

Rendezvous numbers of metric spaces – a potential theoretic approach

By

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Abstract. The present work draws on the understanding how notions of general potential theory – as set up, e.g., by Fuglede – explain existence and some basic results on the “magical” rendezvous numbers. We aim at a fairly general description of rendezvous numbers in a metric space by using systematically the potential theoretic approach.

In particular, we generalize and explain results on invariant measures, hypermetric spaces and maximal energy measures, when showing how more general proofs can be found to them.

1. Introduction. It was proved by O. Gross that for a compact, connected metric space (X, d) there exists a unique number $r = r(X)$ such that for every finite point system $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$ one always finds an $x \in X$ with

$$(1.1) \quad r = \frac{1}{n} \sum_{i=1}^n d(x, x_i).$$

Such a number is called the *rendezvous number* of the metric space X . Since the first result of Gross [12], rendezvous numbers have been attracting much attention and been generalized in many directions: considering weak rendezvous numbers (Thomassen [20]), replacing the metric by some continuous symmetric function (Stadje [18]) or considering instead of the finite average in (1.1) the mean value with respect to some probability measure μ (Elton, Cleary, Morris, Yost [4]). In such abstract investigations various minimax principles play an important role. (See also Morris, Nickolas [14], Nickolas, Yost [15], Stranzen [19]).

Our aim is to put the investigations on the existence and uniqueness of rendezvous numbers in the framework of abstract potential theory, which has been around since the

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60s, but apparently has not gained its due recognition in this field. In this paper we continue [6] with the study of related notions such as invariant measures and maximal energy (Wolf [23], Björck [3]). It turns out that general principles such as existence of capacitary measures or Frostman's equilibrium theorem are accounted for the existence of invariant measures. In the past ten years or so, generalizations to possibly infinite dimensional, hence not locally compact spaces, appeared. In particular, calculation of rendezvous numbers of unit spheres in Banach spaces fascinated many authors. (See, e.g., Baronti, Casini, Papini [2], García-Vázquez, Villa [11], Lin [13], Wolf [21]).

First we spend some words on technicalities and recall the appropriate setting of potential theory in locally compact spaces. For convenience we add $+\infty$ to the set of real numbers, i.e., let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ endowed with its natural topology such that $\overline{\mathbb{R}}_+$ will be compact. Moreover, we will use the notation $\text{conv } E$ for the convex hull of a subset $E \subset \mathbb{R}$ and $\overline{\text{conv}} E$ for the closed convex hull in $\overline{\mathbb{R}}_+$, meaning, for example, $\overline{\text{conv}}(0, +\infty) = [0, +\infty]$.

In the fundamental work of Fuglede [10], general potential theory is presented in locally compact spaces. So unless otherwise stated X will be a locally compact Hausdorff space. Nevertheless we would like to use the developed tools also on metric spaces. To this end the appropriate results will be carried over to this case in Section 3.

On the space X we will consider, a usually fixed, kernel function k in the sense of Fuglede [10, p. 149]. That is, we assume that $k : X \times X \rightarrow \overline{\mathbb{R}}$ is *lower semicontinuous* (l.s.c.) as a two variable function over $X \times X$, and that $-\infty < k(x, y) \leq +\infty$. Moreover, we assume that $k \geq 0$, and that k is symmetric, i.e., $k(x, y) = k(y, x)$ for all $x, y \in X$.

Denote by $\mathfrak{M}(X)$ the family of positive, finite, regular Borel measures on X ; $\mathfrak{M}_1(X)$ will stand for the subset of probability measures. For any $H \subset X$ we let

$$\mathfrak{M}_1(H) := \{\mu \in \mathfrak{M}_1(X) : H \text{ is } \mu\text{-measurable and } \mu(H) = \mu(X)\}.$$

The customary topology on \mathfrak{M} is the vague topology which is the locally convex topology determined by the seminorms $\mu \mapsto |\int_X f d\mu|$, $f \in C_c(X)$. In most places we consider only the family $\mathfrak{M}_1(K)$ of probability measures supported on the same compact set K . In this case, by the Riesz Representation Theorem, $\mathfrak{M}(K)$ is the positive cone of $C(K)'$, and the weak*-topology determined by $C(K)$ and the vague topology coincide.

