$$\begin{aligned}\tilde{\omega}(\tau) &:= \tilde{\omega}(\mathbf{n}, \tau) := \omega(\tilde{\mathbf{n}}, \tau) \\ &:= \sup\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \le \tau\}.\end{aligned} \tag{3.9}$$

Similarly, we define the minimal oscillation of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) minimal oscillation function of $\tilde{\mathbf{n}}$, i.e. as

$$\begin{aligned}\tilde{\Omega}(\tau) &:= \tilde{\Omega}(\mathbf{n}, \tau) := \Omega(\tilde{\mathbf{n}}, \tau) \\ &:= \inf\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \ge \tau\}.\end{aligned} \tag{3.10}$$

By writing "modulus of continuity" we do not mean to say anything like continuity of $\tilde{\mathbf{n}}$. In fact, if for some $\lambda \in \mathbb{R}$ $\tilde{\mathbf{n}}(\lambda)$ is a non-degenerate closed segment, then the left-hand side and right-hand side limits of $\tilde{\mathbf{n}}$ at λ - in the sense of the definition of continuity from the left or right, respectively - are surely different.

We evidently have that the modulus of continuity of $\tilde{\mathbf{n}}$ is subadditive, meaning $\tilde{\omega}(\tau_1 + \tau_2) \leq \tilde{\omega}(\tau_1) + \tilde{\omega}(\tau_2)$, and similarly, that the minimal oscillation of $\tilde{\mathbf{n}}$ is superadditive, meaning $\tilde{\Omega}(\tau_1 + \tau_2) \geq \tilde{\Omega}(\tau_1) + \tilde{\Omega}(\tau_2)$. In fact, a standard property of the modulus of continuity of *any (non-empty valued) multivalued function from* $\mathbb{R}$ *(or from any convex set, in the sense of metric intervals) to* $\mathbb{R}$ is subadditivity, and similarly, minimal oscillation of such a function is superadditive. These properties with non-negativity and non-decreasing property also imply that $\tilde{\omega}(\tau)/\tau$ and $\tilde{\Omega}(\tau)/\tau$ have limits when $\tau \to 0$; moreover, $\lim_{\tau\to 0} \tilde{\omega}(\tau)/\tau = \sup \tilde{\omega}(\tau)/\tau$ and $\lim_{\tau \to 0} \tilde{\Omega}(\tau)/\tau = \inf \tilde{\Omega}(\tau)/\tau$. Note that metric convexity is essential here, so e.g. it is not clear if in $\mathbb{R}^d$ any proper analogy could be established.

Observe that if the curvature of γ exists at $\mathbf{x}_0$, then for the non-empty valued multi-valued function $\mathbf{n}(\mathbf{x})$:="set of values of all outer unit normal vectors of γ at $\mathbf{x}$", we necessarily have $\#\mathbf{n}(\mathbf{x}_0) = 1$ and the curvature can be written as

$$\kappa(\mathbf{x}_0) = \lim_{\mathbf{y}\to\mathbf{x}_0\ \mathbf{v}\in\mathbf{n}(\mathbf{y})} \frac{\arccos\langle \mathbf{n}(\mathbf{x}_0), \mathbf{v}\rangle}{|\mathbf{x}_0 - \mathbf{y}|}, \tag{3.11}$$

where the limit in (3.11) exists with arbitrary choice of $\mathbf{v} \in \mathbf{y}$ and is independent of this choice.

The next two propositions are well-known. We, however, detail their proof in [12] and also in [17] for self-contained presentation, which we do not aim at here.

Proposition 3.1. *Let γ be a planar convex curve. Recall that* (3.7) *and* (3.8) *is the modulus of continuity and the minimal oscillation of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ with respect to chord length, and that* (3.9) *and* (3.10) *stand for the modulus of continuity and the minimal oscillation of $\mathbf{n}(\mathbf{x})$ with respect to arc length. Then for all $\mathbf{x} \in \gamma$ with curvature $\kappa(\mathbf{x}) \in [0, \infty]$ we have*

$$\lim_{\tau\to 0} \frac{\Omega(\tau)}{\tau} = \lim_{\tau\to 0} \frac{\tilde{\Omega}(\tau)}{\tau} \leq \kappa(\mathbf{x}) \leq \lim_{\tau\to 0} \frac{\tilde{\omega}(\tau)}{\tau} = \lim_{\tau\to 0} \frac{\omega(\tau)}{\tau}. \tag{3.12}$$

In the following proposition arccos will denote the branch with values in $[0, \pi]$.

