Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 194 (2015) 108–131

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Carathéodory–Fejér type extremal problems on locally compact Abelian groups

Sándor Krenedits*, Szilárd Gy. Révész

A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Reáltanoda utca 13–15, 1053, Hungary

Received 17 September 2013; received in revised form 4 November 2014; accepted 1 February 2015
Available online 16 February 2015

Communicated by Serguei Denissov

Abstract

We consider the extremal problem of maximizing a point value $|f(z)|$ at a given point $z \in G$ by some positive definite and continuous function f on a locally compact Abelian group (LCA group) G, where for a given symmetric open set $\Omega \ni z$, f vanishes outside Ω and is normalized by $f(0) = 1$.

This extremal problem was investigated in $\mathbb{R}$ and $\mathbb{R}^d$ and for Ω a 0-symmetric convex body in a paper of Boas and Kac in 1945. Arestov, Berdysheva and Berens extended the investigation to $\mathbb{T}^d$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Kolountzakis and Révész gave a more general setting, considering arbitrary open sets, in all the classical groups above. Also they observed, that such extremal problems occurred in certain special cases and in a different, but equivalent formulation already a century ago in the work of Carathéodory and Fejér.

Moreover, following observations of Boas and Kac, Kolountzakis and Révész showed how the general problem can be reduced to equivalent discrete problems of "Carathéodory–Fejér type" on $\mathbb{Z}$ or $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. We extend their results to arbitrary LCA groups.

MSC: primary 43A35; 43A70; secondary 42A05; 42a82

Keywords: Carathéodory–Fejér extremal problem; Pointwise Turán problem; Locally compact Abelian groups; Abstract harmonic analysis; Haar measure; Convolution of functions and of measures; Positive definite functions; Bochner–Weil theorem; Convolution square; Fejér–Riesz theorem

* Corresponding author.
E-mail addresses: krenedits@t-online.hu (S. Krenedits), revesz.szilard@renyi.mta.hu (Sz.Gy. Révész).

http://dx.doi.org/10.1016/j.jat.2015.02.001

1. Introduction

In this work we consider the following fairly general problem.

Problem 1.1. Let $\Omega \subset G$ be a given set in the Abelian group G and let $z \in \Omega$ be fixed. Consider a *positive definite function* $f : G \to \mathbb{C}$ (or $\to \mathbb{R}$), normalized to have $f(0) = 1$ and *vanishing outside of* Ω. How large can then $|f(z)|$ be?

The analogous problem of maximizing $\int_\Omega f$ under the same hypothesis was recently well investigated by several authors under the name of "Turán's extremal problem", although later it turned out that the problem was already considered well before Turán, see the detailed survey [22]. The problem in our focus, in turn, was also investigated on various classical groups (the Euclidean space, $\mathbb{Z}^d$ and $\mathbb{T}^d$ being the most general ones) and was also termed by some as "the pointwise Turán problem", but the paper [13] traced it back to Boas and Kac [3] in the 1940s and even to the work of Carathéodory [4] and Fejér [7] [8, I, p. 869] as early as in the 1910s.

So based on historical reasons to be further explained below, we will term this problem as the *Carathéodory–Fejér type extremal problem on G for z and Ω*. This clearly requires some explanation, since Carathéodory and Fejér worked on their extremal problem well before the notion of positive definiteness was introduced at all.

Positive definite functions on $\mathbb{R}$ were introduced by Mathias in 1923 [16]. For Abelian groups positive definite functions are defined analogously [23, p. 17] by the property that

$$\forall n \in \mathbb{N},\ \forall x_1, \ldots, x_n \in G,\ \forall c_1, \ldots, c_n \in \mathbb{C} \quad \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} f(x_j - x_k) \geqslant 0. \tag{1.1}$$

In other words, positive definiteness of a real- or complex valued function f on G means that for all n and all choice of n group elements $x_1, \ldots, x_n \in G$, the $n \times n$ square matrix $[f(x_j - x_k)]_{j=1,\ldots,n}^{k=1,\ldots,n}$ is a positive (semi-)definite matrix. We will use the notation $f \gg 0$ for a short expression of the positive definiteness of a function $f : G \to \mathbb{C}$ or $G \to \mathbb{R}$.

Perhaps the most well-known fact about positive definite functions is the celebrated *Bochner theorem*, later extended to locally compact Abelian groups (LCA groups for short) in several steps and in this generality termed as the *Bochner–Weil theorem*. This states that a continuous function $f : G \to \mathbb{C}$ on a LCA group G is positive definite if and only if on the dual group $\widehat{G}$ there is an essentially unique (positive) Borel measure $d\mu(\gamma)$ such that f is the inverse Fourier transform of $d\mu$: $f(x) = \int_{\widehat{G}} \gamma(x) d\mu(\gamma)$ $(\forall x \in G)$, see e.g. [23, p. 19].

We will not need this general theorem in its full strength, but only the special case of positive definite sequences, obtained actually in Carathéodory's and Fejér's time, preceding the introduction and general investigation of positive definite functions.

Theorem 1.2 (*Herglotz*)**.** *Let $\psi : \mathbb{Z} \to \mathbb{C}$ be a sequence on $\mathbb{Z}$. Then $\psi \gg 0$ (i.e. ψ is positive definite) if and only if there exists a positive, finite Borel measure μ on $\mathbb{T}$ such that*

$$\psi(n) = \int_{\mathbb{T}} e^{2\pi i n t} d\mu(t) \quad (n \in \mathbb{Z}). \tag{1.2}$$

Furthermore, in case $\operatorname{supp} \psi \subset [-N, N]$ *we have $\psi \gg 0$ if and only if $T(t) := \check{\psi}(t) = \sum_{n=-N}^{N} \psi(n) e^{2\pi i n t} \geqslant 0$ $(t \in \mathbb{T})$, and then $d\mu(t) = T(-t)dt$ and $\psi(n) = \int_{\mathbb{T}} T(t) e^{-2\pi i n t} dt$.*

Proof. The fact that any sequence represented in the form of (1.2) is necessarily positive definite directly follows from the definition, as the reader can easily check. (The same direct verification works in any LCA group, too.)

The existence of such a representation for an arbitrary positive definite sequence on $\mathbb{Z}$ was first proved by Herglotz in [11], preceding the later analogous result of Bochner on $\mathbb{R}$ and the development of the theory of positive definite functions. The general proof on all LCA groups belongs to Weil [27]; for the proof and further details see also [23, 1.4.3].

The special case of finitely supported sequences can be fully proved by a simple direct calculation. □

For further use we also introduce the extremal problems

$$\mathcal{M}(\Omega) := \sup\Big\{a(1) : a : [1, N] \to \mathbb{R},\ N \in \mathbb{N},\ a(n) = 0\ (\forall n \notin \Omega),$$
$$T(t) := 1 + \sum_{n=1}^{N} a(n)\cos(2\pi n t) \geqslant 0\ (\forall t \in \mathbb{T})\Big\}, \tag{1.3}$$

which is called in [13] the *Carathéodory–Fejér type trigonometric polynomial problem* and

$$\mathcal{M}_m(\Omega) := \sup\Big\{a(1) : a : \mathbb{Z}_m \to \mathbb{R},\ a(0) = 1,\ a(n) = 0\ (\forall n \notin \Omega),$$
$$T\left(\frac{r}{m}\right) := \sum_{n \bmod m} a(n)\cos\left(\frac{2\pi n r}{m}\right) \geqslant 0\ (\forall r)\Big\} \tag{1.4}$$

which is termed in [13] as the *Discretized Carathéodory–Fejér type extremal problem.*

Remark 1.3. Obviously we have $\mathcal{M}_m(\Omega) \geqslant \mathcal{M}(\Omega)$, because the restriction on the admissible class of positive definite functions to be taken into account is lighter for the discrete problem: we only need to have $T\left(\frac{r}{m}\right) \geqslant 0$, while for $\mathcal{M}(\Omega)$ the restriction is $T(t) \geqslant 0$ $(\forall t \in \mathbb{T})$.

Let us recall that Carathéodory and Fejér solved the following extremal problem. Let $n \in \mathbb{N}$ be fixed, and assume that the 1-periodic trigonometric polynomial $T : \mathbb{T} \to \mathbb{R}$ of degree (at most) n is *nonnegative*. Under the normalization that the constant term $a(0) = \int_{\mathbb{T}} T = 1$, what is the possible maximum of $a(1)$ (solved already in [4]), and what are the respective extremal polynomials (solved – at all probability independently – in [7])?

Clearly the original Carathéodory–Fejér extremal problem is a special case of the above $\mathcal{M}(\Omega)$ problem—just take $\Omega := [0, n]$, and observe that the possible odd part of T (i.e. the sine series part of the trigonometric expansion) can be neglected, for $a(1)$ is the same for a general $T(x)$ and for $\frac{1}{2}(T(x) + T(-x))$, the even part of T. Here is what they have found.

Theorem 1.4 (*Carathéodory–Fejér*). *We have* $\mathcal{M}([0, n]) = 2\cos\left(\frac{\pi}{n+2}\right)$. *The inequality is sharp, with the only extremal polynomial being*

$$P(x) := P_n(x) := \frac{1}{\sum_{j=0}^{n} \sin^2\left(\frac{(j+1)\pi}{n+2}\right)} \left|\sum_{j=0}^{n} \sin\left(\frac{(j+1)\pi}{n+2}\right) e^{ijx}\right|^2. \tag{1.5}$$

Let us now consider Problem 1.1 on $G := \mathbb{Z}$, with $\Omega := [-n, n]$, but with *real valued* functions (instead of general complex valued ones). Denote a function from the admissible class (that is, a finite sequence of real values on $[-n, n]$) as ψ and assume that $\psi \gg 0$ on $\mathbb{Z}$. As $\widehat{\mathbb{Z}} = \mathbb{T}$, this is equivalent to say (in view of Theorem 1.2) that the trigonometrical polynomial $T(t) := \check{\psi}(t) := \sum_{k=-n}^{n} \psi(k) \exp(2\pi i k t)$ is nonnegative, so also real. It follows that $\overline{T}(t) = T(t)$, that is, $\psi(k) = \psi(-k)$. (Note that positive definiteness of ψ in itself implies that $\psi(k) = \overline{\psi(-k)}$, as is seen from the general introduction below, see (2.2), and so in case ψ is real-valued, we end up with the same relation). Take now $a(0) := \psi(0)$, $a(k) := 2\psi(k)$ $(k = 1, \ldots, n)$. Then the extremal problem translates to the $\mathcal{M}([0, n])$ problem, showing that *for real valued functions* Problem 1.1 on $\mathbb{Z}$ with $\Omega := [-n, n]$ is just the same as the $\mathcal{M}([0, n])$ problem (with a factor 2 between the resulting extremal quantities). Below in Proposition 3.1(ii) we show the easy fact that considering real or complex valued functions does not matter (in this case of sequences on $\mathbb{Z}$)—therefore, we obtain that the original Carathéodory–Fejér extremal problem is a (very) special case of Problem 1.1. This explains our terminology.

2. A short overview of basics about positive definite functions

Definition (1.1) has some immediate consequences,[1] the very first being that $f(0) \geqslant 0$ is nonnegative real (just take $n := 1$, $c_1 := 1$ and $x := 0$).

For any function $f : G \to \mathbb{C}$ the *converse*, or *reversed* function $\widetilde{f}$ (of f) is defined as

$$\widetilde{f}(x) := \overline{f(-x)}. \tag{2.1}$$

E.g. for the characteristic function χ_A of a set A we have $\widetilde{\chi_A} = \chi_{-A}$ (where, as usual, $-A := \{-a : a \in A\}$), because $-x \in A$ if and only if $x \in -A$.

Now let $f : G \to \mathbb{C}$. Then in case f is positive definite we necessarily have

$$f = \widetilde{f}. \tag{2.2}$$

Indeed, take in the defining formula (1.1) of positive definiteness $x_1 := 0$, $x_2 := x$ and $c_1 := c_2 := 1$ and also $c_1 := 1$ and $c_2 := i$: then we get both $0 \leqslant 2f(0) + f(x) + f(-x)$ entailing that $f(x) + f(-x)$ is real, and also that $0 \leqslant 2f(0) + if(x) - if(-x)$ entailing that also $if(x) - if(-x)$ is real. However, for the two complex numbers $v := f(x)$ and $w := f(-x)$ one has both $v + w \in \mathbb{R}$ and $i(v - w) \in \mathbb{R}$ if and only if $v = \overline{w}$.

Next observe that for any positive definite function $f : G \to \mathbb{C}$ and any given point $z \in G$

$$|f(z)| \leqslant f(0), \tag{2.3}$$

and so in particular if $f(0) = 0$ then we also have $f \equiv 0$. Indeed, let $z \in G$ be arbitrary: if $|f(z)| = 0$, then we have nothing to prove, and if $|f(z)| \neq 0$, let $c_1 := 1$, $c_2 := -\overline{f(z)}/|f(z)|$ and $x_1 := 0$, $x_2 := z$ in (1.1); then in view of (2.2) $f(-z) = \overline{f(z)}$, which yields $0 \leqslant 2f(0) + c_2 f(z) + \overline{c_2} f(-z) = 2f(0) - 2|f(z)|$ and (2.3) obtains.