1.1. Potential and energy. Just as in the classical case, the *potential* and *energy* of $\mu \in \mathfrak{M}$ are

$$U^\mu(x) := \int_X k(x, y) d\mu(y), \quad W(\mu) := \int_X \int_X k(x, y) d\mu(y) d\mu(x).$$

Definition 1.1. Let $H \subset X$ be fixed, and $\mu \in \mathfrak{M}_1(X)$ be arbitrary. First put

$$(1.2) \quad Q(\mu, H) := \sup_{x \in H} U^\mu(x), \quad \text{and also} \quad \underline{Q}(\mu, H) := \inf_{x \in H} U^\mu(x).$$

For any two sets $H, L \subset X$ we define the quantities

$$(1.3) \quad q(H, L) := \inf_{\mu \in \mathfrak{M}_1(H)} Q(\mu, L) \text{ and } \underline{q}(H, L) := \sup_{\nu \in \mathfrak{M}_1(H)} \underline{Q}(\nu, L).$$

Definition 1.2. For $\mu \in \mathfrak{M}_1$, recalling Fuglede [10], we write

$$(1.4) \quad W(\mu) := \sup_{K \subset X} W(\mu_K),$$

with μ_K denoting the trace of μ on the set K . The Wiener energy (reciprocal capacity) of any set $H \subset X$ is

$$w(H) := \inf_{\mu \in \mathfrak{M}_1(H)} W(\mu) = \inf_{\substack{\mu \in \mathfrak{M}_1(H) \\ \text{supp } \mu \in H}} W(\mu) = \inf_{K \subset H} w(K),$$

where for the last forms see [10, (2), p. 152].

We remind that Fuglede [10, (1), p. 152] defines the so-called “uniform” and “de la Vallée-Poussin” energies $U(\mu) := Q(\mu, X)$ and $V(\mu) := Q(\mu, \text{supp } \mu)$ and their counterparts $u(H)$ and $v(H)$ for subsets of $H \subset X$. In [5] their relationship to the Chebyshev constant (see below) and transfinite diameter is studied. However, we will not need these special cases, in the following.

We will use the following statement from [10, Lemma 2.2.1] or [6, Lemma 1.5].

Lemma 1.3. *Let $H, L \subset X$. The functions below are lower semicontinuous*

- a) $\mathfrak{M}_1(H) \times L \ni (\mu, x) \mapsto U^\mu(x) (= \int k(x, y) d\mu(y))$,
- b) $\mathfrak{M}_1(H) \ni \mu \mapsto Q(\mu; L) (= \sup_{x \in L} U^\mu(x))$.

1.2. Chebyshev constants.

Definition 1.4. For arbitrary $H, L \subset X$ we define the (general) n^{th} Chebyshev constant of L with respect to H as

$$M_n(H, L) := \sup_{w_1, \dots, w_n \in H} \inf_{x \in L} \frac{1}{n} \left(\sum_{k=1}^n k(x, w_k) \right),$$

and the (general) n^{th} dual Chebyshev constant of L relative to H as

$$\overline{M}_n(H, L) := \inf_{w_1, \dots, w_n \in H} \sup_{x \in L} \frac{1}{n} \left(\sum_{j=1}^n k(x, w_j) \right).$$

The n^{th} Chebyshev constant of H is $M_n(H) := M_n(H, H)$ and the n^{th} dual Chebyshev constant of H is $\overline{M}_n(H) := \overline{M}_n(H, H)$.

The following proposition is proved by showing that the respective sequences are quasimonotonous, hence Fekete's lemma (see [8], or also [5], [6]) applies.

Proposition 1.5. *For any $H, L \subset X$, the Chebyshev constants $M_n(H, L)$ and $\overline{M}_n(H, L)$ converge, more precisely*

$$\sup_{n \in \mathbb{N}} M_n(H, L) = \lim_{n \rightarrow \infty} M_n(H, L) \text{ and } \inf_{n \in \mathbb{N}} \overline{M}_n(H, L) = \lim_{n \rightarrow \infty} \overline{M}_n(H, L).$$

The limits whose existence is assured by the previous proposition are denoted by $M(H, L)$ and $\overline{M}(H, L)$ (and for $H = L$ also by $M(H)$, $\overline{M}(H)$), respectively.

1.3. Rendezvous intervals. We define the (weak) rendezvous number(s) and average distance number(s) of the space X , or even of subsets of X . Again, for good reasons (explained in more detail in [6]) we define these notions in dependence of two sets as variables.

Definition 1.6. For arbitrary subsets $H, L \subset X$ the n^{th} (weak) rendezvous set of L with respect to H is

$$(1.5) \quad R_n(H, L) := \bigcap_{w_1, \dots, w_n \in H} \overline{\text{conv}} \left\{ p_n(x) := \frac{1}{n} \sum_{j=1}^n k(x, w_j) : x \in L \right\},$$

$$R_n(H) := R_n(H, H).$$

Correspondingly, one defines

$$(1.6) \quad R(H, L) := \bigcap_{n=1}^{\infty} R_n(H, L), \quad R(H) := R(H, H).$$

Similarly, one defines the (weak) average set of L with respect to H as

$$(1.7) \quad A(H, L) := \bigcap_{\mu \in \mathfrak{M}_1(H)} \overline{\text{conv}} \{ U^\mu(x) : x \in L \}, \quad A(H) := A(H, H).$$

Remark 1.7. Denoting the interval

$$(1.8) \quad A(\mu, L) := [\underline{Q}(\mu, L), \overline{Q}(\mu, L)] = \overline{\text{conv}} \{ U^\mu(x) : x \in L \},$$

we see that $R_n(H, L)$, $R(H, L)$ and $A(H, L)$ are all of the form $\bigcap_{\mu} A(\mu, H)$, with μ ranging over all averages of n Dirac measures at points of H , over all measures finitely supported in H and having only rational probabilities, and over all of $\mathfrak{M}(H)$, respectively, see [6].