Proposition 3.2. *Let γ be a closed convex curve, and* (3.7) *and* (3.8) *be the modulus of continuity and the minimal oscillation of the (in general, multi-valued) unit normal vector function $\mathbf{n}(\mathbf{x})$.*

(i) If the curvature exists and is bounded from above by κ_0 all over γ, then there exists a bound $\tau_0 > 0$ so that for any two points $\mathbf{x}, \mathbf{y} \in \gamma$ with $|\mathbf{x} - \mathbf{y}| \leq$

$\tau \le \tau_0$ we must have $\omega(\mathbf{n},\tau) < \pi/2$ and $\arccos\langle \mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y})\rangle \le \kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$. Thus we also have $\omega(\mathbf{n},\tau) \le \kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$ for $\tau \le \tau_0$.

(ii) If the curvature $\kappa(\mathbf{x})$ exists (linearly, that is, according to arc length parametrization) almost everywhere, and is bounded from below by κ_0 (linearly) almost everywhere on γ, then for any two points $\mathbf{x}, \mathbf{y} \in \gamma$ with $|\mathbf{x}-\mathbf{y}| \ge \tau$ and for all $\mathbf{u} \in \mathbf{n}(\mathbf{x})$, $\mathbf{v} \in \mathbf{n}(\mathbf{y})$ we have $\arccos\langle \mathbf{u}, \mathbf{v}\rangle \ge \kappa_0\tau$ and hence $\Omega(\mathbf{n},\tau) \ge \kappa_0\tau$.

Rotations of $\mathbb{C} = \mathbb{R}^2$ about the origin O by the counterclockwise measured (positive) angle φ will be denoted by U_φ, that is,

$$U_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.13}$$

We denote T the reflection to the y-axis, i.e. the linear mapping defined by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 3.3 (**Mangled n-gons**). Let $2 \le k \in \mathbb{N}$ and put $n = 4k-4$, $\varphi^* := \frac{\pi}{2k}$. We define the *standard mangled n-gon* as the convex n-gon

$$M_k := \mathrm{con}\,\{A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_{2k-1}, A_{2k+1}, \dots, A_{3k-1}, A_{3k+1}, \dots, A_{4k-1}\}, \tag{3.14}$$

of $n = 4k-4$ vertices with

$$A_m := \left(\sum_{j=1}^{m} \cos(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \cos(\ell k \varphi^*), \sum_{j=1}^{m} \sin(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \sin(\ell k \varphi^*) \right), \tag{3.15}$$

where $m \in \{1, \dots, 4k\} \setminus \{k, 2k, 3k, 4k\}$. That is, we consider a regular $4k$-gon of unit sides, but cut out the middle "cross-shape" (i.e., the union of two rectangles which are the convex hulls of two opposite sides of the regular $4k$-gon, these pairs of opposite sides being perpendicular to each other) and push together the left over four quadrants (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k-1}$ consecutively to join the remaining sides of the polygon. Observe that taking $A_0 := O$, the same formula (3.15) is valid also for $A_0 := O = A_{4k} = A_{4k-1}$ and $A_{\ell k} = A_{\ell k-1}$, $\ell = 1,2,3,4$, showing how the vertices of the regular $4k$-gon were moved into their new positions.)

Now let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0, \pi/4]$ be arbitrary. Take $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$, so that $\varphi^* := \frac{\pi}{2k} \ge \varphi$.