Therefore, all positive definite functions are bounded and $\|f\|_\infty = f(0)$. That is an important property which makes the analysis easier: in particular, we immediately see that the answer to our extremal problem formulated in Problem 1.1 cannot exceed 1.

[1] These properties are basic and well-known, see e.g. [23, Section 1.4.1] We prove them just for being self-contained, as they are easy.

Note also that similar elementary calculations show that continuity of a positive definite function on a LCA group holds if and only if the function is continuous at 0 c.f. [23, (4), p. 18] This we will not use, however.

For LCA groups, characters play a fundamental role, so it is of relevance to mention that all characters $\gamma \in \widehat{G}$ of a LCA group G are positive definite. To see this one only uses the multiplicativity of the characters to get

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\overline{c_k}\gamma(x_j-x_k) = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j\gamma(x_j)\overline{c_k\gamma(x_k)} = \left|\sum_{j=1}^{n} c_j\gamma(x_j)\right|^2 \geqslant 0$$

for all choices of $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $x_j \in G$ $(j = 1, \ldots, n)$. Similarly, for any $f \gg 0$ and character $\gamma \in \widehat{G}$ also the product $f\gamma \gg 0$ since for all choices of $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $x_j \in G$ $(j = 1, \ldots, n)$ applying (1.1) with $a_j := c_j\gamma(x_j)$ in place of c_j $(j = 1, \ldots, n)$ gives

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\overline{c_k}\gamma(x_j-x_k)f(x_j-x_k) &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j\gamma(x_j)\overline{c_k\gamma(x_k)}f(x_j-x_k)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j\overline{a_k}f(x_j-x_k) \geqslant 0\\ &\quad \Big(a_j := c_j\gamma(x_j)\ (j = 1, \ldots, n)\Big).\end{aligned} \tag{2.4}$$

It is equally easy to see directly from the definition that for a positive definite function f also $\overline{f} \gg 0$, $f^{\sharp}(x) := f(-x) \gg 0$ and $\Re f \gg 0$, and that for $f, g \gg 0$ and $\alpha, \beta > 0$ also $\alpha f + \beta g \gg 0$.

Perhaps the most fundamental tool in topological groups is the Haar measure, which is a non-negative regular and translation invariant Borel measure μ_G, existing and being unique up to a positive constant factor in any LCA group, see [23, p. 1,2] with a full proof. As a direct consequence of uniqueness, we also have $\mu_G(E) = \mu_G(-E)$ for all Borel measurable set E, [23, 1.1.4].

Following standard notations, in particular that of Rudin, we simply write dx, dy, dz, etc. in place of $d\mu_G(x), d\mu_G(y), d\mu_G(z)$, etc. Throughout the sequel we will consider the convolution of functions with respect to the Haar-measure μ_G, that is

$$(f \star g)(x) := \int_G f(y)g(x-y)dy = \int_G f(x+z)g(-z)dz \tag{2.5}$$

defined for all functions $f, g \in L^1(\mu_G)$, or pairs of functions $f \in L^p(\mu_G)$, $g \in L^q(\mu_G)$ with $1/p + 1/q = 1$, see e.g. [23, p. 3]. Convolution is commutative and associative on any LCA group, see [23, 1.6.1. Theorem].

We will consider convolution of (bounded, complex valued, regular Borel) measures and convolution of such measures and functions as well. Rudin defines convolution of bounded regular measures μ and λ in [23, 1.3.1] with reference to the product measure $\mu \times \lambda$ on $G^2 = G \times G$: to each Borel set $E \subset G$ the derived set $E' := \{(x, y) \in G^2 : x + y \in E\}$ is constructed and then $\mu \star \lambda(E) := \mu \times \lambda(E')$. In particular this also means that for $E \subset G$ a Borel set we have – see [23, (1) p. 17] – the formula

$$\mu \star \lambda(E) = \int_G \mu(E-y)d\lambda(y). \tag{2.6}$$

With this construction, convolution of any two (bounded, regular, complex valued) measures is defined and yields another such measure, moreover, convolution is commutative and associative on any LCA group G, [23, 1.3.2 Theorem].

Note that the definition easily implies that the convolution of two measures, supported in sets S and T, resp., will vanish outside $S+T$: for an open set E, disjoint from $S+T$, and any $y \in T$, the set $E-y$ is disjoint from S and thus $\mu(E-y)=0$ resulting in $\mu \star \lambda(E)=0$. In particular, for any subgroup H of G, which contains $S+T$, $(\mu \star \lambda)|_H$ is essentially the same measure than $\mu \star \lambda$, the latter vanishing identically outside the subgroup H, moreover, $\mu|_H \star \lambda|_H$, (interpreting the convolution in H) agrees with $(\mu \star \lambda)|_H$.

It is also easy to see, as is remarked in [23, (4), p. 15], that one can equivalently define convolution of measures by the relation

$$\int_G f d(\mu \star \lambda) := \int_G \int_G f(x+y) d\mu(x) d\lambda(y)$$
$$= \int_G \int_G f(x+y) d\lambda(y) d\mu(x) \quad (f \in L^\infty(G)). \tag{2.7}$$

Indeed, let the set of complex valued continuous functions with compact support be denoted as $C_0(G)$: then, by the Riesz representation theorem, the set $M(G)$ of all regular Borel bounded (i.e. of finite total variation) measures is the topological dual of $C_0(G)$ and thus can be written as $M(G) \cong C_0^*(G)$. Now if $\mu, \lambda \in M(G)$, then their convolution $\mu \star \lambda$ is defined according to (2.7) for all $f \in C_0(G)$, which then extends easily also to all $f \in L^\infty(G)$. Note that (2.6) can be regarded as the special case of $f = \chi_E$, for $\mu(E-y) = \int_G \chi_{E-y}(x) d\mu(x) = \int_G \chi_E(y+x) d\mu(x)$.

Convolution of functions can then be regarded as a special case of convolution of measures, for $f \star g$ is the density function w.r.t. μ_G of the measure $\nu \star \sigma$ with $d\nu := f d\mu_G$ and $d\sigma := g d\mu_G$. Also convolutions of measures with functions or functions with measures can be obtained the same way. It is easy to see that for any $f \in L^1(\mu_G)$ and $\nu \in M(G)$ we have the formula

$$f \star \nu(x) = \nu \star f(x) = \int_G f(x-y) d\nu(y). \tag{2.8}$$

Another way to obtain this is to approximate f by simple functions and then use linearity and (2.6) for each characteristic functions. It is then immediate that the formula extends to $L^\infty(G)$, too.

Also, analogously to (2.1) the *converse measure* $\widetilde{\mu}(x) := \overline{\mu}(-x)$ (i.e. $\widetilde{\mu}(E) := \overline{\mu}(-E)$) is defined to any $\mu \in M(G)$. Then if $\phi \in C_0(G)$, then $\int_G \phi d(\widetilde{\mu \star \nu}) = \int_G \phi(-x) d\overline{(\mu \star \nu)} = \int_G \int_G \phi(-x-y) d\overline{\mu}(x) d\overline{\nu}(y) = \int_G \int_G \phi(x+y) d\widetilde{\mu}(x) d\widetilde{\nu}(y) = \int_G \phi d(\widetilde{\nu} \star \widetilde{\mu})$, so that $\widetilde{\mu \star \nu} = \widetilde{\mu} \star \widetilde{\nu}$.

For further use let us record here a few concrete formulae with convolutions. By (2.7) for any $u, v \in G$ the formula $\delta_u \star \delta_v = \delta_{u+v}$ holds true (where $\delta_u \in M(G)$ denotes the Dirac measure (unit point mass) at $u \in G$): for $\int_G \phi d(\delta_u \star \delta_v) = \int_G \int_G \phi(x+y) d\delta_u(x) d\delta_v(y) = \phi(u+v) = \int_G \phi d\delta_{u+v}$. Also, if $\phi \in L^\infty(G)$ and $u \in G$, then we have in view of (2.8)

$$\delta_u \star \phi(x) = \int_G \phi(x-y) d\delta_u(y) = \phi(x-u). \tag{2.9}$$

If for some Borel measurable A we put $\phi := \chi_A$, we obtain similarly

$$\delta_u \star \chi_A(x) = \chi_A(x-u) = \chi_{A+u}(x). \tag{2.10}$$

As (2.5) holds for all L^1 functions, it also holds for χ_A, χ_B with A, B Borel measurable sets with finite measure, yielding

$$\begin{aligned}\chi_A \star \chi_B(x) &= \int_G \chi_A(y)\chi_B(x-y)dy = \int_G \chi_A \chi_{(x-B)} d\mu_G = \int_G \chi_{A\cap(x-B)} d\mu_G \\ &= \mu_G(A \cap (x-B)).\end{aligned} \tag{2.11}$$

The same obtains also from calculating the measure convolution $\mu_G|_A \star \mu_G|_B = \chi_A \mu_G \star \chi_B \mu_G$.

In this paper a particular role is played by the case of $G = \mathbb{Z}$, where $\mu_{\mathbb{Z}} = \#$ is just the counting measure, and thus all locally finite measures ν are absolutely continuous and can as well be represented by their "density function" $\varphi_\nu(k) := \nu(\{k\})$, and conversely, any function φ defines the respective measure ν_φ with $\nu_\varphi(\{k\}) := \varphi(k)$, i.e. $\nu_\varphi = \sum_{k\in\mathbb{Z}} \varphi(k)\delta_k$; moreover, clearly $L^1(\mathbb{Z}) = \ell^1 \cong M(\mathbb{Z})$. In particular for $\varphi, \psi \in \ell^1$, $\rho := \varphi \star \psi$ is the density function of the measure $\tau \in M(G)$ with $\tau = \nu \star \sigma$ and with $\nu := \varphi_\nu d\#$, $\sigma := \psi d\#$ being the measures with density φ, ψ respectively.

We will need the next well-known assertion (which we will use only in the special case of compactly supported step functions, however).

Lemma 2.1. *Let G be any LCA group, μ_G its Haar measure, and $f \in L^2(\mu_G)$ be arbitrary. Then the "convolution square" of f exists, moreover, it is a continuous positive definite function, that is, $f \star \widetilde{f} \gg 0$ and belongs to $C(G)$.*

Proof. This can be found in [23, Section 1.4.2(a)]. □

Although it is very useful when it holds, in general this statement cannot be reversed. Even for classical Abelian groups, it is a delicate question when a positive definite continuous function has a "convolution root" in the above sense. For a nice survey on the issue see e.g. [6]. We will however be satisfied with a very special case, where this converse statement is classical.

Lemma 2.2. (i) *Let $\psi : \mathbb{Z} \to \mathbb{C}$ be a finitely supported positive definite sequence. Then there exists another sequence $\theta : \mathbb{Z} \to \mathbb{C}$, also finitely supported, such that $\theta \star \widetilde{\theta} = \psi$. Moreover, if* $\operatorname{supp}\psi \subset [-N, N]$, *then we can take* $\operatorname{supp}\theta \subset [0, N]$.

(ii) *If $\psi : \mathbb{Z}_m \to \mathbb{C}$, $\psi \gg 0$ on $\mathbb{Z}_m$, then there exists $\theta : \mathbb{Z}_m \to \mathbb{C}$ with $\theta \star \widetilde{\theta} = \psi$.*

Note the slight loss of precision in (ii)—it does not provide also localization, i.e. we cannot bound the support of θ in terms of a control of the support of ψ. This is natural, for the same finitely supported sequence can be positive definite on $\mathbb{Z}_m$ more easily than on $\mathbb{Z}$, as the equivalent restriction $T(2\pi n/m) \geqslant 0$ can be satisfied more easily than $T(t) \geqslant 0$ $(\forall t \in \mathbb{T})$. However, here the support remains finite anyway, which is the only essential fact we need in our arguments below.

Proof of Part (i). Here we invoke the special case of Bochner's Theorem as formulated in Theorem 1.2 to get that $T(t) := \check{\psi}(t) \geqslant 0$ $(\forall t \in \mathbb{T} = \widehat{\mathbb{Z}})$. Since ψ is finitely supported, T is a 1-periodic trigonometrical polynomial (with complex coefficients $\psi(k)$).

Let n stand for $\deg T$, so that $\operatorname{supp}\psi \subset [-N, N]$ translates to $n \leqslant N$. Write $T(t)$ in its trigonometric form as $T(t) = \sum_{k=0}^N a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$ with $a_k := \psi(k) + \psi(-k)$ and $b_k = (\psi(k) - \psi(-k))/i$. A glance at (2.2) yields $\psi(-k) = \overline{\psi(k)}$, so that then $a_k = 2\Re\psi(k)$ and $b_k = 2\Im\psi(k)$, whence in its trigonometrical form T must have real coefficients $a_k, b_k \in \mathbb{R}$ for all $0 \leqslant k \leqslant N$. This is of course obvious also from the usual trigonometric version of the coefficient formulas: $a_0 = \int_{\mathbb{T}} T(x)dx$, $b_0 = 0$, $a_k = 2\int_{\mathbb{T}} T(x)\cos(2\pi kx)dx$, $b_k = 2\int_{\mathbb{T}} T(x)\sin(2\pi kx)dx$ $(k = 1, \ldots, N)$.

Now the well-known classical theorem of L. Fejér and F. Riesz, see [26, Theorem 1.2.1], [7], or [8, I, p. 845], applies: there exists another trigonometrical polynomial $P(t)$ of degree n and with complex coefficients – more precisely, an algebraic polynomial $p(z)$ of degree n with $P(t) = p(e^{2\pi it})$ – such that $T(t) = |P(t)|^2$.