Remark 1.8. If k is a continuous kernel – in particular when it is a metric – and L is compact, then it suffices to take conv instead of $\overline{\text{conv}}$, since then together with k also

$U^\mu(x)$ is continuous for any μ . Thus for compact subsets L of metric spaces a real number $r \in \mathbb{R}_+$ belongs to $R(H, L)$ if and only if for any $x_1, \dots, x_n \in H$ ($n \in \mathbb{N}$) we always have points $y, z \in L$ satisfying

$$(1.9) \quad \frac{1}{n} \sum_{j=1}^n k(y, x_j) \leq r \quad \text{and} \quad \frac{1}{n} \sum_{j=1}^n k(z, x_j) \geq r,$$

which is the usual definition of *weak rendezvous numbers* in metric spaces.

Moreover, in case the set L is connected, this is further equivalent to the existence of a “rendezvous point” $x \in L$ with

$$(1.10) \quad \frac{1}{n} \sum_{j=1}^n k(x, x_j) = r.$$

In particular, for compact and connected L in a metric space (or in a locally compact space with continuous kernel k) the rendezvous set $R(H, L)$ consists of a *unique* point, say $R(H, L) = \{r(H, L)\}$, if this latter property (1.10) is satisfied only for $r = r(H, L)$.

Remark 1.9. If k is only l.s.c., also potentials are l.s.c., which entails that they take on their infimum over compact sets. Thus for compact L the first half of the above equivalent formulation (1.9) remains valid even for general kernels. However, for the second part we must already write that “ $\forall s < r \exists z \in L$ such that $\frac{1}{n} \sum_{j=1}^n k(z, x_j) > s$ ”. Such modification of the formulation is necessary also when we consider sets $L \subset X$ which are not compact, or when we are discussing the case when $+\infty \in R(H, L)$. Clearly, in our settings $R_n(H, L)$, $R(H, L)$ and $A(H, L)$ are subsets of $[0, \infty]$, but note that traditionally rendezvous numbers or average numbers are considered only among the reals.

With the above notions at hand, the following description of various rendezvous sets is easy to see, cf. [6].

Proposition 1.10. *For arbitrary subsets $H, L \subset X$ we have*

$$(1.11) \quad \begin{aligned} R_n(H, L) &= [M_n(H, L), \overline{M}_n(H, L)], & R_n(H) &= [M_n(H), \overline{M}_n(H)], \\ R(H, L) &= [M(H, L), \overline{M}(H, L)], & R(H) &= [M(H), \overline{M}(H)], \\ A(H, L) &= [\underline{q}(H, L), q(H, L)], & A(H) &= [\underline{q}(H), q(H)]. \end{aligned}$$

Remark 1.11. Note that intervals appearing in Proposition 1.10 may indeed be empty, meaning, for example, that $q(H, L) < \underline{q}(H, L)$, cf. [6] and also Theorem 2.4 below.

2. General results on rendezvous numbers. The following theorem, known as Frostman’s theorem in the classical case, shows the relationship between the potential of a capacity (energy-minimizing) measure and the Wiener energy of a set. See [10, Theorem 2.4], or [6].

Theorem 2.1 (Fuglede). *Let k be a positive, symmetric kernel and $K \subset X$ be a compact set with $w(K) < +\infty$. Every $\mu \in \mathfrak{M}_1(K)$ having minimal energy ($W(\mu) = w(K)$) satisfies*

$$(2.1) \quad U^\mu(x) \geq w(K) \text{ for nearly every } x \in K,$$

$$(2.2) \quad U^\mu(x) \leq w(K) \text{ for every } x \in \text{supp } \mu,$$

$$(2.3) \quad U^\mu(x) = w(K) \text{ for } \mu\text{-almost every } x \in X.$$

Remark 2.2. In case k is continuous, or even if only it is bounded on $K \times K$, there can be no sets of finite measure but infinite energy. Therefore, the exceptional set in (2.1) (which refers to probability measures of $\mathfrak{M}_1(K)$) must be void, and (2.1) holds everywhere.

The following results are recalled from [6].