Then we write $M(\varphi) := M_k$, and, moreover, we also define

$$M(\mathbf{x}, \alpha, \varphi, \tau) := M(\mathbf{x}, \alpha, \varphi^*, \tau) := U_\alpha(\tau M_k) + \mathbf{x}, \tag{3.16}$$

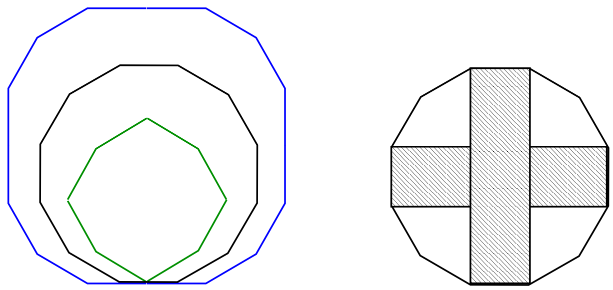

FIGURE 4. Left: The regular, mangled and fattened n-gons with $n = 12$. For $n \to \infty$ the sizes are all closer and closer to each other and the same circle. Right: The shaded "middle cross" of the regular n-gon is either cut out (in the mangled) or doubled (in the fattened) n-gon.

that is, the copy shifted by $\mathbf{x}$ of the $4k-4$-gon obtained by dilating $M(\varphi) = M_k$ from $O = A_0 = A_{4k-1}$ with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \in (\pi/6, \pi/4]$, then $k = 2$, $\varphi^* = \pi/4$, $n = 4$, and M_2 is just a unit square, its side lines having direction tangents ± 1 and having its lowest vertex at O. It is the left over part, pushed together, of a regular octagon of unit side length, when the middle cross-shape is removed from its middle.

It is easy to see that the inradius $\rho(\varphi)$ and the circumradius $R(\varphi)$ of $M(\varphi) = M(\varphi^*) = M_k$ are

$$\begin{cases} r(\varphi) &= \frac{1}{2}\left\{\cot\frac{\pi}{4k} - \sqrt{2}\cos\left(\frac{1-(-1)^k}{8k}\pi\right)\right\}, \\ R(\varphi) &= \frac{1}{2}\left\{\cot\frac{\pi}{4k} - 1\right\}, \end{cases} \qquad \left(k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor\right), \tag{3.17}$$

respectively.

Similarly to the *mangled n-gons* M_k, we also define the *fattened n-gons* F_k.

Definition 3.4 (**Fattened n-gons**). Let $k \in \mathbb{N}$ and put $n = 4k$, $\varphi^* := \frac{\pi}{2k}$. We first define the *standard fattened n-gon* as the convex n-gon

$$F_k := \operatorname{con}\{A_1, \ldots, A_{k-1}, A_k, A_{k+1}, \ldots, A_{4k-1}, A_{4k}\}, \tag{3.18}$$

of $n = 4k$ vertices with

$$A_m := \left(\sum_{j=1}^{m}\cos(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor}\cos(\ell k\varphi^*), \sum_{j=1}^{m}\sin(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor}\sin(\ell k\varphi^*)\right). \tag{3.19}$$

That is, we consider a regular $4k$-gon , but fatten the middle "cross-shape" to twice as wide, and move the four quadrants to the corners formed by this width-doubled cross (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k-1}+2(A_{\ell k}-A_{\ell k-1})$ consecutively to join the remaining sides of the polygon). Observe that $A_{4k-1} = (-1,0)$ and $A_{4k}=(1,0)$.

Let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0,\pi)$ be arbitrary. Now we take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$, whence $\varphi^* := \frac{\pi}{2k} \leq \varphi$.

Then we write $F(\varphi) := F_k$, and, moreover, we also define

$$F(\mathbf{x},\alpha,\varphi,\tau) := F(\mathbf{x},\alpha,\varphi^*,\tau) := U_\alpha\left(\tau F_k\right) + \mathbf{x}, \tag{3.20}$$

that is, the copy shifted by $\mathbf{x}$ of the $4k$-gon obtained by dilating $F(\varphi) = F_k$ from O with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \geq \pi/2$, then $k=1$, $\varphi^* = \pi/2$, $n=4$, and F_4 is just the square spanned by the vertices (1,0), (1,2), (-1,2), (-1,0) and having sides of length 2.