However, $|P|^2 = P \cdot \overline{P}$ and by the well-known properties of the Fourier transform, this means that there exists a finitely supported $\theta : \mathbb{Z} \to \mathbb{C}$, (the coefficient sequence of P; whence actually it can be written as $\theta : [0, n] \to \mathbb{C}$, otherwise vanishing) such that $\check{\theta}(t) = P(t)$ (and thus also $\check{\widetilde{\theta}} = \overline{P}$) and $\psi = \theta \star \widetilde{\theta}$. Note that $\operatorname{supp}\theta = [0, n] \subset [0, N]$, as needed. □

Proof of Part (ii). Consider $\widehat{\psi}(\nu) := \frac{1}{m}\sum_{j=0}^{m-1} \psi(j)\exp(-2\pi i\frac{j\nu}{m})$ which gives rise the representation $\psi(n) = \sum_{\nu \bmod m} \widehat{\psi}(\nu)\exp(2\pi i\frac{n\nu}{m})$ (Fourier inversion on $\mathbb{Z}_m$).

First, observe that $\widehat{\psi}(\nu) \geqslant 0$ for all $\nu \in \mathbb{Z}_m$, for by definition (1.1) of positive definiteness we must have with $x_j := j \in \mathbb{Z}_m$ and $c_j := \frac{1}{m}\exp(-2\pi i\frac{j\nu}{m})$ the inequality $0 \leqslant \sum_{j\in\mathbb{Z}_m}\sum_{j'\in\mathbb{Z}_m} \psi(j-j')\frac{1}{m}\exp(-2\pi i\frac{j\nu}{m})\frac{1}{m}\exp(2\pi i\frac{j'\nu}{m}) = \sum_{k\in\mathbb{Z}_m}\psi(k)\frac{1}{m}\exp(-2\pi i\frac{k\nu}{m}) = \widehat{\psi}(\nu)$.

Second, take $\widehat{\theta}(\nu) := \frac{1}{\sqrt{m}}\sqrt{\widehat{\psi}(n)}e^{i\varphi_\nu}$ $(\forall\nu \in \mathbb{Z}_m)$, with arbitrary real $\varphi_\nu \in [-\pi, \pi)$. This gives rise to $\theta(n) := \sum_{\nu\in\mathbb{Z}_m}\widehat{\theta}(\nu)\exp(2\pi i\frac{n\nu}{m})$. Then we obtain

$$\begin{aligned}
\theta \star \widetilde{\theta}(n) &:= \sum_{k\in\mathbb{Z}_m} \theta(k)\overline{\theta(k-n)} \\
&= \sum_{k\in\mathbb{Z}_m}\sum_{\nu\in\mathbb{Z}_m}\widehat{\theta}(\nu)\exp\left(2\pi i\frac{k\nu}{m}\right)\sum_{\mu\in\mathbb{Z}_m}\overline{\widehat{\theta}(\mu)}\exp\left(2\pi i\frac{(n-k)\mu}{m}\right) \\
&= \sum_{\nu\in\mathbb{Z}_m}\widehat{\theta}(\nu)\sum_{\mu\in\mathbb{Z}_m}\overline{\widehat{\theta}(\mu)}\exp\left(2\pi i\frac{n\mu}{m}\right)\sum_{k\in\mathbb{Z}_m}\exp\left(2\pi i\frac{k\nu}{m}\right)\exp\left(-2\pi i\frac{k\mu}{m}\right) \\
&= \sum_{\nu\in\mathbb{Z}_m}|\widehat{\theta}(\nu)|^2\exp\left(2\pi i\frac{n\nu}{m}\right)m = \sum_{\nu\in\mathbb{Z}_m}\widehat{\psi}(\nu)\exp\left(2\pi i\frac{n\nu}{m}\right) = \psi(n).
\end{aligned}$$

Clearly, here we have found a convolution squareroot θ, but it is not guaranteed here that $\operatorname{supp}\theta \subset [0, N]$ or at least $[-N, N]$, even if $\operatorname{supp}\psi \subset [-N, N]$. On the other hand this can still suffice, as $\mathbb{Z}_m$, whence all supports, are a priori finite, hence compact. □

3. Function classes and variants of the Carathéodory–Fejér type extremal problem

Already the above introductory discussion exposes the fact that Problem 1.1 may have various interpretations depending on how we define the exact class of positive definite functions what we consider, and also on what topology we use on G, if any (which determines what functions may be continuous, Borel measurable, compactly supported, Haar summable, etc.). Fixing the meaning of positive definiteness as in (1.1), similarly to [14], in principle we may consider many different function classes and corresponding extremal quantities. With respect to f "living" in Ω only, three immediate possibilities are that $f(x) = 0$ $(\forall x \notin \Omega)$, that $\operatorname{supp} f \subset \Omega$ and that $\operatorname{supp} f \Subset \Omega$ (the latter notation standing for compact inclusion). For "nicety" of the function f one may combine conditions of belonging to $C(G)$ (continuous functions), $L^1(G)$ (summable functions), $L^1_{\mathrm{loc}}(G)$ (locally summable functions), etc.

In case of the analogous "Turán problem" one maximizes the integral $\int_G f d\mu_G$ rather than just a fixed point value $|f(z)|$. In this question considerations of various classes are more delicate, and although several formulations were shown to be equivalent, see [14, Theorem 1], the authors call attention to cases of deviation as well. In the Carathéodory–Fejér extremal problem, however, we will find that the solution is largely indifferent to any choice of these classes, a somewhat unexpected corollary of our general approach. So instead of formally introducing all kind of function classes and corresponding extremal quantities, let us restrict to the two extremal cases, that is the possibly widest and smallest function classes, and define here only

$$\mathcal{F}_G^{\#}(\Omega) := \{f : G \to \mathbb{C} : f \gg 0,\ f(0) = 1,\ f(x) = 0\ \forall x \notin \Omega\}, \tag{3.1}$$

$$\mathcal{F}_G^{c}(\Omega) := \{f : G \to \mathbb{C} : f \gg 0,\ f(0) = 1,\ f \in C(G),\ \operatorname{supp} f \Subset \Omega\}. \tag{3.2}$$

Let us note, once again, that the first formulation is absolutely free of any topological or measurability structure of the group G. On the other hand, equipping G with the discrete topology the latter gives back a formulation close to the former but with restricting f to have finite support only.

The respective "Carathéodory–Fejér constants" are then

$$\mathcal{C}_G^{\#}(\Omega, z) := \sup\left\{|f(z)| : f \in \mathcal{F}_G^{\#}(\Omega)\right\}, \qquad \mathcal{C}_G^{c}(\Omega, z) := \sup\left\{|f(z)| : f \in \mathcal{F}_G^{c}(\Omega)\right\}. \tag{3.3}$$

In view of (2.3) giving that for $f \gg 0 \|f\|_\infty = f(0)$, the *trivial estimate* or *trivial (upper) bound* for the Carathéodory–Fejér constants $\mathcal{C}_G^{\#}(\Omega, z)$ and $\mathcal{C}_G^{c}(\Omega, z)$ is thus simply $f(0) = 1$.

As for a lower estimation, in the most classical cases it is easy to show that there exists a (real-valued) $f \in \mathcal{F}_G^{c}(\Omega)$ with $f(z) \geqslant 1/2$, so $\mathcal{C}_G^{c}(\Omega, z) \geqslant 1/2$. We will work out this for the general case, too, in Proposition 3.2, as later this may be instructive for comprehending the proofs of our main results. However, preceding it we discuss another issue.

By the above general definition, for $G = \mathbb{Z}$ and $G = \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ the Carathéodory–Fejér constants (3.3) with $z := 1$ – and denoting by H the fundamental set in place of Ω in this case and writing $\mathcal{F}_{\mathbb{Z}_m}(H) := \mathcal{F}_{\mathbb{Z}_m}^{\#}(H) = \mathcal{F}_{\mathbb{Z}_m}^{c}(H)$ – are

$$\begin{aligned}
\mathcal{C}^{\#}(H) &:= \mathcal{C}_{\mathbb{Z}}^{\#}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}}^{\#}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C},\ \varphi \gg 0,\ \varphi(0) = 1,\ \operatorname{supp} \varphi \subset H\}, \\
\mathcal{C}^{c}(H) &:= \mathcal{C}_{\mathbb{Z}}^{c}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}}^{c}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C},\ \varphi \gg 0,\ \varphi(0) = 1,\ \operatorname{supp} \varphi \subset H,\ \#\operatorname{supp} \varphi < \infty\}, \\
\mathcal{C}_m(H) &:= \mathcal{C}_{\mathbb{Z}_m}^{\#}(H, 1) = \mathcal{C}_{\mathbb{Z}_m}^{c}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}_m}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z}_m \to \mathbb{C},\ \varphi \gg 0,\ \varphi(0) = 1,\ \operatorname{supp} \varphi \subset H\}.
\end{aligned} \tag{3.4}$$

At this point also the issue whether we consider functions $f : G \to \mathbb{C}$ or just real valued functions, occurs naturally.

Note that in case of maximization of the integral $\int_\Omega f$ in place of the single function value $|f(z)|$ (that is, in case of the "Turán problem") the paper [14] easily concludes that even in the generality of LCA groups the restriction to real valued functions does not change the extremal quantity. Indeed, $S := \operatorname{supp} f \Subset \Omega$ is always symmetric (for $f \gg 0$ implies $f = \widetilde{f}$) and so $\int_S f = \int_{(-S)} \widetilde{f} = \int_S \overline{f}$, whence $\int_S f = \int_S \Re f$, too.

Here, in the point value maximization question Problem 1.1, the versions of the extremal question in various function classes are generally easier to compare and remain equivalent.

However, comparing the variants with real- or complex valued functions become more interesting and in fact the results differ in some cases, while they remain equivalent for others. In this preliminary section we consider only the fundamental cases of $\mathbb{Z}$ and $\mathbb{Z}_m$ for various $m \in \mathbb{N}$. For a more concise notation first let us write similarly to the complex valued case

$$\mathcal{K}^{\#}_G(\Omega, z) := \sup_{\varphi\in\mathcal{F}^{\#\mathbb{R}}_G(\Omega)} |\varphi(z)|, \qquad \mathcal{K}^{c}_G(\Omega, z) := \sup_{\varphi\in\mathcal{F}^{c\mathbb{R}}_G(\Omega)} |\varphi(z)|, \tag{3.5}$$
$$\mathcal{K}^{\#}(H) := \mathcal{K}^{\#}_{\mathbb{Z}}(H, 1), \qquad \mathcal{K}^{c}(H) := \mathcal{K}^{c}_{\mathbb{Z}}(H, 1),$$
$$\mathcal{K}_m(H) := \mathcal{K}_{\mathbb{Z}_m}(H, 1) := \sup_{\varphi\in\mathcal{F}^{\mathbb{R}}_{\mathbb{Z}_m}(H)} |\varphi(1)|,$$

where naturally we write for any group, (and so in particular for $G = \mathbb{Z}$ and $G = \mathbb{Z}_m$)

$$\mathcal{F}^{\#\mathbb{R}}_G(\Omega) := \{\varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}^{\#}_G(\Omega)\}, \qquad \mathcal{F}^{c\mathbb{R}}_G(\Omega) := \{\varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}^{c}_G(\Omega)\}.$$

Proposition 3.1. *We have the following.*

(i) $\mathcal{M}(H) = 2\mathcal{K}^c(H)$ *and for all* $m \in \mathbb{N}$ $\mathcal{M}_m(H) = 2\mathcal{K}_m(H)$.
(ii) *We have* $\mathcal{K}^c(H) = \mathcal{C}^c(H)$ *and* $\mathcal{K}^{\#}(H) = \mathcal{C}^{\#}(H)$.
(iii) *For all* $m \in \mathbb{N}$, $\cos(\pi/m)\mathcal{C}_m(H) \leqslant \mathcal{K}_m(H) \leqslant \mathcal{C}_m(H)$.
(iv) ***[Ruzsa]*** *If* $4 \leqslant m \in \mathbb{N}$, *then in general* (iii) *is the best possible estimate with both inequalities being attained for some symmetric subset* $H \subset \mathbb{Z}_m$.
(v) *If* $m = 2, 3$, *then for any admissible* H *we must have* $H = \mathbb{Z}_m$ *and thus* $\varphi(x) \equiv 1$ *shows* $\mathcal{C}_m(H) = \mathcal{K}_m(H) = 1$.

Proof. As regards (i), $\mathcal{M}(H) = 2\mathcal{K}^c(H)$ is quite easy and was already discussed in the final part of Section 1. The analogous relation $\mathcal{M}_m(H) = 2\mathcal{K}_m(H)$ $(m \in \mathbb{N})$ is seen the same way.

It remains to compare the respective extremal quantities for the cases of real- and complex valued functions. The obvious direction is that $\mathcal{K}^c(H) \leqslant \mathcal{C}^c(H)$, $\mathcal{K}^{\#}(H) \leqslant \mathcal{C}^{\#}(H)$; and also $\mathcal{K}_m(H) \leqslant \mathcal{C}_m(H)$ for all $m \in \mathbb{N}$.