Theorem 2.3. *Let $H, L \subset X$, then*

$$(2.4) \quad M(H, L) \leq \underline{q}(H, L) \leq q(L, H) \leq \overline{M}(L, H).$$

If $L \subset X$ is compact, then

$$(2.5) \quad M(H, L) = \underline{q}(H, L) = q(L, H).$$

If $K \subset X$ is compact and k is continuous, then

$$(2.6) \quad q(L, K) = \overline{M}(L, K).$$

Theorem 2.4. *Let X be a locally compact Hausdorff space, $\emptyset \neq H \subset L \subset X$ be arbitrary, and let k be any nonnegative, symmetric kernel on X . Then the intervals $R_n(H, L)$, $R(H, L)$ and $A(H, L)$ are nonempty.*

Remark 2.5. In general, $A(H) \subsetneq R(H)$ is possible, see [6, Remark 6.4]. Also, the rendezvous intervals can be “almost empty”: consider, e.g., $R_n(\mathbb{R}, \mathbb{R}) = \{+\infty\}$. This and Remarks 1.8 and 1.9 already explain the slightly disturbing situation that some papers state that “there is no rendezvous number” for cases where we find one. However, not only $+\infty$ can show up in the closure of intervals for the definition of rendezvous numbers, hence not only $+\infty$ can be a rendezvous number for us while does not exist for other authors. See [7] for the cases of ℓ_p spaces.

Theorem 2.6. *Let X be any locally compact Hausdorff topological space, k be any l.s.c., nonnegative, symmetric kernel function, and $K \subset X$ be compact. Then $A(K)$ consists of one single point. Furthermore, if k is continuous and K is compact, then even $R(K)$ consists of only one point.*

Theorem 2.7. *If k is continuous and L is compact, we have $R(H, L) = A(H, L)$ for all $H \subset X$.*

In general, the theory of rendezvous numbers seems to be flourishing in the context of metric spaces instead of locally compact spaces with a Fuglede-type kernel. The latter theory is more general regarding the kernel, but is a bit restrictive in requiring local compactness of the underlying space. This gap is filled by indicating that the above potential theoretical approach works even for metric spaces, even if not locally compact. That leads us to the next section.

3. Rendezvous numbers for metric spaces. Note that for a nonnegative, Borel measurable (e.g., a continuous or l.s.c.) function $f : X \rightarrow \mathbb{R}$ and a (positive, finite) Borel measure μ the integral $\int f d\mu$ may be defined as a Lebesgue integral. Thus the potential $U^\mu(x) := \int_X d(x, y) d\mu(y)$ – and hence all related notions, considered previously – are defined (cf. Section 1). Further, keeping the notations from Section 1, we have that $\mu \in \mathfrak{M}_1(H)$ implies that H is μ -measurable.

These remarks are already sufficient to define the Chebyshev constants, rendezvous intervals and to show the equalities (1.11) as well as Theorem 2.3 and 2.4 in the metric space setting. We will further elaborate on this matter in [7] regarding normed spaces. Now we cover the theory of rendezvous numbers of metric spaces.

Gross' result on the existence of a rendezvous number was generalized by G. Elton to general Borel probability measures in place of finite convex combinations of Dirac measures. Note that for continuous kernels on compact sets the closure can be skipped from Definition 1.6 (as in that case potentials are continuous, and a continuous image of a compact set is always closed). On the other hand, Thomassen [20] extended the result to not necessarily connected spaces by considering so-called *weak rendezvous numbers*, which is equivalent to applying the convex hull in the definition, cf. Remark 1.8. Hence in our notation merging Gross', Thomassen's and Elton's theorems corresponds to the following result.

Theorem 3.1 (Gross–Thomassen–Elton). *Let (X, d) be a compact metric space. Then we have $A(X) = R(X) = \{r(X)\}$. Furthermore, there exist probability measures $\mu, \nu \in \mathfrak{M}_1$ with the property that*

$$(3.1) \quad U^\mu(x) \leq r(X) \leq U^\nu(y) \quad (\forall x, y \in X).$$

Remark 3.2. As mentioned a couple of times above, by compactness and continuity here we have exactly the same result even if closure is skipped from Definition 1.6; furthermore, if the space X is connected, then neither is any need for convex hull.

A further extension is due to Stadje [18], who essentially obtained the assertion of Theorem 2.6 concerning $R(X)$. He in fact assumed connectedness, but this assumption is easily removed when considering weak rendezvous numbers, i.e., convex hulls of values attained by the respective potential functions; also, he considered only $R(X)$, and not $A(X)$. We see that all these results follow from Theorem 2.6.

Note that Elton did not publish his result, but references to his work [4], [14] mention that he proved his statement even for continuous, nonnegative and symmetric functions f (in place of the metric d) over compact connected Hausdorff topological spaces. In any case, his results are now included in the following.