Observe that using the usual Minkowski addition, we can represent the connections of these deformed n-gons and the regular n-gon easily. Write Q_n for the regular n-gon placed symmetrically to the y-axis but above the x-axis with $O \in \partial Q_n$ a midpoint (hence not a vertex) of a side of Q_n. (This position is uniquely determined.) Also, denote the standard square as $S := Q_4 := \operatorname{con}\{(1/2,0);(1/2,1);(-1/2,1);(-1/2,0)\}$. Then we have $M_k + S = Q_{4k}$ and $Q_{4k} + S = F_k$.

It is also easy to see that the inradius $\mathfrak{r}(\varphi)$ and the circumradius $\mathfrak{R}(\varphi)$ of $F(\varphi) = F(\varphi^*)$ are

$$\mathfrak{r}(\varphi) = \frac{1}{2}\cot\frac{\pi}{4k} + \frac{1}{2} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{3.21}$$

and

$$\mathfrak{R}(\varphi) = \begin{cases} \frac{1}{2\sin\frac{\pi}{4k}} + \frac{1}{\sqrt{2}} & \text{if } 2 \mid k \\ \sqrt{\frac{1}{2} + \frac{1}{4\sin^2\frac{\pi}{4k}} + \frac{1}{\sqrt{2}}\cot\frac{\pi}{4k}} & \text{if } 2 \mid k \end{cases} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{3.22}$$

respectively.

The actual values of the above in- and circumradii in (3.17), (3.21), (3.22) are not important, but observe that for $\varphi \to 0$, or, equivalently, for $k \to \infty$, we have the asymptotic relation $r(\varphi) \sim R(\varphi) \sim \mathfrak{r}(\varphi) \sim \mathfrak{R}(\varphi) \sim \frac{1}{\varphi}$.

4. Discrete versions of the Blaschke Rolling Ball Theorems

Our further results will all be derived from various extensions and strengthening of the Blaschke Theorem. In this section we skip the tedious, elaborate proofs, to be found in [12] and also in [17]. However, we list the geometry results, which may be of independent interest. At least we know of other useful

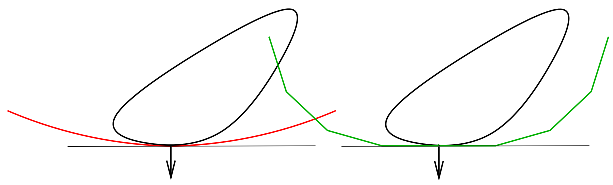

FIGURE 5. Outscribed circles and n-gons at a boundary point as provided by the smooth and discrete dual Blaschke Theorems.

applications of them in analytic problems, so it may prove to be useful elsewhere for others, too.

Theorem 4.1. *Let $K \subset \mathbb{C}$ be a convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\omega(\mathbf{n}, \tau) \le \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

An even stronger version can be proved considering the modulus of continuity $\tilde{\omega}$ with respect to arc length. We thank this sharpening to Endre Makai, who kindly called our attention to this possibility and suggested the crucial Lemma for the proof.

Theorem 4.2. *Let $K \subset \mathbb{C}$ be a planar convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\tilde{\omega}(\tau) \le \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

Finally, as in case of the classical Blaschke theorems, there is a dual version of all these considerations. The result is this.

Theorem 4.3. *Let $K \subset \mathbb{C}$ be a (planar) convex body and $\tau > 0$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\Omega(\mathbf{n}, \tau) \ge \varphi$. Take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is normal to γ at $\mathbf{x}$, then $F(\mathbf{x}, \alpha, \varphi, \tau) \supset K$.*

This is the version we actually make use of in this work, see Figure **4**.

5. Extensions of the Blaschke Rolling Ball Theorem

As the first corollaries, we can immediately deduce the classical Blaschke theorems. We denote by $D(\mathbf{x}, r)$ the closed disc of center $\mathbf{x}$ and radius r.