For proving some estimate in the other direction, let now $G = \mathbb{Z}_m$ or $\mathbb{Z}$ and $\psi \in \mathcal{F}^c_G(H)$ be arbitrary. Let further $\gamma_t \in \widehat{G}$ be the character belonging to the parameter t, i.e. $\gamma_t(k) := \exp(2\pi i t k)$, where $t \in \mathbb{T}$ in case $G = \mathbb{Z}$, and $t := j/m$ with $j \in \mathbb{Z}_m$ in case $G = \mathbb{Z}_m$.

As said above, together with ψ, also $\psi\gamma_t \gg 0$ and even $\varphi := \Re\{\psi\gamma_t\} \gg 0$ – see (2.4) and around – while belonging to the same function class $\mathcal{F}^c_G(H)$, as we also have $\varphi(0) = \psi(0) = 1$ and $\operatorname{supp}\varphi \subset \operatorname{supp}\psi =: S$. Moreover, being real-valued, even $\varphi \in \mathcal{F}^{c,\mathbb{R}}_G(H)$. Thus $\mathcal{K}^c_G(H, 1) \geqslant \sup_{\psi\in\mathcal{F}^c_G(H),\,\gamma_t\in\widehat{G}} \Re\{\psi(1)\gamma_t(1)\} \geqslant \sup_{\psi\in\mathcal{F}^c_G(H)} \min_{\alpha\in\mathbb{T}} \sup_{\gamma_t\in\widehat{G}} \Re\{|\psi(1)|e^{2\pi i\alpha}\gamma_t(1)\} = \mathcal{C}^c_G(H, 1) \cdot \min_{\alpha\in\mathbb{T}} \sup_{\gamma_t\in\widehat{G}} \Re\{e^{2\pi i\alpha}\gamma_t(1)\} = \mathcal{C}^c_G(H, 1) \cdot \min_{\alpha\in\mathbb{T}} \sup_{\gamma_t\in\widehat{G}} \cos(2\pi(\alpha + t))$.

With the choice of $t := -\alpha$ this latter estimate gives for $G = \mathbb{Z}$ that $\mathcal{K}^c(H) \geqslant \mathcal{C}^c(H)$, and by a completely analogous computation with $\psi \in \mathcal{F}^{\#}_{\mathbb{Z}}(H)$ we also find $\mathcal{K}^{\#}(H) \geqslant \mathcal{C}^{\#}(H)$. As the converse inequalities are obvious, these furnish (ii).

Furthermore, for $G = \mathbb{Z}_m$ we can always choose $t := -[m\alpha + 1/2]/m$ – where $[\cdot]$ stands for integer part – and thus obtain $\mathcal{K}_m(H) \geqslant \mathcal{C}_m(H) \min_{\alpha\in\mathbb{T}} \cos(2\pi(\alpha - [m\alpha + 1/2]/m))$. The function of the variable α within the minimum symbol is obviously periodic mod $1/m$ and can be written with $\xi = m\alpha$ as $\cos\left(\frac{2\pi}{m}(\xi - [\xi + 1/2])\right)$. Obviously, this achieves a minimum exactly where the argument of cos is maximal in absolute value, i.e. for $\xi = m\alpha = j + 1/2$, $(j \in \mathbb{Z})$, where we get $\cos(\pi/m)$. Thus $\mathcal{K}_m(H) \geqslant \mathcal{C}_m(H)\cos(\pi/m)$, so also (iii) obtains.

To find an example of equality $\mathcal{K}_m(H) = \mathcal{C}_m(H)$ is trivial, as $H := \mathbb{Z}_m$ suffices. To obtain the other extreme, for $4 \leqslant m \in \mathbb{N}$ we use a construction communicated to us by Ruzsa. Namely, we take $H := \{-1, 0, 1\} \subset \mathbb{Z}_m$, compute the extremal quantity $\mathcal{K}_m(H)$ and then compare it to $\mathcal{C}_m(H)$ as follows.

To start with, we prove $\mathcal{K}_m(H) = 1/2$ for an arbitrary $m \geqslant 4$. First, for any positive definite real sequence ψ supported on $\{-1, 0, 1\}$ with $\psi(0) = 1$, by $\widetilde{\psi} = \psi$ we must have $\psi(-1) = \psi(1)$. Second, take now $x_j := j \pmod m$ and $c_j = (-1)^j$ $(j = 1, \ldots, m)$. Then we will get from the definition (1.1) that $m - 2m\psi(1) \geqslant 0$, so $\psi(1) \leqslant 1/2$. Third, the real sequence $1/2, 1, 1/2$ on H is positive definite according to Lemma 2.1, because it is the convolution square of the function $\theta : \mathbb{Z}_m \to \mathbb{R}$ defined as $\theta(0) := \theta(1) := 1/\sqrt{2}$ and $\theta(k) := 0$ for all $k \not\equiv 0, 1 \bmod m$.

Now let us find a lower estimate for the value of $\mathcal{C}_m(H)$ for $m \geqslant 4$ *even*. We consider the function $\psi(0) = 1$, $\psi(\pm 1) = r \exp(\pm\pi i/m)$, where $r > 0$ is a parameter, and $\psi(k) := 0$ for all $k \notin H$. Here $\psi(-1) = \overline{\psi(1)}$ is needed to satisfy $\widetilde{\psi} = \psi$; otherwise, ψ can be arbitrary on H as long as it remains positive definite. So we want $\psi \gg 0$, so $\psi \in \mathcal{F}_{\mathbb{Z}_m}(H)$, while r maximal possible. Obviously, any suitable value of r, whence also the maximal one, still cannot exceed $\mathcal{C}_m(H)$.

Let us compute the Fourier transform $\widehat{\psi}(n) = \int_{\mathbb{Z}_m} \psi(k) e^{-2\pi i kn/m} d\mu_{\mathbb{Z}_m}(k)$ now. We get $\widehat{\psi}(n) = \sum_{k \bmod m} \psi(k) e^{-2\pi i kn/m} = 1 + re^{-\pi i(2n-1)/m} + re^{\pi i(2n-1)/m} = 1 + 2r\cos((2n-1)\pi/m)$. This remains nonnegative, for all $n \mod m$ if and only if $r \leqslant 1/(2\cos(\pi/m))$; if r equals this bound, then for $n := m/2, m/2+1$ $\widehat{\psi}(m/2) = \widehat{\psi}(m/2+1) = 0$. (Here it is essential that m is even!) So now we find that to keep the Fourier transform – that is the scalar product with all characters – nonnegative, it is necessary and sufficient that $r \leqslant 1/(2\cos(\pi/m))$.

In fact it is a very special, trivial instance of the general Bochner–Weil theorem that $\widehat{\psi}(n) \geqslant 0$ $(\forall n \mod m)$ is further equivalent to positive definiteness of the sequence ψ on $\mathbb{Z}_m$. For this particular case of the general theorem let us note that characters are positive definite, and so are their (finite) positive linear combinations as said above, therefore also any function on $\mathbb{Z}_m$ with nonnegative Fourier transform. To see the converse statement, that is $\widehat{\psi} \geqslant 0$ if $\psi \gg 0$, we can fix any k mod m, take $n := m$, $c_j := e^{-2\pi i jk/m}$, $x_j := j \bmod m$ $(j = 1, \ldots, m)$ and compute

$$0 \leqslant \sum_{j=1}^{m} \sum_{\ell=1}^{m} e^{2\pi i(\ell-j)k/m} \psi(j-\ell) = m \sum_{a \bmod m} e^{-2\pi i ak/m} \psi(a) = m\widehat{\psi}(k).$$

So we find that $\mathcal{C}_m(H)$ is at least the maximal r in the above construction, which reaches $r = 1/(2\cos(\pi/m))$. It also follows that for m even and at least 4, $\mathcal{C}_m(H) \geqslant \mathcal{K}_m(H)/\cos(\pi/m)$.

Let now $m > 4$ be *odd*. For a similar construction as given above for even m, we now choose $\psi(0) := 1$, $\psi(\pm 1) := r\exp(\pm 2\pi i[m/2]/m)$ and $\psi(k) := 0$ for all $k \not\equiv 0, \pm 1 \bmod m$. Again, we use the Bochner characterization that $\psi \gg 0$ on $\mathbb{Z}_m$ if (and only if) $\widehat{\psi} \geqslant 0$ on $\widehat{\mathbb{Z}_m} = \mathbb{Z}_m$. This means that for k mod m we must have $0 \leqslant \widehat{\psi}(k) = \sum_{\ell \bmod m} \psi(\ell)\exp(-2\pi i k\ell/m) = 1 + r\exp(2\pi i([m/2]-k)/m) + r\exp(2\pi i(k-[m/2])/m) = 1 + 2r\cos(2\pi(k-[m/2])/m) = 1 - 2r\cos(\pi(2k+1)/m)$, which holds true for all k mod m if and only if its minimum, with respect to k, satisfies nonnegativity, that is, when $0 \leqslant 1 - 2r\cos(\pi/m)$ and thus $r \leqslant 1/\{2\cos(\pi/m)\}$. This leads to exactly the same estimate as before, that is, $\mathcal{C}_m(H) \geqslant \mathcal{K}_m(H)/\cos(\pi/m)$. Therefore, (iv) obtains.

In view of $0, 1 \in H$ and H being symmetric, for both $m = 2$ and $m = 3$ $H = \mathbb{Z}_m$ is clear, whence also (v) is obvious. The proposition is proved. $\square$

Now we can formulate the already announced lower estimation with the somewhat more precise form containing the same lower estimate even with real functions.

Proposition 3.2. *For any LCA group G, $z \in G$ and $0, \pm z \in \Omega \subset G$ open set we have $\mathcal{C}_G^c(\Omega, z) \geqslant 1/2$, moreover, there exists a real-valued function $f \in \mathcal{F}_G^{c\mathbb{R}}(\Omega)$ with $f(z) \geqslant 1/2$.*

Proof. Basically, we want to utilize the fact that the measure $\nu := 2\delta_0 + \delta_z + \delta_{-z}$ is a positive definite Borel measure. Instead of formally defining the notion of positive definiteness of measures, let us remark here that clearly $\nu = \sigma \star \widetilde{\sigma}$ with $\sigma := \delta_0 + \delta_z$ (and, as is easy to see, $\widetilde{\sigma} = \delta_0 + \delta_{-z}$). From this starting point we then wish to construct a positive definite, real-valued, continuous function F, compactly supported within Ω, and with $F(z) \geqslant \frac{1}{2}F(0)$.

As $0, \ \pm z \in \Omega \subset G$ and Ω is open, in the locally compact group G there exists an open set $U \ni 0, z, -z$ with its compact closure $\overline{U} \Subset \Omega$. Next we take another open neighborhood V of 0 satisfying $V - V, V - V - z, V - V + z \subset U$. Such a V exists for all three functions $(x, y) \to x - y$, $(x, y) \to x - y - z$, $(x, y) \to x - y + z$ are continuous from $G \times G$ to G mapping 0 to $0, -z, z$, respectively, while all these images $0, \pm z$ lie in U. (Or, saying it a bit differently: this is equivalent to $V - V \subset U' := U \cap (U + z) \cap (U - z)$, which is still an open neighborhood of 0 and is thus such that there is V with $V \times V \to U'$ under $(x, y) \to x + y$.)

So formally with the characteristic function χ_V of V we now take $\Phi := \chi_V + \chi_{V+z} = \chi_V \star (\delta_0 + \delta_z)$ and accordingly $\widetilde{\Phi} = \widetilde{\chi_V} + \widetilde{\chi_{V+z}} = \chi_{-V} + \chi_{-V-z}$, so that using (2.11)

$$\begin{aligned}
F(x) := \Phi \star \widetilde{\Phi}(x) &= (\chi_V + \chi_{V+z}) \star (\chi_{-V} + \chi_{-V-z})(x) \\
&= \chi_V \star \chi_{-V}(x) + \chi_V \star \chi_{-V-z}(x) + \chi_{V+z} \star \chi_{-V}(x) + \chi_{V+z} \star \chi_{-V-z}(x) \\
&= \mu_G(V \cap (x + V)) + \mu_G(V \cap (x + z + V)) \\
&\quad + \mu_G((V + z) \cap (x + V)) + \mu_G((V + z) \cap (x + z + V)) \\
&= 2\mu_G(V \cap (x + V)) + \mu_G(V \cap (x - z + V)) + \mu_G(V \cap (x + z + V)).
\end{aligned}$$

By Lemma 2.1, $F \gg 0$, F is continuous, and obviously $\operatorname{supp} F \subset \overline{\operatorname{supp} \Phi + \operatorname{supp} \widetilde{\Phi}} = \overline{(V \cup (V + z)) + ((-V) \cup (-V - z))} = \overline{(V - V) \cup (V - V - z) \cup (V - V + z)} \subset \overline{U} \Subset \Omega$ by construction, so $F \in C_0(G)$, too and in fact $F/F(0) \in \mathcal{F}_G^c(\Omega)$. Moreover, F is real-valued, too.