Theorem 3.3. *Let X be a locally compact Hausdorff space, k a symmetric, l.s.c., non-negative kernel, and $\emptyset \neq H \subset X$ be arbitrary, while $\emptyset \neq K \subseteq X$ be compact subsets of X . Then there exists $\mu \in \mathfrak{M}_1(K)$ with the property that*

$$(3.2) \quad U^\mu(x) \leq q(K, H) \quad (\forall x \in X)$$

and for all $\epsilon > 0$ there exists $v \in \mathfrak{M}_1(K)$ with

$$(3.3) \quad \underline{q}(K, H) - \epsilon \leq U^v(y) \quad (\forall y \in X).$$

Moreover, if the kernel k is continuous and bounded on $K \times H$, then we have

$$(3.4) \quad \underline{q}(K, H) \leq U^v(y) \quad (\forall y \in X).$$

Proof. By definition, there exist $\mu_n \in \mathfrak{M}_1(K)$ with $Q(\mu_n, H) \leq q(K, H) + 1/n$. Since $\mathfrak{M}_1(K)$ is weak*-compact by compactness of K , there exists a subnet \mathcal{N} of these measures converging to some $\mu \in \mathfrak{M}_1(K)$. In view of lower semicontinuity (see Lemma 1.3b)), $q(K, H) \geq \liminf_{\mathcal{N}} Q(\mu_n, H) \geq Q(\mu, H)$, hence the assertion (3.2).

Inequality (3.3) is just the definition.

To prove (3.4) consider the “dual” kernel $\ell := C - k$ whenever k is continuous and bounded by some constant C . Then ℓ is nonnegative, symmetric and l.s.c., and the first part applies. It is easy to check that to any measure $v \in \mathfrak{M}_1(K)$ the potentials with respect to k and ℓ are related¹⁾ by $U_k^v = C - U_\ell^v$, while $\underline{q}_k(K, H) = C - q_\ell(K, H)$. \square

Note that we did not assume H to be compact. However, in case we have a pair of compact sets K, L , then a continuous kernel is necessarily bounded on $K \times L$ and thus (3.4) follows. In particular, for $K = L$ and a continuous kernel Elton’s result is obtained using also $\underline{q}(K) = q(K)$, i.e., the last part of Theorem 2.6.

4. Invariant measures and rendezvous numbers. Following Morris and Nickolas [14], but extending the notion from $H = L = X$ to arbitrary subsets $H, L \subset X$, and from metrics d to arbitrary kernels k , we call a measure $\mu \in \mathfrak{M}_1(H)$ *k-invariant* (on L), if the respective potential integral is constant:

$$(4.1) \quad U_k^\mu(x) := \int_X k(x, y) d\mu(y) \equiv \text{const.} \quad (\text{for all } x \in L).$$

¹⁾We will use subscript notation for the kernel. For instance, we write U_k^μ to emphasize that the potential U^μ is understood with respect to the kernel k .

Saying only that μ is k -invariant refers to the central case $L = X$. Then an extension of the result of Morris and Nickolas [14] to general kernels k sounds as follows.

Theorem 4.1. *Assume that X is a locally compact Hausdorff topological space and k is a nonnegative, l.s.c., symmetric kernel function. Let $\emptyset \neq H \subset L \subset X$ be arbitrary and assume that there exists a measure $\mu \in \mathfrak{M}_1(H)$ which is k -invariant on L . Then we have*

$$A(H, L) = A(\mu, L).$$

Furthermore, if k is continuous and L is compact, then we even have

$$R(H, L) = A(\mu, L).$$

Proof. Note that $A(\mu, L)$ being the (convex closure of the) set of values of $U^\mu(x)$ when x runs L , invariance immediately implies that $\#A(\mu, L) = 1$. Hence only (non-empty) existence of (1.7) is needed to conclude $A(H, L) = A(\mu, L)$: this follows from Theorem 2.4. To obtain the last assertion from this, it suffices to refer to Theorem 2.7. \square

Corollary 4.2 (Morris–Nickolas). *Let (X, d) be a compact (connected) metric space. Assume that there exists a d -invariant measure $\mu_0 \in \mathfrak{M}_1(X)$. Then we have*

$$A(X) = R(X) = \{r(X)\} = A(\mu_0, X), \quad U^{\mu_0}(x) \equiv r(X) \quad (\forall x \in X).$$

5. Maximal energy and rendezvous numbers. Wolf [23] presents a theory of rendezvous numbers and maximal (i.e., maximal energy) measures on compact connected metric spaces (X, d) . Let us revise these results in this section. Following Björck [3], Wolf says that a probability measure $\mu_1 \in \mathfrak{M}_1(X)$ is *maximal*, and that the space has *maximal energy* $E(X)$, if

$$(5.1) \quad E(X) := E_d(X) := \sup_{\mu \in \mathfrak{M}_1(X)} W_d(\mu) = W_d(\mu_1).$$

By weak*-compactness of $\mathfrak{M}_1(X)$, existence of μ_1 is obvious. Wolf proves that $r(X) \leq E(X)$, and also gives examples when $r(X) < E(X)$.

Theorem 5.1 (Wolf). *Let (X, d) be a compact metric space. Then*

- (i) $r_d(X) \leq E_d(X)$.
- (ii) *If $r_d(X) = E_d(X)$, then there exists some d -invariant measure in $\mathfrak{M}_1(X)$.*

In his proof in [23, pp. 396–397] Wolf uses properties of metrics rather heavily. Here we extend the result first proving the following.