Proof of Theorem 3.1. Let τ_0 be the bound provided by (i) of Proposition **3**.2. Under the condition, we find (with $\omega(\mathbf{n}, \tau) < \pi/2$)

$$\omega(\mathbf{n}, \tau) \leq \frac{\kappa_0 \tau}{\cos(\omega(\mathbf{n}, \tau))} =: \varphi(\tau) \qquad (\tau \leq \tau_0). \tag{5.1}$$

Let us apply Theorem **4**.1 for the boundary point $\mathbf{x} \in \gamma$ with normal vector $\mathbf{n}(\mathbf{x}) = (\sin\alpha, -\cos\alpha)$. If necessary, we have to reduce τ so that the hypothesis $\varphi(\tau) \leq \pi/4$ should hold. We obtain that the congruent copy $U_\alpha(\tau M_k) + \mathbf{x}$ of τM_k is contained in K, where $k = \lfloor \pi/2\varphi(\tau) \rfloor$. Note that $U_\alpha(\tau M_k) + \mathbf{x} \supset D(\mathbf{z}, \tau r(\varphi(\tau)))$, where $\mathbf{z} = \mathbf{x} - \tau R(\varphi(\tau))\mathbf{n}(\mathbf{x})$. When $\tau \to 0$, also $\varphi(\tau) \to 0$, therefore also $\omega(\mathbf{n}, \tau) \to 0$ in view of (5.1), and we see

$$\lim_{\tau \to 0} (\tau R(\varphi(\tau))) = \lim_{\tau \to 0} (\tau r(\varphi(\tau))) = \lim_{\tau \to 0} \frac{\tau}{\varphi(\tau)} = \lim_{\tau \to 0} \frac{\cos(\omega(\mathbf{n}, \tau))}{\kappa_0} = \frac{1}{\kappa_0}.$$

Note that we have made use of $\omega(\mathbf{n}, \tau) \to 0$ in the form $\cos(\omega(\mathbf{n}, \tau)) \to 1$. It follows that $D(\mathbf{x} - \frac{1}{\kappa_0}\mathbf{n}(\mathbf{x}), \frac{1}{\kappa_0}) \subset K$, whence the assertion. $\square$

Note that in the above proof of Theorem **3**.1 we did not assume C^2-boundary, as is usual, but only the existence of curvature and the estimate $\kappa(\mathbf{x}) \leq \kappa_0$. So we found the following stronger corollary (still surely well-known).

Corollary 5.1. *Assume that $K \subset \mathbb{R}^2$ is a convex domain with boundary curve γ, that the curvature κ exists all over γ, and that there exists a positive constant $\kappa_0 > 0$ so that $\kappa \leq \kappa_0$ everywhere on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $D_R \subset K$.*

Similarly, one can deduce also the "dual" Blaschke theorem, i.e. Lemma**1**.3, in a similarly strengthened form. In fact, the conditions can be relaxed even further, as was shown by Strantzen, see [3, Lemma 9.11]. Our discrete approach easily implies Strantzen's strengthened version, originally obtained along different lines.

Corollary 5.2 (**Strantzen**). *Let $K \subset \mathbb{R}^2$ be a convex body with boundary curve γ. Assume that the (linearly) a.e. existing curvature κ of γ satisfies $\kappa \geq \kappa_0$ (linearly) a.e. on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $K \subset D_R$.*

Proof. Now we start with (ii) of Proposition **3**.2 to obtain $\Omega(\tau) \geq \kappa_0 \tau$ for all τ. Put $\varphi := \varphi(\tau) := \kappa_0 \tau$. Clearly, when $\tau \to 0$, then also $\varphi(\tau) \to 0$ and $k := \lceil \pi/(2\varphi(\tau)) \rceil \to \infty$. Take $\mathbf{n}(\mathbf{x}) = (\cos\alpha, \sin\alpha)$ and apply Theorem **4**.3 to obtain $U_\alpha(\tau F_k) + \mathbf{x} \supset K$ for all $\tau > 0$. Observe that $D_\varphi := D((0, \mathfrak{r}(\varphi)), \mathfrak{R}(\varphi)) \supset F_k$, hence $U_\alpha(\tau D_\varphi) + \mathbf{x} \supset K$. In the limit, since $\mathfrak{r}(\varphi(\tau)) \sim \mathfrak{R}(\varphi(\tau)) \sim 1/(\varphi(\tau)) = 1/(\kappa_0 \tau)$, we find $D(\mathbf{x} - (1/\kappa_0)\mathbf{n}, 1/\kappa_0) \supset K$, for any $\mathbf{n} \in \mathbf{n}(\mathbf{x})$, that implies the statement. □