Finally, denote $\alpha := \mu_G(V)$, $\beta := \mu_G(V \cap (z + V))$ and $\gamma := \mu_G(V \cap (2z + V))$. Then $F(0) = 2\mu_G(V) + \mu_G(V \cap (V - z)) + \mu_G(V \cap (z + V)) = 2\alpha + 2\beta$, and similarly $F(z) = 2\beta + \alpha + \gamma$. It follows that $F(z)/F(0) = 1/2 + (\beta + \gamma)/(2\alpha + 2\beta) \geqslant 1/2$, as we wanted. □

Note that the construction also shows that if the group order $o(z)$ of z is $o(z) = 2$, i.e. $2z = 0$, then $\gamma = \alpha$ and $F(z)/F(0) = 1$, i.e. $\mathcal{K}_G(\Omega, z) = 1$ taking into account the trivial estimate from above, too.

We have noted in Proposition 3.1(v) that in $\mathbb{Z}_2$, when $m = 2$ (and thus in particular $o(1) = 2$) and also in $\mathbb{Z}_3$, the trivial choice of $f \equiv 1$ proves $\mathcal{C}_{\mathbb{Z}_2}(H, z) = 1$, $\mathcal{C}_{\mathbb{Z}_3}(H, z) = 1$. Now we obtained also this in quite a larger generality.

Remark 3.3. This proposition is not used in the later proofs, but only gives some preliminary information on the size of the extremal constants in question. We have presented the above proof to preview in a simpler case the methods and ingredients of our main arguments later.

However, as a referee pointed out to us, there is a natural way to arrive at the same result by use of the later theorems given in Section 6.

Namely, it would have been easy to prove the stated result for cyclic groups, for in $\mathbb{Z}$ or any $\mathbb{Z}_m$ (with $m \geqslant 3$), the function f supported in $\{-1, 0, 1\}$ and defined by $f(0) = 1$, $f(\pm 1) = 1/2$ is easily seen to be positive definite. (E.g. it is the convolution square of the characteristic function of $\{0, 1\}$, normalized by a factor $1/2$.) Given this, the later results of Section 6 directly imply that the same estimates extend to arbitrary LCA groups, too.

Next let us mention a continuity-type result.

Proposition 3.4. *Let $H \subset \mathbb{Z}$ be a fixed symmetric finite set containing* 0 *and* 1. *Then*

$$\lim_{m\to\infty} \mathcal{K}_m(H) = \lim_{m\to\infty} \mathcal{C}_m(H) = \mathcal{C}^c(H). \tag{3.6}$$

Proof. Note that for a *finite* set $H \subset \mathbb{Z}$, for large enough m we also have $H \subset (-m/2, m/2]$, which makes it possible to interpret H unambiguously as a subset of $\mathbb{Z}_m$, too.

Consider first only the statement that $\lim_{m\to\infty} \mathcal{K}_m(H) = \mathcal{K}^c(H)$, that is, restrict to real-valued positive definite functions only. Note that even the existence of the limit must be proved.

Since we deal with $m \to \infty$, we can assume $m > 2 \max H$. Then obviously $\mathcal{F}^{c\mathbb{R}}_{\mathbb{Z}}(H) \subset \mathcal{F}^{\mathbb{R}}_{\mathbb{Z}_m}(H)$, hence $\mathcal{K}^c(H) \leqslant \mathcal{K}_m(H)$, as was remarked for the cosine formulation already in Remark 1.3. Whence $\mathcal{K}^c(H) \leqslant \liminf_{m\to\infty} \mathcal{K}_m(H)$.

For an estimate from the other direction, let $\varepsilon > 0$ be arbitrarily fixed, and let $\varphi_m \in \mathcal{F}^{\mathbb{R}}_{\mathbb{Z}_m}(H)$ be such that $\varphi_m(1) \geqslant (1-\varepsilon)\mathcal{K}_m(H)$. We can then define the corresponding extension $\psi_m : \mathbb{Z} \to \mathbb{R}$ with $\psi_m|_H = \varphi_m|_H$ and $\psi_m|_{\mathbb{Z}\setminus H} = 0$. Then $\check{\psi}_m(k/m) := \sum_{j\in H} \varphi_m(j) e^{2\pi i jk/m} \geqslant 0$ $(k \in \mathbb{Z})$ (even if positive definiteness of φ_m on $\mathbb{Z}_m$ does not imply $\check{\psi}_m(t) \geqslant 0$ for all $t \in \mathbb{T}$).

Now first we select a subsequence (m_ℓ) of the indices with $\psi_{m_\ell}(1) \to \limsup_{m\to\infty} \mathcal{K}_m(H)$. By the Bolzano–Weierstrass Theorem we can select a further subsequence $(m_{\ell_n}) \subset (m_\ell)$ – which for simplicity we shall denote as (m_n) from now on – such that the function sequence (ψ_{m_n}) converges: $\psi_{m_n}(j) \to \psi(j)$ for every $j \in H$ with some finite value $\psi(j)$. (Here for the application of the Bolzano–Weierstrass Theorem it is essential that $\#H < \infty$, and also that by positive definiteness and normalization of $\varphi_m \in \mathcal{F}^{\mathbb{R}}_{\mathbb{Z}_m}(H)$ we necessarily have $|\psi_m(j)| = |\varphi_m(j)| \leqslant \varphi_m(0) = 1$.) Note that the limit values are also even $(\psi(-j) = \psi(j))$, as by positive definiteness and having real values this property holds for all φ_m by (2.2).

Next let $\eta > 0$ be arbitrary, and assume that for $n > N$ we already have $|\psi_{m_n}(j) - \psi(j)| < \eta$. For an arbitrary $t \in \mathbb{T}$ let us choose a suitable $k_n \in \mathbb{Z}$ with $|k_n/m_n - t| < 1/m_n$: then we find

$$\check{\psi}(t) = \sum_{j\in H} \psi(j) e^{2\pi i jt} \geqslant \sum_{j\in H} \psi_{m_n}(j) e^{2\pi i jt} - \#H\eta \geqslant \check{\psi}_{m_n}\left(\frac{k_n}{m_n}\right) - \#H\frac{2\pi}{m_n} - \#H\eta$$

because $|e^{2\pi i t} - e^{2\pi i s}| = |e^{2\pi i(t-s)} - 1| = 2|\sin(\pi(t-s))| \leqslant 2\pi|t-s|$. However, $\varphi_{m_n} \gg 0$ (on $\mathbb{Z}_m$) implies $\widetilde{\psi_{m_n}}\left(\frac{k_n}{m_n}\right) = \widetilde{\varphi_{m_n}}\left(\frac{k_n}{m_n}\right) \geqslant 0$, and thus we are led to

$$\check{\psi}(t) \geqslant -\#H\left(\frac{2\pi}{m_n} + \eta\right).$$

Letting $n \to \infty$ and noting that $\eta > 0$ was arbitrary yields $\check{\psi}(t) \geqslant 0$, which shows $\psi \gg 0$ on $\mathbb{Z}$ in view of Theorem 1.2. Furthermore, clearly $\psi(0) = 1$ and $\operatorname{supp} \psi \subset H$, hence $\psi \in \mathcal{F}^{c\mathbb{R}}_{\mathbb{Z}}(H)$, while by construction $\psi(1) = \lim_{n\to\infty} \psi_{m_n}(1) \geqslant (1-\varepsilon) \limsup_{m\to\infty} \mathcal{K}_m(H)$.

So it follows that $\mathcal{K}^c(H) \geqslant (1-\varepsilon) \limsup_{m\to\infty} \mathcal{K}_m(H)$, and this holding for any arbitrarily fixed $\varepsilon > 0$, we even have $\mathcal{K}^c(H) \geqslant \limsup_{m\to\infty} \mathcal{K}_m(H)$. Combining this with $\liminf_{m\to\infty} \mathcal{K}_m(H) \geqslant \mathcal{K}^c(H)$, recorded already in the very beginning of the proof, furnishes $\lim_{m\to\infty} \mathcal{K}_m(H) = \mathcal{K}^c(H)$.

Note that according to Proposition 3.1(ii), $\mathcal{K}^c(H) = \mathcal{C}^c(H)$. However, the positive sequences $\mathcal{K}_m(H)$ and $\mathcal{C}_m(H)$ must be equivalent regarding convergence in view of Proposition 3.1(iii), so the first, and hence also the last equality of (3.6) holds true. □

Proposition 3.5. *We have*

$$\mathcal{C}^\#(H) = \mathcal{C}^c(H). \tag{3.7}$$

Therefore, taking into account also Proposition 3.1(i) and (ii), we can put

$$\mathcal{C}(H) := \mathcal{K}^c(H) = \mathcal{K}^\#(H) = \mathcal{C}^c(H) = \mathcal{C}^\#(H) = \frac{1}{2}\mathcal{M}(H)$$

for any $H \subset \mathbb{Z}$.

Proof. Clearly the supremum is taken on a smaller set in $\mathcal{C}^c(H)$, hence $\mathcal{C}^c(H) \leqslant \mathcal{C}^\#(H)$.

Conversely, let $\varphi \in \mathcal{F}^\#_{\mathbb{Z}}(H)$ and let us consider the representation, given by Theorem 1.2 of Herglotz: $\varphi(n) = \int_{\mathbb{T}} e^{2\pi i n t} d\nu(t)$, with ν a positive regular Borel measure on $\mathbb{T}$.

Let $N \in \mathbb{N}$ be arbitrary. Then we can consider $\psi := \psi_N := \varphi \cdot \Delta_N$, where $\Delta_N(n) := (1 - \frac{|n|}{2N+1})_+$, and so in particular Δ_N and ψ_N have finite support.

First let us observe that Δ_N is positive definite. This follows from Lemma 2.1 writing

$$\begin{aligned}\left(\chi_{[-N,N]} \star \chi_{[-N,N]}\right)(n) &= \int_{\mathbb{Z}} \chi_{[-N,N]}(n-j)\chi_{[-N,N]}(j)d\mu_{\mathbb{Z}}(j) \\ &= \sum_{|j|,|n-j|\leqslant N} 1 = (2N+1)\Delta_N(n).\end{aligned}$$

Also, it easily follows from the fact that $\Delta_N(n) = \widecheck{F_N}(n) = \int_{\mathbb{T}} e^{2\pi i n t} F_N(t)dt$, where $F_N(t) := \frac{1}{2N+1}\left(\frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\right)^2 \geqslant 0$ is the classical Fejér kernel, providing the positive representation of Herglotz described in Theorem 1.2. Note that this means that $\Delta_N(n)$ is just the Fourier transform (i.e. the sequence of Fourier coefficients) of F_N.

Now the Herglotz-type positive representation for ψ_N obtains from the usual rules of convolutions and the above: $\psi_N(n) = (\widecheck{F_N \star \nu})(n) = \int_{\mathbb{T}} e^{2\pi i n t}\left(\int_{\mathbb{T}} \frac{1}{2N+1}\left(\frac{\sin(\pi(2N+1)(t-s))}{\sin(\pi(t-s))}\right)^2 d\nu(s)\right) dt$. That is, $\psi_N := \varphi \cdot \Delta_N \gg 0$, too.

Since now $\psi_N(1) = \varphi(1)(1 - \frac{1}{2N+1})$, clearly $\mathcal{C}^c(H) \geqslant \sup_N\{|\psi_N(1)|\} = |\varphi(1)|$, and as this holds for all possible $\varphi \in \mathcal{F}^\#_{\mathbb{Z}}(H)$, we get $\mathcal{C}^c(H) \geqslant \mathcal{C}^\#(H)$ concluding the proof. □

Kolountzakis and Révész prove in [13, Section 2, p. 404] that in $\mathbb{R}^d$ and for an unbounded symmetric open set Ω the bounded parts $\Omega_N := \Omega \cap B_N$, where $B_N = NB$ and $B \subset \mathbb{R}^d$ is the unit ball, approximate Ω in such a way that $\mathcal{C}^c_{\mathbb{R}^d}(\Omega_N) \to \mathcal{C}^c_{\mathbb{R}^d}(\Omega)$ as $N \to \infty$. (The argument is essentially the same as the one above for Proposition 3.5.) Analogously, $\mathcal{C}^c(\Omega_N) \to \mathcal{C}^c(\Omega)$ also in the group $\mathbb{Z}$. These seem to suggest that a limiting argument should give Proposition 3.4 even if $\Omega \subset \mathbb{Z}$ is infinite. However, this is false.

Remark 3.6. For any $\varepsilon > 0$ there exists an infinite set $H \subset \mathbb{Z}$, sparse enough to have $\mathcal{C}^c(H) \leqslant 1/2 + \varepsilon$ but still containing a copy of $\mathbb{Z}_m$ for every $m \in \mathbb{N}$, and hence having $\mathcal{C}_m(H) = 1$, the maximal possible value.

In fact, $H := \{0, \pm 1, \pm N, \pm(N+1), \pm(N+2), \ldots\}$ has $\mathcal{C}^c(H) = 1/(2\cos\frac{\pi}{N+2})$, see [13, Theorem 4.4 (iii)].

This underlies the importance of carefully distinguishing between the cases when we work in $\mathbb{Z}$ or in any $\mathbb{Z}_m$, which explains why we formulated separately the two, otherwise rather similar theorems in Section 6.

4. Some previous work on Carathéodory–Fejér type extremal problems

For general domains in arbitrary dimension d the problem was formulated in [13]. With our above notations and general definition we can now recall it simply as follows.

Problem 4.1 (*Boas–Kac-Type Pointwise Extremal Problem for the Space*)**.** Find $\mathcal{K}^c_{\mathbb{R}^d}(\Omega, z)$.