Theorem 5.2. *Let $\emptyset \neq K \subset L \subset X$ be arbitrary sets and k be a nonnegative, l.s.c., symmetric kernel. Then we have*

$$(5.2) \quad \min R(K, L) \geq w(K).$$

In particular, if k is continuous and K is compact, then the set K has a unique rendezvous number $r(K)$, and we have

$$(5.3) \quad r(K) \geq w(K).$$

Furthermore, if $r(K) = w(K)$, then there exists some k -invariant measure in $\mathfrak{M}_1(K)$.

Proof. Existence of rendezvous numbers $A(K, L) \subset R(K, L)$ are provided by Theorem 2.4, and we also know $R(K, L) = [M(K, L), \overline{M}(K, L)]$ (see Proposition 1.10). At this point (5.2) follows from the fact that $M(K) \geq w(K)$ and that $M(K, L) \geq M(K)$ (see [5], [6]).

According to Theorem 2.6, continuity of k on the compact set K implies uniqueness of the rendezvous numbers $A(K) = R(K) = \{r(K)\}$, giving the second part of the statement.

Furthermore, let now $r(K) = w(K)$ be assumed. Since k is continuous and K is compact, in this case $w(K) < +\infty$ is obvious.

Take now a probability measure $\mu \in \mathfrak{M}_1(K)$ minimizing $Q(\mu, K)$, i.e., with $Q(\mu, K) = q(K) = r(K)$. Such a measure exists, because $\mu \mapsto Q(\mu, K)$ is l.s.c. in view of Lemma 1.3 b), and $\mathfrak{M}_1(K)$ is weak*-compact. For any such μ we have

$$\begin{aligned} w(K) &\leq W(\mu) = \int_K U^\mu(x) d\mu(x) \leq \sup_{x \in K} U^\mu(x) \\ &= Q(\mu, K) = r(K) = w(K), \end{aligned}$$

so equality must hold throughout. Hence μ minimizes also $W(\mu)$ (it is a capacitary measure). For this μ the inequality (2.1) of Theorem 2.1 holds, moreover, it holds *everywhere* on K by Remark 2.2. But then $w(K) \leq U^\mu(x) \leq Q(\mu, K) = r(K) = w(K)$, hence $U^\mu(x) = w(K)$ holds for all $x \in K$, and μ is seen to be a k -invariant measure. \square

Now we are in the position to deduce Wolf's theorem as an easy corollary.

Proof of Wolf's Theorem 5.1. Let $k := \text{diam}(X) - d$, which is then a continuous, symmetric, nonnegative kernel function. By the previous Theorem 5.2, $r_k(X) \geq w_k(X)$. Also, $E_d(X) = \text{diam}(X) - w_k(X)$ is immediate. In view of Theorem 2.6, uniqueness of the rendezvous numbers hold both with respect to d and k , and thus we have $r_k(X) = q_k(X) = \underline{q}_k(X)$ and also $r_d(X) = q_d(X) = \underline{q}_d(X)$. Definition 1.1 immediately yields $q_k(X) = \text{diam}(X) - \underline{q}_d(X)$ and $q_d(X) = \text{diam}(X) - \underline{q}_k(X)$, which show (i).

Let us now assume $E_d(X) = r_d(X)$, i.e., $r_k(X) = w_k(X)$. As then a k -invariant measure $\mu \in \mathfrak{M}_1(X)$ exists, and obviously $U_k^\mu(x) = \text{diam}(X) - U_d^\mu(x)$, the very same measure is also d -invariant and even (ii) follows.

Wolf also treats the converse question: when does the existence of a d -invariant measure imply the equality of the maximal energy and the rendezvous number? He uses the following notion.

Definition 5.3. A metric space (X, d) is called *hypermetric*, if for all finite collections of points $x_i \in X$ ($i = 1, \dots, n$) and real scalars c_i ($i = 1, \dots, n$) with $\sum_{i=1}^n c_i = 0$, we have

$$(5.4) \quad \sum_{i=1}^n \sum_{j=1}^n c_i c_j d(x_i, x_j) \leq 0.$$

Wolf discusses how the notion proves to be useful, a number of well-known spaces being hypermetric spaces; for the details see [23]. Here we only aim at revealing the potential theoretic background even of this notion and the corresponding converse result of Wolf.