6. Further results for non-flat convex domains

The above Theorem **1**.3 was formulated with very precise constants. In particular, it gives a good description of the "inverse Markov factor"

$$M(E_b) := \inf_{p \in \mathcal{P}_n(E_b)} M(p),$$

when n is fixed and $b \to 0$. In this section we aim at a precise generalization of Theorem **1**.3 using appropriate geometric notions. Our argument stems out of the notion of "circular sets", used in [8] and going back to Turán's work. This approach can indeed cover the full content of Theorem **1**.3. Moreover, the geometric observation and criteria we present will cover a good deal of different, not necessarily smooth domains. First let us have a recourse to Theorem **1**.5.

Theorem 6.1. *Let $K \subset \mathbb{C}$ be any convex domain with C^2-smooth boundary curve $\partial K = \Gamma$ having curvature $\kappa(\zeta) \geq \kappa$ with a certain constant $\kappa > 0$ and for all points $\zeta \in \Gamma$. Then $M(K) \geq (\kappa/2)n$.*

Proof. The proof hinges upon geometry in a large extent. For this smooth case we use Blaschke's Rolling Ball Theorem, i.e. Lemma **1**.3. This means, with our definition above, that if the curvature of the boundary curve of a twice differentiable convex body exceeds $1/R$, then the convex body is R-circular. From this an application of Theorem **1**.4 yields the assertion. □

So now it is worthy to calculate the curvature of ∂E_b.

Lemma 6.1. *Let E_b be the ellipse with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Consider its boundary curve Γ_b. Then at any point of the curve the curvature is between b and $1/b^2$.*

Proof. Now we depart from arc length parameterization and use for $\Gamma_b := \partial E_b$ the parameterization $\gamma(\varphi) := (\cos(\varphi), b\sin(\varphi))$. Then we have

$$\kappa(\gamma(\varphi)) = \frac{|\dot{\gamma}(\varphi) \times \ddot{\gamma}(\varphi)|}{|\dot{\gamma}(\varphi)|^3} \ ,$$

that is,

$$\begin{aligned}\kappa(\gamma(\varphi)) &= \frac{|(-\sin\varphi, b\cos\varphi) \times (-\cos\varphi, -b\sin\varphi)|}{|(-\sin\varphi, b\cos\varphi)|^3} \\ &= \frac{b\sin^2\varphi + b\cos^2\varphi}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} \\ &= \frac{b}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} \ .\end{aligned}$$

Clearly, the denominator falls between $(b^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2} = b^3$ and $(\sin^2\varphi+\cos^2\varphi)^{3/2} = 1$, and these bounds are attained, hence $\kappa(\gamma(\varphi)) \in [b, 1/b^2]$ whenever $b \le 1$. □

Proof of Theorem 1.3. The curvature of Γ_b at any of its points is at least b according to Lemma **6**.1. Hence $M(E_b) \ge (b/2)n$ in view of Theorem **6**.1, and Theorem **1**.3 follows. □

However, not only smooth convex domains can be proved to be circular. E.g. it is easy to see that if a domain is the intersection of finitely many R-circular domains, then it is also R-circular. The next generalization is not that simple, but is still true.

Lemma 6.2 (**Strantzen**). *Let the convex domain K have boundary $\Gamma = \partial K$ with angle function $\alpha_\pm$ and let $\kappa > 0$ be a fixed constant. Assume that $\alpha_\pm$ satisfies the curvature condition $\kappa(s) = \alpha'(s) \ge \kappa$ almost everywhere. Then K is $R = 1/\kappa$-circular.*

Proof. This result is essentially the far-reaching, relatively recent generalization of Blaschke's Rolling Ball Theorem by Strantzen, i.e. Corollary **5**.2 above. The only slight alteration from the standard formulation in [3], suppressed in the above quotations, is that Strantzen's version assumes $\kappa(t) \ge \kappa$ wherever the curvature $\kappa(t) = \alpha'(t)$ exists (so almost everywhere for sure), while above we stated the same thing for almost everywhere, but not necessarily at every points of existence. This can be overcome by reference to the subdifferential version, too. □

Now we are in an easy position to prove Theorem **1**.11.