Problem 4.2 (*Turán-Type Pointwise Extremal Problem for the Torus*)**.** Find $\mathcal{K}^c_{\mathbb{T}^d}(\Omega, z)$.

As is easy to see, c.f. [13, Remark 1.4], if $\Omega \subset [-1/2, 1/2]$ and is thus can also be considered as subset of both $\mathbb{T}^d$ and of $\mathbb{R}^d$, then $\mathcal{K}^c_{\mathbb{T}^d}(\Omega, z) \geqslant \mathcal{K}^c_{\mathbb{R}^d}(\Omega, z)$, always.

The extremal value in Problem 4.1 was estimated together with its periodic analogue Problem 4.2 in the work [2] in dimension $d = 1$ for an interval $\Omega := (-h, h)$. Note that Boas and Kac have already solved the interval (hence dimension $d = 1$) case of Problem 4.1 in [3], a fact which seems to have been unnoticed in [2].

These problems are not only analogous, but also related to each other, and, in fact, Problem 4.1 is only a special, limiting case of the more complex Problem 4.2, see [13, Theorem 6.6]. On the other hand, Boas and Kac have already observed, that Problem 4.1 (dealt with for $\mathbb{R}$ in [3]) is connected to trigonometric polynomial extremal problems. In particular, from the solution to the interval case they deduced the value $\mathcal{M}([0, n]) = 2\cos\frac{\pi}{n+2}$ of the original extremal problem due to Carathéodory [4] and Fejér [7] or [8, I, p. 869]. They also established a connection (see [3, Theorem 6]) what corresponds to the one-dimensional case of the first part of [13, Theorem 2.1].

Our results will extend these results together with the until now most general results of [13], comprising all these and much more. So first let us record these results here.

Theorem 4.3 (*Kolountzakis–Révész*)**.** *In $\mathbb{R}^d$ and for any $z \in \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ an open, symmetric neighborhood of $\mathbf{0} \in \mathbb{R}^d$, we have with $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$ the relation*

$$\mathcal{K}^{\#}_{\mathbb{R}^d}(\Omega, z) = \mathcal{K}^c_{\mathbb{R}^d}(\Omega, z) = \mathcal{C}(H(\Omega, z)). \tag{4.1}$$

If $\Omega \subset \mathbb{T}^d$ is an open symmetric neighborhood of $\mathbf{0} \in \mathbb{T}^d$, and the order of z is infinite (i.e. z has no torsion), then we have with $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$

$$\mathcal{K}^{\#}_{\mathbb{T}^d}(\Omega, z) = \mathcal{K}^c_{\mathbb{T}^d}(\Omega, z) = \mathcal{C}(H(\Omega, z)). \tag{4.2}$$

Finally, if the order of $z \in \mathbb{T}^d$ is $o(z) = m$, then with $H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$ we have

$$\mathcal{K}^{\#}_{\mathbb{T}^d}(\Omega, z) = \mathcal{K}^c_{\mathbb{T}^d}(\Omega, z) = \mathcal{K}_m(H_m(\Omega, z)). \tag{4.3}$$

Actually, the above can be collected from [13, Theorem 2.1] and [13, Theorem 2.4]. The most important aspect of it is perhaps the understanding that the above point-value extremal problems depend only on the set $H(\Omega, z)$ and the order of z itself, and are in fact equivalent to the trigonometric polynomial extremal problems given in (1.3) and (1.4). In other words, the result carries over all information (value, estimates etc.), possibly known about the corresponding equivalent special problem in $\mathbb{Z}$ or in $\mathbb{Z}_m$, to the given more general problem in G. Until that work the equivalence remained unclear in spite of the fact that, e.g., Boas and Kac found ways to deduce the solution of the trigonometric extremal problem (1.3) from their results on

Problem 4.1. Kolountzakis and Révész also obtained a clear picture of the limiting relation between torus problems and space problems, formulated above as Problems 4.1 and 4.2, and parallel to this, between the finitely conditioned trigonometric polynomial extremal problem (1.4) and the positive definite trigonometric polynomial extremal problem (1.3). Furthermore, the investigation was extended to arbitrary (symmetric open) sets $\Omega \subset \mathbb{R}^d$ or $\mathbb{T}^d$, dropping the condition of convexity of Ω.

Let us remark, however, that even with the above equivalence result, the actual calculation of the extremal values may still take considerable work and innovation, see e.g. [12]. For the numerous applications see the original paper [13] and Refs. [2,3,21,20,18].

Ending this section, let us recall that investigation of the so-called Turán-type problems started with keeping an eye on number theoretic applications and connected problems. The interesting papers of Gorbachev and Manoshina [9,10] mention [15] and character sums; applications to van der Corput sets were mentioned by several authors and in particular by Ruzsa [24]. Here we recall another question of a number theoretic relevance, open for at least two decades by now, and also mentioned in [13].

Problem 4.4. Determine $\Lambda(n) := \sup\{\mathcal{M}(H)/2 : 1 \in H \subseteq \mathbb{N}, |H| = n\}$.

We only know (c.f. [17]) $1 - \frac{5}{(n+1)^2} \leqslant \Lambda(n) \leqslant 1 - \frac{0.5}{(n+1)^2}$. The question is relevant to the Beurling theory of generalized primes, see [19].

5. A discussion of the extremal problems in cyclic groups

For gaining some familiarity with the topic, here we briefly discuss Carathéodory–Fejér type extremal problems on *cyclic* groups. Depending on order, the cyclic group can be either infinite (when it is torsion-free) or finite (when it has torsion). Consider first the case of a torsion-free group, which is then algebraically isomorphic to $\mathbb{Z}$, so we can assume $G = \mathbb{Z}$, however, perhaps with some different (non-discrete) topology.

A particular case is when $\Omega := [-n, n] \subset \mathbb{Z}$, with the discrete topology on $\mathbb{Z}$, and $z = 1$ is the generator of the group $\mathbb{Z}$. Then by the classical result of Carathéodory and Fejér, c.f. Theorem 1.4, we have $\mathcal{K}([-n, n]) = \cos\left(\frac{\pi}{n+2}\right)$. Historically, one of the first extensions of this result occurred when several authors observed, how this can be extended to estimation of *arbitrary* coefficients $a(k)$ of the nonnegative trigonometric (cosine) polynomial in (1.3), i.e. to arbitrary points $z = k$ with $k \leqslant n$. Lukács (mentioned in [8, II. p. 346]), Szegő [25], M. Krafft (mentioned in [5, p. 646]) and Egerváry and Szász [5] all found the following (see also Fejér [8, II. p. 346]).

Theorem 5.1 (*Egerváry–Szász, Krafft, Lukács, Szegő*)**.** $\mathcal{C}([-n, n], k) = \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$.

Note that this is analogous to the result of Boas and Kac [3], formulated for $\mathbb{R}^d$.

We give the proof in order to facilitate the reader both to our notations and to the basic idea of the upcoming main results. The proof is presented from the dual point of view here. That is, we do not consider nonnegative trigonometric polynomials on $\mathbb{T}$ (as the classical arguments run), but positive definite functions (sequences) on $\mathbb{Z}$. As said above, these approaches are equivalent.

Proof. First, assume that $f \gg 0$ is a function from $\mathcal{F}^c([-n, n]) = \mathcal{F}^\#([-n, n])$, where here the finiteness of $\Omega := [-n, n]$ guarantees the equality of these function classes. We need to show $f(k) \leqslant \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$. Take now the function $g := f|_{k\mathbb{Z}}$, i.e. $g : k\mathbb{Z} \to \mathbb{C}$ $g(j) := f(j)$. Obviously, g is positive definite on the subgroup $G' := k\mathbb{Z}$ of $G := \mathbb{Z}$, as in general restriction to a subgroup is always positive definite, too. Indeed, the restriction needs to satisfy the same definitive inequalities (1.1) only for $x_1, \ldots, x_n \in G'$, which obviously hold true in view of $x_j - x_\ell \in G'$. Furthermore, $g(0) = f(0) = 1$ and g is supported on the set $\Omega' := \Omega \cap k\mathbb{Z} = \{j = k\ell \in k\mathbb{Z} : j \in \Omega\}$. So we obtain $g \in \mathcal{F}^c(\Omega') = \mathcal{F}^\#(\Omega')$. As $\mathbb{Z} \cong k\mathbb{Z}$, with the isomorphism provided by $\phi : \ell \to k\ell$, changing now the point of view we can consider $h := g \circ \phi : \mathbb{Z} \to \mathbb{C}$, which is again positive definite, and, moreover, $h \in \mathcal{F}^c(H) = \mathcal{F}^\#(H)$ with $H := H(\Omega, k) := \{\ell \in \mathbb{Z} : k\ell \in \Omega\} = [-[\frac{n}{k}], [\frac{n}{k}]]$. It remains to see that $h(1) = g(\phi(k)) = f(k)$, whence $f(k) \leqslant \mathcal{C}(H) = \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$.

Observe that here we have proved an important part of the statement, namely the *upper estimation* $\mathcal{C}([-n, n], k) \leqslant \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$, which was found without any essential use of particulars of the original group $\mathbb{Z}$ or the (original) topology on the group. So in fact we may (and will, see the proof of Theorem 6.1) get the upper estimation $\mathcal{C}_G^c(\Omega, k) \leqslant \mathcal{C}(H(\Omega, z))$ almost verbatim for an arbitrary LCA group G, any open, symmetric set $\Omega \subset G$ and $z \in \Omega$.

For the converse inequality let now $h := P_{[n/k]}$ be the extremal function of Theorem 1.4 in (1.5) with degree $[n/k]$. Then $h(1) = \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$. Arguing backwards, we can again take $g := h \circ \phi^{-1}$ and record that $g : k\mathbb{Z} \to \mathbb{C},\ g \in \mathcal{F}^c(\Omega')$.

So the proof hinges upon finding an extension of g to the whole group $\mathbb{Z}$ keeping its positive definiteness and staying in $\mathcal{F}^c(\Omega)$. In case of $\mathbb{Z}$ and $k\mathbb{Z}$, this is obvious, because the *trivial extension*

$$f(j) := \begin{cases} g(j) & \text{if } k|j \\ 0 & \text{if } k \not| j \end{cases} \tag{5.1}$$

is already such an extension. Indeed, for any system of points x_j one can divide the points into congruence classes: the inequality (1.1) is then satisfied for all subsystems within each of the congruence classes, hence for the sum of terms with $x_j \equiv x_\ell$, while for points from different classes the difference $x_j - x_\ell \notin k\mathbb{Z}$ and is thus providing $f(x_j - x_\ell) = 0$. Furthermore, $f(0) = g(0) = h(0) = 1$, f is supported in $\phi^{-1}(H) = \Omega' \subset \Omega$, and f is continuous, too, for any function is continuous in the discrete topology. That is, we found $f \in \mathcal{F}^c(\Omega)$ with $f(k) = h(1) = \cos\left(\frac{\pi}{[\frac{n}{k}]+2}\right)$, and the inequality is thus sharp. $\square$

Let us observe the key ideas in this argument. First, it is a fully general fact that restriction to a subgroup keeps positive definiteness of a function, so this can be applied in all groups. Moreover, in a topological group, restricting a function to the subgroup, equipped with the derived topology, will naturally keep continuity of the function. Therefore, there is no difficulty with the *upper estimation* of the extremal constant by subgroup restrictions.

However, to prove *precise equality* is harder: if we take an extremal (or, an approximately extremal) function for the $\mathcal{C}(H)$ problem, and somehow transplant it back to a subgroup of G, then we still need to extend it to the whole of G preserving both continuity and also compact support inside Ω.

What subgroups to use then? The most spurious idea is just to take the subgroup generated by the selected point $z \in G$, i.e. to consider the generated cyclic subgroup $Z := \langle z \rangle$. This works in the above situation, and this is what we will pursue later in the general case. Above, we had $z := k \in \mathbb{Z}$, and the order of z was $o(z) = \infty$, too. If so, we always have, as above, the isomorphism ϕ between Z and $\mathbb{Z}$, and we can thus transform the consideration to the group $\mathbb{Z}$.

The sharpness, however, requires something more. It is still true that the trivial extension (5.1) of some positive definite function $g = h \circ \phi^{-1}$ (say the extension built from the extremal h) will be again positive definite, normalized again by $f(0) = 1$. The level set $\{f \neq 0\}$ will also be subset of $\Omega' \subset \Omega$ by construction. However, we also need both *continuity* of f and supp $f \Subset \Omega$ for having $f \in \mathcal{C}^c(\Omega)$.

Now these properties of f are provided "for free" if the subgroup G', from which we try to extend g, is itself open (and hence also closed) in the group G, for then also supp $f \Subset G'$, and continuity is obvious. Conversely, if G' is not open, then f in (5.1) *cannot be continuous*, since $f(0) = 1$ would require a full neighborhood of 0 with function values >0, i.e. within the subgroup G' where this may occur.

Therefore, it seems that we may be forced to consider larger subgroups, then just $Z = \langle z \rangle$. If we are to take into account any topological properties, then the minimal thing we may think of is to take $M := \overline{Z}$, the closure of Z. It is at least a closed subgroup of G, so first working within M and then extending somehow the extremal function, say, seems to be a hopeful strategy. Groups, topologically generated by one single element, are called *monothetic groups* [23, p. 39]. According to [23, 2.3.2. Theorem], then either $M = Z$ with Z discrete and isomorphic to $\mathbb{Z}$ – so we are back to the already settled nice situation – or M is compact.