Observe that by the denseness of convex linear combinations of Dirac measures in \mathfrak{M}_1 for the weak*-topology (see, e.g., [16, Proposition 2.1.2, page 52] and the Kreĭn–Milman Theorem) and in view of continuity of d , (5.4) implies that we also have

$$(5.5) \quad W_d(\mu - \nu) = \int_X \int_X d(x, y) d(\mu - \nu)(x) d(\mu - \nu)(y) \leq 0 \quad (\mu, \nu \in \mathfrak{M}_1).$$

Translating this property to a property of the “dual kernel” $k := m - d$, where $m := \max_{X \times X} d$ is constant (in fact, the diameter), we get

$$(5.6) \quad W_k(\mu - \nu) = \int_X \int_X k(x, y) d(\mu - \nu)(x) d(\mu - \nu)(y) \geq 0 \quad (\mu, \nu \in \mathfrak{M}_1).$$

This property is almost identical with the notion of (positive) definiteness, having great importance in potential theory, see [10, p. 151]. Fuglede calls a kernel k (positive) definite, if for any signed regular Borel measure $\sigma \in \mathfrak{M}^\pm(X)$ one has $W_k(\sigma) \geq 0$. This is slightly more stringent, than (5.6), where only $\|\sigma^+\| = \|\sigma^-\|$ is considered, but (5.6) will suffice in the next argument.

Theorem 5.4. Let $\emptyset \neq K \subset L \subset X$ be arbitrary sets. Assume (5.6) and that $A(K, L) = \{a(K, L)\}$. If there is a probability measure $\mu_0 \in \mathfrak{M}_1(K)$ which is k -invariant on L , then we have $a(K, L) = w(K)$ and U^{μ_0} is constant $w(K)$ (on L).

Proof. Since $A(K, L) = \{a(K, L)\}$, we have that $a(K, L) \in A(\mu, L)$ for all $\mu \in \mathfrak{M}_1(K)$, so $U^{\mu_0}(x) = a(K, L)$ for all $x \in L$.

Applying (5.6) for $\nu := \mu_0$ and arbitrary $\mu \in \mathfrak{M}_1(K)$, we obtain

$$\begin{aligned} 0 &\leq W(\mu - \mu_0) = \int_K (U^\mu(y) - a(K, L)) d(\mu - \mu_0)(y) \\ &= \int_K U^\mu(y) d(\mu - \mu_0)(y) \\ (5.7) \quad &= W(\mu) - \int_K U^{\mu_0}(x) d\mu(x) = W(\mu) - a(K, L) \quad (\mu \in \mathfrak{M}_1), \end{aligned}$$

hence for all $\mu \in \mathfrak{M}_1(K)$ we have $W(\mu) \geq a(K, L)$. Taking infimum over all $\mu \in \mathfrak{M}_1(K)$ yields $w(K) \geq a(K, L)$. On the other hand, (5.2) of Theorem 5.2 and $R(K, L) \supseteq A(K, L)$ yield $a(K, L) \geq w(K)$, hence $a(K, L) = w(K)$, and also $U^{\mu_0} \equiv w(K)$. \square

Using that for continuous kernels one always has the uniqueness of the rendezvous numbers and the equality $A(K, L) = R(K, L)$, we arrive to the following corollary.

Corollary 5.5. *Let $\emptyset \neq K$ be a compact set and k a continuous kernel. Assume (5.6) (or its equivalent discrete form, analogous to (5.4)). If there is a probability measure $\mu_0 \in \mathfrak{M}_1(K)$ which is k -invariant on K , then we have $r(K) = w(K)$ and U^{μ_0} is constant $w(K)$ (on K).*

Corollary 5.6 (Wolf). *Let the compact metric space (X, d) be hypermetric. If there is a d -invariant probability measure $\mu \in \mathfrak{M}_1(X)$, then we have $r(X) = E(X)$. Furthermore, the potential of the d -invariant measure is constant $r(X)$ and is of maximal energy.*

Proof. Note that for compact X , the inequalities (5.4) and (5.5) are equivalent. Again we consider the continuous, symmetric, nonnegative kernel function $k := \text{diam}(X) - d$. By Theorem 5.4, $r_k(X) = w_k(X)$, and so $r_d(X) = E_d(X)$ follows. Moreover, $U_k^\mu(x) \equiv w_k(X)$ implies the rest of the statement. \square

Question 5.7. Does there exist a true invariant measure for, e.g., the unit sphere S_{ℓ_p} ?

Question 5.8. Do we have an Elton-type “separation theorem” even in not locally compact spaces? In normed spaces?

Definition 5.9. A measure $\mu \in \mathfrak{M}_1(H)$ is termed “ ε -quasi-invariant on L ” if $\sup_L U^\mu - \inf_L U^\mu \leq \varepsilon$.

Question 5.10. If the rendezvous number is unique, do we have a (quasi-) converse: Do there exist at least ε -quasi-invariant measures?

This is interesting as there is way to conclude the argument of Theorem 5.4 from the very existence of such ε -quasi-invariant measures.