Proof of Theorem 1.11. The proof follows from a combination of Theorem **1**.4 and Lemma **6**.2. □

Let us illustrate the strengths and weaknesses of the above results on the following instructive examples, suggested to us by J. Szabados (personal communication). Consider for any $1 < p < \infty$ the ℓ_p unit ball

$$B^p := \{(x,y) : |x|^p + |y|^p \leq 1\}, \qquad \Gamma^p := \partial B^p = \{(x,y) : |x|^p + |y|^p = 1\}. \tag{6.1}$$

Also, let us consider for any parameter $0 < b \leq 1$ the affine image ("ℓ_p-ellipse")

$$B^p_b := \{(x,y) : |x|^p + |y/b|^p \leq 1\}, \qquad \Gamma^p_b := \partial B^p_b = \{(x,y) : |x|^p + |y/b|^p = 1\}. \tag{6.2}$$

By symmetry, it suffices to analyze the boundary curve $\Gamma := \Gamma^p_b$ in the positive quadrant. Here it has a parametrization $\Gamma(x) := (x, y(x))$, where $y(x) = b\,(1-x^p)^{1/p}$. As above, the curvature of the general point of the arc in the positive quadrant can be calculated and we get

$$\kappa(x) = \frac{(p-1)bx^{p-2}(1-x^p)^{1/p-2}}{\left(1 + b^2x^{2p-2}(1-x^p)^{2/p-2}\right)^{3/2}} \tag{6.3}$$

For $p > 2$, the curvature is continuous, but it does not stay off 0: e.g. at the upper point $x = 0$ it vanishes. Therefore, neither Theorem **6**.1 nor Theorem **1**.11 can provide any bound, while Theorem **1**.9 provides an estimate, even if with a small constant: here $d(B) = 2$, $w(B) = 2b$, and we get $M(B) \geq 0.00015bn$.

When $p = 2$, we get back the disk and the ellipses: the curvature is minimal at $\pm ib$, and its value is b there, hence $M(B) \geq (b/2)n$, as already seen in Theorem **1**.3. On the other hand Theorem **1**.9 yields only $M(B) \geq 0.00015bn$ also here.

For $1 < p < 2$ the situation changes: the curvature becomes infinite at the "vertices" at $\pm ib$ and ± 1, and the curvature has a positive minimum over the curve Γ. When $b = 1$, it is possible to explicitly calculate it, since the role of x and y is symmetric in this case and it is natural to conjecture that minimal curvature occurs at $y = x$; using geometric-arithmetic mean and also the inequality between power means (i.e. Cauchy-Schwartz), it is not hard to compute $\min \kappa(x,y) = (p-1)2^{1/p-1/2}$, (which is the value attained at $y = x$). Hence Theorem **1**.11 (but not Theorem **6**.1, which assumes C^2-smoothness, violated here at the vertices!) provides $M(B^p) \geq (p-1)2^{1/p-3/2}n$, while Theorem **1**.9 provides, in view of $w(B^p) = 2^{3/2-1/p}$, something like $M(B^p) \geq 0.0003\,2^{-1/2-1/p}n \geq 0.0001n$, which is much smaller until p comes down very close to 1.

For general $0 < b < 1$ we obviously have $d(B) = 2$, $(\sqrt{2}b <)2b/\sqrt{1+b^2} < w(B) < 2b$, and Theorem **1**.9 yields $M(B) \geq 0.0001bn$ independently of the value of p.