With compact monothetic groups, however, we already face with a large variety of groups, including p-adic groups, Lie groups, subgroups of the powers of the torus, and their direct products e.g. Even if M is compact, the description, due to Kakutani, says that the dual group of M is a subgroup of $\mathbb{T}_d$, the torus equipped with the discrete topology, see e.g. [23, 2.3.3. Theorem]. In this generality it is difficult to say anything. In fact the only thing we can really do is the already fully general approach in the forthcoming two sections.

Of course, one can still try to consider the cyclic group $Z := \langle z \rangle$ itself. If it is not at least closed (within G), then we lose one more important feature: Z will not be locally compact any more. We can just say that Z is cyclic, but otherwise it has an essentially arbitrary group topology, inherited from the topological structure of the original group G. In particular, let us reflect back to the original Carathéodory–Fejér problem. If we pose the question on $\mathbb{Z}$, but $\mathbb{Z}$ equipped with another topology than the discrete one, then the first observation is that Ω *cannot be finite* any more. Indeed, if Ω is open with $0 \in \Omega$, then a sequence of points n_j, tending to 0 will be contained in Ω after some index j_0. Thus also the family of admissible sets Ω greatly depend on the topology itself.

One may still think that the obstacle is artificial: we can extend the definitions to arbitrary (say, symmetric) sets $\Omega \ni 0$. However, if we make it precise, the extremal Problem 1.1 involves defining a class of positive definite functions: $\mathcal{C}^c(\Omega)$, etc. Now if Ω can be arbitrary, or if the topology is not locally compact, then these definitions will split, and the nice identifications of extremal constants are lost. In general, we can always look for the class of all positive definite functions, irrespective of any further continuity restrictions, and then we can analyze the situation on $\mathbb{Z}$ as if having the discrete topology at the outset. But then to come back to the original group G will force us to consider even there the class of all positive definite functions, continuous or not, Haar measurable or not, and thus the discrete topology on G, which may have less to do with the original question. The only thing we gain is already trivial: with the discrete topology, restricting to the subgroup generated by z provides an equivalent question. As said above, this is trivial, since then all points, whence all subgroups are both open and closed, and any extension including (5.1) will be continuous.

So let us return to the original topology of G, with the inherited non-discrete one on $\mathbb{Z}$. Is it always easy to manage with an arbitrary (Hausdorff) topology on $\mathbb{Z}$, compatible with the group structure? The answer is rather to the negative. First, a result of Ajtai, Havas and Komlós [1] says that *every* Abelian group admits a bad (group-)topology, namely every infinite Abelian group can be provided with a Hausdorff group topology, in which the continuous characters do not separate the elements of the group. Furthermore, they also showed that on $\mathbb{Z}$ even a (Hausdorff, group compatible) topology can be given with the trivial character $\chi_0 \equiv 1$ being the only continuous character. Note that local compactness, on the other hand, does guarantee a much larger $\widehat{G}$, separating the points of G; and that on $\mathbb{Z}$ local compactness implies the discrete topological structure, so either we have the usual discrete topology of $\mathbb{Z}$, or we have a topology, which is not locally compact.

Therefore, considering a cyclic (sub-)group Z of G in itself, with the only assumption on its topology that it is a Hausdorff group topology, is too general. We need to take into account that the topology is inherited from a locally compact group G, and that the corresponding supporting set Ω is a symmetric, open set of G, generating a corresponding set $\Omega' := \Omega \cap Z$ and also $H = H(\Omega, z) := \phi^{-1}(\Omega') \subset \mathbb{Z}$. The forthcoming results will show that then we *can* solve the extension question – even if *not* with the trivial extension (5.1) – *preserving both continuity and positive definiteness*, and *having compact support within* Ω. This is the crux of the proof in Section 7, where the construction of course essentially uses *both the openness of* Ω *and that the topology of* G *is local compact*, as well as the information that the function, to be extended, has a *finite support*. Ensuring that we may indeed restrict to the case of finite support also needs some considerations.

Let us now also consider the case of torsion, i.e. when $o(g) = M < \infty$. This is the case of a finite cyclic group—$G = \mathbb{Z}_M$, $g = 1 \mod M$. Now if $z \in \Omega \subset G$ and $z = g^a$, then there is a subgroup of G generated by z: $G^* := \langle z \rangle \leqslant G$. If $(a, M) = 1$, then we have in fact $G^* = G \backsimeq \mathbb{Z}_M$ again, while in general $G^* \backsimeq \mathbb{Z}_m$ with $m := o(z) = M/(a, M)$.

The technical difficulty in computing even the simplest-looking extremal quantity arises from the fact that the description of $\Omega \subset G$ can considerably change when representing the whole group or the generated subgroup as it is generated by z. A (more or less) known case is when $\Omega = (-p, p) \subset \mathbb{Z}_M$, but $z = p - 1$. Although the original set $\Omega = (-p, p)$ is the nicest possible, and $Z := \langle z \rangle = G$, the change of generator $g = 1$ to $z = p - 1$ transforms the very nice set $\Omega := (-p, p)$ to the set $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$, which is just another subset of $\mathbb{Z}$, the description of which is relatively complicated.

Let us summarize here the content of Lemma 1 and Theorems 5 and 6 from [12] to describe the solution. First, an auxiliary polynomial is introduced. Namely, for arbitrary $r, q \in \mathbb{N}$ with $(r, q) = 1$, $r \leqslant q/2$, denote by $f_{r,q}(x) = 1 + 2\sum_{k=1}^{r-1} \widehat{f_{r,q}}(k) \cos(2\pi kx)$ the solution of the "first discretized Fejér problem on $\frac{1}{q}\mathbb{Z}_q$" (as posed in formula (12) of [12]), which of course has the properties that $f_{r,q}(x)$ is an even, nonnegative (on $\frac{1}{q}\mathbb{Z}_q$!) trigonometric polynomial of degree $r - 1$ satisfying also $0 < \widehat{f_{r,q}}(k) \leqslant 1$ $(k = 0, 1, \ldots, r - 1)$. V.I. Ivanov and A.I. Ivanov describe $f_{r,q}$ in somewhat more detail in [12], also explaining that it is a discrete approximation of the classical Fejér polynomial on $\frac{1}{q}\mathbb{Z}_q$ in a precise sense of approximating the zeros of the latter on the discrete points of $\frac{1}{q}\mathbb{Z}_q$. With this polynomial we have the following result.

Theorem 5.2 (*V.I. Ivanov and A.V. Ivanov*). *For all* $p \in \mathbb{N}$, $(p, q) = 1$ *and for all* h *with* $\lceil qh \rceil = p+1$*– i.e.* $p/q < h \leqslant (p+1)/h$*– the Carathéodory–Fejér extremal quantities (3.5) for the set* $\Omega := (-h, h)$ *and point* $z = p/q$ *are*

$$\mathcal{K}^{\#}_{\mathbb{T}}\left((-h,h),\frac{p}{q}\right) = \mathcal{K}^{c}_{\mathbb{T}}\left((-h,h),\frac{p}{q}\right) = \begin{cases} \dfrac{1}{2\cos(\pi/q)}\dfrac{f_{p,q}(0)}{f_{p,q}(1/q)} & \text{if } q \text{ is odd,} \\ \dfrac{f_{p,q}(0)}{2f_{p,q}(1/q)} & \text{if } q \text{ is even.} \end{cases} \tag{5.2}$$

Another example is the case when $G = \mathbb{T}^d$, and the fundamental set is $\Omega = (-1/2, 1/2)^d = \mathbb{T}^d \setminus \{\mathbf{x} : \exists j, x_j = 1/2\}$. Note that this extends the case (investigated by Arestov, Berdysheva and Berens in [2]), when $G := \mathbb{T}$ and $z \in \mathbb{T}$ is a rotation, rational or irrational, with $\Omega = \mathbb{T} \setminus \{1/2\}$. Let us recall the result [13, Theorem 5].

Theorem 5.3 (*Kolountzakis–Révész*). *Let $\Omega = (-\frac{1}{2}, \frac{1}{2})^d \in \mathbb{T}^d$. Then we have*

(i) $\mathcal{K}_{\mathbb{T}}((-\frac{1}{2}, \frac{1}{2})^d, z) = 1$ *if* $z \notin \mathbb{Q}^d$.
Moreover, if $z \in \mathbb{Q}^d$, $z = (\frac{p_1}{q_1}, \dots, \frac{p_d}{q_d})$ *with* $(p_j, q_j) = 1,\ q_j = 2^{s_j} t_j\ (s_j \in \mathbb{N}),\ t_j$ *odd for* $j = 1, \dots, d$ *and* $m := [q_1, \dots, q_d] = 2^s t,\ t$ *odd, then we have either*
(ii) $1 \leqslant s = s_1 = \cdots = s_d$, *and then* $\mathcal{K}_{\mathbb{T}}((-\frac{1}{2}, \frac{1}{2})^d, z) = \frac{1}{2}(1 + \cos \frac{2\pi}{m})$, *or*
(iii) $s = 0$ *or* $\exists j,\ 1 \leqslant j \leqslant d$ *with* $s_j < s$ *and then* $\mathcal{K}_{\mathbb{T}}((-\frac{1}{2}, \frac{1}{2})^d, z) = 1$.

Not many other cases are known exactly, and the exact computation is usually nontrivial. For further details see [13]. Still, in many cases good estimates can be obtained either directly or by duality, using the duality type result of [17].

6. Formulation of the main results

For points $z \in G$ with infinite order the problem becomes equivalent to the trigonometric polynomial extremal problem of the sort (1.3).

Theorem 6.1. *Let G be any locally compact Abelian group and let $\Omega \subset G$ be an open (symmetric) neighborhood of* 0*. Let also $z \in \Omega$ be any fixed point with $o(z) = \infty$, and denote $H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$. Then we have*

$$\mathcal{C}^c_G(\Omega, z) = \mathcal{K}^c_G(\Omega, z) = \mathcal{C}^\#_G(\Omega, z) = \mathcal{K}^\#_G(\Omega, z) = \mathcal{C}(H(\Omega, z)). \tag{6.1}$$

Remark 6.2. For G any locally compact Abelian group, $\Omega \subset G$ any open (symmetric) neighborhood of 0, and $z \in \Omega$ any fixed point with $o(z) = \infty$, we have $\mathcal{C}^c_G(\Omega, z) = \mathcal{K}^c_G(\Omega, z) = \mathcal{C}^\#_G(\Omega, z) = \mathcal{K}^\#_G(\Omega, z)$, the common value of which can thus be denoted simply by $\mathcal{K}_G(\Omega, z)$ or $\mathcal{C}_G(\Omega, z)$.

If $z \in G$ is cyclic (has torsion), the situation is analogous: then Problem 1.1 reduces to a well-defined discrete problem of the sort (1.4).

Theorem 6.3. *Let G be any locally compact Abelian group and let $\Omega \subset G$ be an open (symmetric) neighborhood of* 0*. Let also $z \in \Omega$ be any fixed point with $o(z) = m < \infty$, and denote $H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$. Then we have*

$$\begin{aligned} &\mathcal{C}^\#_G(\Omega, z) = \mathcal{C}^c_G(\Omega, z) = \mathcal{C}_m(H_m(\Omega, z)) \quad \textit{and} \\ &\mathcal{K}^\#_G(\Omega, z) = \mathcal{K}^c_G(\Omega, z) = \mathcal{K}_m(H_m(\Omega, z)). \end{aligned} \tag{6.2}$$

Remark 6.4. For G any locally compact Abelian group, $\Omega \subset G$ any open (symmetric) neighborhood of 0, and $z \in \Omega$ any fixed point with $o(z) < \infty$, we still have $\mathcal{C}^c_G(\Omega, z) = \mathcal{C}^\#_G(\Omega, z)$ and $\mathcal{K}^c_G(\Omega, z) = \mathcal{K}^\#_G(\Omega, z)$, the common value of which can thus be denoted by $\mathcal{C}_G(\Omega, z)$ and $\mathcal{K}_G(\Omega, z)$, respectively.

Note that this also holds true if $o(z) = \infty$, furthermore, than $\mathcal{C}_G(\Omega, z) = \mathcal{K}_G(\Omega, z)$ according to Remark 6.2. We will use these notations in the last section.

7. Proofs of the main results

Proof of Theorem 6.1. We present the argument only for $\mathcal{C}_G^\#(\Omega, z) = \mathcal{C}_G^c(\Omega, z) = \mathcal{C}^c(H(\Omega, z))$ (i.e. the complex case), for the real variant $\mathcal{K}_G^\#(\Omega, z) = \mathcal{K}_G^c(\Omega, z) = \mathcal{K}^c(H(\Omega, z))$ is completely similar. Once these real and complex variants are proved, a combination of Proposition 3.1(ii) and Proposition 3.5 gives that the right hand sides of these equalities are all equal.

To simplify the notation somewhat, we will write throughout this proof $H := H(\Omega, z)$.