Proposition 5.11. *Let X be any (not necessarily locally compact) Hausdorff topological space, and $H, L \subset X$ be arbitrary with $A(H, L) \neq \emptyset$. Assume that for all $\varepsilon > 0$ there exists some ε -quasi-invariant measure on L from $\mathfrak{M}_1(H)$. Take any sequence $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$) together with the corresponding measures $\mu_n \in \mathfrak{M}_1(H)$, ε_n -quasi-invariant on L , and consider any values ρ_n attained by the respective potentials U^{μ_n} on L . We then have $\rho_n \rightarrow a(H, L)$ as $n \rightarrow \infty$, where the average number exists uniquely, i.e., $A(H, L) = \{a(H, L)\}$.*

Proof. By ε_n -quasi-invariance, $A(\mu_n, L) \subset [\rho_n - \varepsilon_n, \rho_n + \varepsilon_n]$. As the intersection of the sets $A(\mu_n, L)$ contains $A(H, L)$, the intersection must be nonempty by condition. Therefore, the intersection is a diameter 0 nonempty subset – that is, a single point – of \mathbb{R} . However, as this set $\{\rho\}$ contains the nonempty set $A(H, L)$, we conclude $\rho = a(H, L)$. It is clear that $\rho_n \rightarrow \rho$ as $n \rightarrow \infty$. \square

Remark 5.12. The analogue of the above proposition for the rendezvous numbers also hold, where $R(H, L)$, $r(H, L)$ replace $A(H, L)$ and $a(H, L)$ respectively.

References

- [1] V. ANAGNOSTOPOULOS and SZ. GY. RÉVÉSZ, Polarization constants for products of linear functionals over \mathbb{R}^2 and \mathbb{C}^2 and Chebyshev constants of the unit sphere. *Publ. Math. Debrecen* **68** (1–2), 75–83 (2006).
- [2] M. BARONTI, E. CASINI and P. L. PAPINI, On average distances and the geometry of Banach spaces. *Nonlinear Anal. Theory Methods Appl.* **42A**(3), 533–541 (2000).
- [3] G. BJÖRCK, Distributions of positive mass, which maximize a certain generalized energy integral. *Ark. Mat.* **3**, 255–269 (1958).
- [4] J. M. CLEARY, S. A. MORRIS and D. YOST, Numerical geometry – numbers for shapes. *Amer. Math. Monthly* **93**, 260–275 (1986).
- [5] B. FARKAS and B. NAGY, Transfinite diameter, Chebyshev constant, and capacity on locally compact spaces. *Alfréd Rényi Ins. Preprint Series, Hung. Acad. Sci.* **7/2004**.
- [6] B. FARKAS and SZ. GY. RÉVÉSZ, Potential theoretic approach to rendezvous numbers. *Monatsh. Math.* To appear.
- [7] B. FARKAS and SZ. GY. RÉVÉSZ, Rendezvous numbers in normed spaces. *Bull. Austral. Math. Soc.* **72**, 423–440 (2005).
- [8] M. FEKETE, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.* **17**, 228–249 (1923).
- [9] J. B. G. FRENK, G. KASSAY and J. KOLUMBÁN, On equivalent results in minimax theory. *European J. Oper. Res.* **157**(1), 46–58 (2004).
- [10] B. FUGLEDE, On the theory of potentials in locally compact spaces. *Acta Math.* **103**, 139–215 (1960).
- [11] J. C. GARCÍA-VÁZQUEZ and R. VILLA, The average distance property of the spaces $\ell_\infty^n(\mathbb{C})$ and $\ell_1^n(\mathbb{C})$. *Arch. Math.* **76**, 222–230 (2001).
- [12] O. GROSS, The rendezvous value of a metric space. In: *Advances in Game Theory*. *Ann. Math. Studies.* **52**, 49–53 Princeton, (1964).
- [13] P. K. LIN, The average distance property of Banach spaces. *Arch. Math.* **68**, 496–502 (1997).
- [14] S. A. MORRIS and P. NICKOLAS, On the average distance property of compact connected metric spaces. *Arch. Math.* **40**, 459–463 (1983).
- [15] P. NICKOLAS and D. YOST, The average distance property for subsets of euclidean spaces. *Arch. Math.* **50**, 380–384 (1988).
- [16] P. MEYER–NIEBERG, *Banach lattices*. Berlin-Heidelberg-New York 1991.
- [17] SZ. GY. RÉVÉSZ and Y. SARANTOPOULOS, Plank problems, polarization, and Chebyshev constants. *J. Korean Math. Soc.* **41**(1), 157–174 (2004).
- [18] W. STADJE, A property of compact, connected spaces. *Arch. Math.* **36**, 275–280 (1981).
- [19] J. STRANZEN, An average distance result in Euclidean n -space. *Bull. Austral. Math. Soc.* **26**(3), 321–330 (1982).
- [20] C. THOMASSEN, The rendezvous number of a symmetric matrix and a compact connected metric space. *Amer. Math. Monthly* **107**(2), 163–166 (2000).
- [21] K. WOLF, On the average distance property in finite dimensional real Banach spaces. *Bull. Austral. Math. Soc.* **51**, 87–101 (1994).

- [22] K. WOLF, On the average distance property of spheres in Banach spaces. Arch. Math. **62**, 338–344 (1994).
- [23] K. WOLF, On the average distance property and certain energy integrals. Ark. Mat. **35**, 387–400 (1997).

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