Now $\min \kappa$ can be estimated within a constant factor (actually, when $b \to 0$, even asymptotically precisely) the following way. On the one hand, taking $x_0 := 2^{-1/p}$ leads to $\kappa(x_0) = (p-1)b2^{1+1/p}/(1+b^2)^{3/2} < b(p-1)2^{1+1/p}$, hence

$\min \kappa(x < b(p-1)2^{1+/p}$. Note that when $b \to 0$, we have asymptotically $\kappa(x - 0) \sim b(p-1)2^{1+/p}$. On the other hand denoting $\xi := x^p$ and $\beta := 2/p - 1 \in (0,1)$, from (6.3) we get

$$\frac{(p-1)b}{\kappa(x)} = [\xi(1-\xi)]^\beta \left[\xi^{1-\beta} + b^2(1-\xi)^{1-\beta}\right]^{3/2}$$
$$\leq 2^{-2\beta} \left[(\xi + (1-\xi))^{1-\beta}(1 + (b^2)^{1/\beta})^\beta\right]^{3/2},$$

with an application of geometric-arithmetic mean inequality in the first and Hölder inequality in the second factor. In general we can just use $b < 1$ and get

$$\kappa(x) \geq (p-1)b2^{2\beta}\left[1 + b^{2/\beta}\right]^{-3\beta/2} \geq (p-1)b2^{\beta/2} = (p-1)b2^{1/p-1/2},$$

within a factor $2^{3/2}$ of the upper estimate for $\min \kappa$.

Therefore, inserting this into Theorem **1**.11 as above, we derive $M(B_b^p) \geq (p-1)b2^{1/p-3/2}n$.

In all, we see that Theorems **6**.1 (essentially due to Erőd) and **1**.11 usually (but not always, c.f. the case $p \approx 1$ above !)) give better constants, when they apply. However, in cases the curvature is not bounded away from 0, we can retreat to application to the fully general Theorem **1**.9, which, even if with a small absolute constant factor, but still gives a precise estimate even regarding dependence of the constant on geometric features of the convex domain. According to Theorem **1**.10, this latter phenomenon is not just an observation on some particular examples, but is a general fact, valid even for not necessarily convex domains.

7. Further remarks and open problems

In the case of the unit interval also Turán type L^p estimates were studied, see [23] and the references therein. It would be interesting to consider the analogous question for convex domains on the plane. Note that already Turán remarked, see the footnote in [21, p.141], that on D an L^p version holds, too. Also note that for domains there are two possibilities for taking integral norms, one being on the boundary curve and another one of integrating with respect to area. It seems that the latter is less appropriate and convenient here.

In the above we described a more or less satisfactory answer of the problem of inverse Markov factors for convex domains. However, Levenberg and Poletsky showed that star-shaped domains already do not admit similar inverse Markov factors. A question, posed by V. Totik, is to determine exact order of the inverse Markov factor for the "cross" $C := [-1,1] \cup [-i,i]$; clearly, the point is not in the answer for the cross itself, but in the description of the inverse Markov factor for some more general classes of sets.

Another question, still open, stems from the Szegő extension of the Markov inequality, see [20], to domains with sector condition on their boundary. More precisely, at $z \in \partial K$ K satisfies the *outer sector condition* with $0 < \beta < 2$, if there exists a small neighborhood of z where some sector $\{\zeta : \arg(\zeta - z) \in (\theta, \beta\pi + \theta)\}$ is disjoint from K. Szegő proved, that if for a domain K, bounded by finitely many smooth (analytic) Jordan arcs, the supremum of β-values satisfying outer sector conditions at some boundary point is $\alpha < 2$, then $\|P'\| \ll n^{\alpha}\|P\|$ on K. Then Turán writes: "Es ist sehr wahrscheinlich, daß auch den Szegőschen Bereichen $M(p) \geq cn^{1/\alpha}$...", that is, he finds it rather likely that the natural converse inequality, suggested by the known cases of the disk and the interval (and now also by any other convex domain) holds also for general domains with outer sector conditions.

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We are indebted to Gábor Halász for his generous contribution of an essential idea, explained at the beginning of §**2** above. For this contribution, in fact, we consider Theorem **1**.9 a joint result with him.

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