As we trivially have $\mathcal{C}_G^\#(\Omega, z) \geqslant \mathcal{C}_G^c(\Omega, z)$, to derive the equality of these quantities of the complex setup and $\mathcal{C}(H)$ – which is the common value of $\mathcal{C}^\#(H) = \mathcal{C}^c(H)$ in view of Proposition 3.5 – it suffices to prove $\mathcal{C}_G^\#(\Omega, z) \leqslant \mathcal{C}(H)(= \mathcal{C}^\#(H) = \mathcal{C}^c(H)) \leqslant \mathcal{C}_G^c(\Omega, z)$ only.

So we are to prove only two inequalities, the first being that $\mathcal{C}_G^\#(\Omega, z) \leqslant \mathcal{C}^\#(H)$. Let us take any $f \in \mathcal{F}_G^\#(\Omega)$, and consider the subgroup $Z := \langle z \rangle \leqslant G$.

Observe that $g := f|_Z \gg 0$ on Z, for if the defining requirements (1.1) hold for all selections of the $x_j \in G$, then obviously they must also hold for all values chosen from Z. So this way we have defined a function $g \in \mathcal{F}_Z^\#((\Omega \cap Z))$. Finally, let us remark that the natural isomorphism $\eta : \mathbb{Z} \to Z$, which maps according to $\eta(k) := kz$, carries over g, defined on $Z \leqslant G$, to a function $\psi := g \circ \eta$, which is therefore positive definite on $\mathbb{Z}$, has normalized value $\psi(0) = g(0) = f(0) = 1$, and $\operatorname{supp} \psi \subset H$ for $\operatorname{supp} g \subset \operatorname{supp} f \Subset \Omega$.

From here we read that $|f(z)| \leqslant \sup \left\{ |\psi(1)| : \psi \in \mathcal{F}_{\mathbb{Z}}^\#(H) \right\} = \mathcal{C}^\#(H)$. Taking $\sup_{f \in \mathcal{F}_G^\#(\Omega)}$ on the left hand side concludes the proof of the first part.

It remains to show the inequality $\mathcal{C}^c(H) \leqslant \mathcal{C}_G^c(\Omega, z)$.

Let $\psi \in \mathcal{F}_{\mathbb{Z}}^c(H)$, so also positive definite and of finite support $S \subset H$, say. We also define the respective measure $\nu := \sum_{k \in S} \psi(k) \delta_{kz}$, where δ_s is the Dirac measure, concentrated at $s \in G$. By definition of H, $S = \operatorname{supp} \nu$ is a finite subset of $H := H(\Omega, z) := Z \cap \Omega$.

In view of Lemma 2.2(i), to ψ there exists another sequence $\theta : \mathbb{Z} \to \mathbb{C}$ of finite support $Q := \operatorname{supp} \theta$ such that $\psi = \theta \star \widetilde{\theta}$ (on $\mathbb{Z}$, with the convolution understood in $\mathbb{Z}$). Let us define the measure $\sigma := \sum_{k \in Q} \theta(k) \delta_{kz}$. Note that Q, though, is not necessarily included in H, therefore, the finite subset $\operatorname{supp} \sigma \subset Z$ is not necessarily a subset of Ω. Nevertheless, $\psi = \theta \star \widetilde{\theta}$ means that for each $k \in \mathbb{Z}$ we have $\psi(k) = \sum_{m \in Q} \theta(m) \widetilde{\theta}(k - m) = \sum_{m \in Q} \theta(m) \overline{\theta(m - k)}$ and so this vanishes outside $S \subset H$. Computing the convolution of measures now in G – and using the above-mentioned rule that $\delta_u \star \delta_v = \delta_{u+v}$ – we find

$$\sigma \star \widetilde{\sigma} = \sum_{m \in Q} \sum_{j \in Q} \theta(m) \overline{\theta(j)} \delta_{mz} \delta_{-jz} = \sum_{k \in \mathbb{Z}} \left(\sum_{m \in Q} \theta(m) \overline{\theta(m - k)} \right) \delta_{kz}$$
$$= \sum_{k \in S} \psi(k) \delta_{kz} = \nu, \tag{7.1}$$

which is supported in $S \subset Z \cap \Omega$.

Now our construction is the following. For a compact neighborhood W of 0 (to be chosen suitably later), the function $g := \chi_W \star \sigma$ is a compactly supported step function, hence is in $L^2(\mu_G)$, moreover, it has converse $\widetilde{g} = \widetilde{\chi_W \star \sigma} = \widetilde{\sigma} \star \chi_{-W}$ and thus the "convolution square" $f := g \star \widetilde{g}$, positive definite and continuous according to Lemma 2.1, will be just

$$f := g \star \widetilde{g} = \chi_W \star \sigma \star \widetilde{\sigma} \star \chi_{-W} = \chi_W \star \chi_{-W} \star \sigma \star \widetilde{\sigma} = \chi_W \star \chi_{-W} \star \nu = \sum_{k \in S} f_k,$$
$$\text{with } f_k(x) := \psi(k) \left(h \star \delta_{kz} \right)(x) = \psi(k) h(x - kz), \ h := \chi_W \star \chi_{-W}, \tag{7.2}$$

using also (2.9). Clearly $\operatorname{supp} f_k = \operatorname{supp} h + kz \subset W - W + kz$, which is a compact set itself in view of compactness of W. So because of finiteness of S we also find that $\cup_{k\in S}(W - W + kz)$ is compact, whence $\operatorname{supp} f \subset \cup_{k\in S} \operatorname{supp} f_k \subset \cup_{k\in S}(W - W + kz)$ shows that also $\operatorname{supp} f$ is compact.

If we choose now disjoint open neighborhoods $U_k \subset \Omega$ of $kz \in \Omega$ for each $k \in S$, by continuity of $(x, y) \to x - y + kz$ from $G \times G \to G$ we can take a compact neighborhood W_k of 0 with $W_k - W_k + kz \Subset U_k$, so intersecting the finitely many W_k for all $k \in S$ we arrive at a $W^* := \cap_{k\in S} W_k$, compact neighborhood of 0, such that $W^* - W^* + kz \Subset U_k \subset \Omega$ ($\forall k \in S$). Therefore if we choose some appropriate $W \Subset W^*$, then also $\operatorname{supp} f \Subset \cup_{k\in S} U_k \subset \Omega$. In all, for any such choice of W we arrive at $\operatorname{supp} f \Subset \Omega$, as needed. Our last condition on W will be that we want $kz \in W - W$ for a $k \in S$ only if $k = 0$, i.e. we require $W - W \cap \{kz : k \in S\} = \{0\}$. So it suffices to fix some open neighborhood $V \subset G$ of 0 such that $V \subset G\backslash\{kz : k \in S, k \neq 0\}$, then choose a compact neighborhood $W' \subset G$ of 0 satisfying $W' - W' \subset V$ (which can again be done according to the continuity of $(x, y) \to x - y$), and then take $W := W' \cap W^*$.

So we arrive at $f \gg 0$, $f \in C_0(G)$, $\operatorname{supp} f \Subset \Omega$, with $\operatorname{supp} f \subset \cup_{k\in S} \operatorname{supp} f_k$ and $\operatorname{supp} f_k \Subset (W - W + kz)$. It remains to compute the function values of f at 0 and at z. First, as $\operatorname{supp} h \subset W - W \subset V$, h vanishes on all kz with $k \in S\backslash\{0\}$ by construction, so from $f(0) = \sum_{k\in S} \psi(k)h(-kz)$ we get $f(0) = \psi(0)h(0) = 1 \cdot \chi_W \star \chi_{-W}(0) = \mu_G(W)(> 0)$, according to (2.11). Second, completely similarly we have $f(z) = \sum_{k\in S} \psi(k)h(z - kz) = \psi(1)\mu_G(W)$.

In all, we can take $F := \frac{1}{\mu_G(W)} f$, which then has $F(0) = 1$, too, and hence $F \in \mathcal{F}^c_G(\Omega)$, moreover, $F(z) = \psi(1)$, hence $|\psi(1)| \leqslant \mathcal{C}^c_G(\Omega, z)$. Having this for all $\psi \in \mathcal{F}^c_{\mathbb{Z}}(H)$, taking supremum yields $\mathcal{C}^c(H) \leqslant \mathcal{C}^c_G(\Omega, z)$, whence the theorem. □

Proof of Theorem 6.3. The proofs of the complex and real variants are almost identical to the preceding one, once we carefully change all references from $\mathbb{Z}$ to $\mathbb{Z}_m$, $\mathcal{C}^c(H)$ to $\mathcal{C}_m(H)$ and $\mathcal{F}^c_{\mathbb{Z}}(\Omega, z)$ to $\mathcal{F}_{\mathbb{Z}_m}(\Omega, z)$, and using Lemma 2.2(ii) instead of (i); and similarly in the real case, while noting that $Z := \langle z\rangle$ is only a finite subgroup with $Z \cong \mathbb{Z}_m$, so the natural isomorphism $\eta(k) := kz$ acts between $\mathbb{Z}_m$ and Z now. However, here we do not have the equality of the extremal quantities $\mathcal{C}_m(H)$ and $\mathcal{K}_m(H)$, as in this case only the estimates of Proposition 3.1(iii) hold. Therefore, the real and complex cases here split as formulated separately in (6.2). We spare the reader from further details of the proof. □

8. Final remarks

In view of Theorems 6.1 and 6.3, the connection between the real and complex cases in $\mathbb{Z}$, found in Proposition 3.1(ii) and (iii), extends to all LCA groups. That is, we obtain

Corollary 8.1. *Let G be any locally compact Abelian group and let $\Omega \subset G$ be an open (symmetric) neighborhood of* 0*. Let also $z \in \Omega$ be any fixed point with $o(z) = m$, and denote $H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$. Then we have*

$$\cos(\pi/m)\mathcal{C}_G(\Omega, z) \leqslant \mathcal{K}_G(\Omega, z) \leqslant \mathcal{C}_G(\Omega, z). \tag{8.1}$$

Also the "$m = \infty$" case holds true giving for torsion-free elements $z \in \Omega$ the equality

$$\mathcal{K}_G(\Omega, z) = \mathcal{C}_G(\Omega, z). \tag{8.2}$$

Let us recall that when $\mathcal{M}(H)$ or $\mathcal{C}(H)$ is known for a certain $H \subset \mathbb{Z}$, then further cases can be obtained via the following duality result.

Lemma 8.2 (*See [17]*). *Let $H \subseteq \mathbb{Z}$ be arbitrary with $\{-1, 0, 1\} \subset H$. Then denoting $H^\star := (\mathbb{N} \backslash H) \cup \{-1, 0, 1\}$ we have $\mathcal{M}(H)\mathcal{M}(H^\star) = 2$.*

Remark 8.3. By Proposition 3.1(i), it is equivalently formulated as $\mathcal{K}(H)\mathcal{K}(H^\star) = \frac{1}{2}$.

Remark 8.4. The analogous finite dimensional duality relation in $\mathbb{Z}_m$ is much easier, essentially trivial to obtain along the lines of [17]. It gives with $H_m^\star := (\mathbb{Z}_m \backslash H) \cup \{-1, 0, 1\}$

$$\mathcal{M}_m(H)\mathcal{M}_m(H^\star) = 2 \quad \text{or, equivalently} \quad \mathcal{K}_m(H)\mathcal{K}_m(H^\star) = \frac{1}{2},$$

taking into account Proposition 3.1(i) again.

It is explained in [13] in the context of the groups $\mathbb{R}^d$ and $\mathbb{T}^d$ that description of the $\mathcal{C}_G(\Omega, z)$ problems by the Carathéodory–Fejér type extremal problems on $\mathbb{Z}$ or $\mathbb{Z}_m$ automatically extends these duality results to the more general situation. Regarding Problem 1.1 we have

Corollary 8.5. *For any LCA group G, open set $\Omega \subseteq G$ and $z \in \Omega$ we have $\mathcal{K}_G(\Omega)\mathcal{K}_G(\Omega^\star) = \frac{1}{2}$ where Ω^* is any symmetric open set with $Z \cap \Omega \cap \Omega^* = \{0, z, -z\}$ and $(\Omega \cup \Omega^*) \supset Z$, with $Z := \langle z \rangle$, i.e. $\{kz : k \in \mathbb{Z}\}$ or $\{kz : k \in \mathbb{Z}_m\}$, respectively.*

Remark 8.6. When $\mathcal{C}_G(\Omega, z) = \mathcal{K}_G(\Omega, z)$ and $\mathcal{C}_G(\Omega^\star, z) = \mathcal{K}_G(\Omega^\star, z)$ – in particular when $o(z) = \infty$ – we have the analogous formula $\mathcal{C}_G(\Omega, z)\mathcal{C}_G(\Omega^\star, z) = 2$ for the complex quantities. However, it fails whenever $\mathcal{C}_G(\Omega, z) = \mathcal{K}_G(\Omega, z)$ or $\mathcal{C}_G(\Omega^\star, z) = \mathcal{K}_G(\Omega^\star, z)$ does so.

This of course covers the corresponding results for $\mathbb{R}^d$ and $\mathbb{T}^d$ given in [13, Corollary 4.7].

It would be interesting – perhaps by a direct argument extending that in [17] – to derive this duality result without relying on Theorems 6.3 and 6.1.

Acknowledgments

This research was supported in part by the Hungarian National Foundation for Scientific Research, Project #'s K-100461, NK-104183 and K-109789.

References

[1] M. Ajtai, I. Havas, J. Komlós, Every group admits a bad topology, in: P. Erdős, et al. (Eds.), Studies in Pure Mathematics—A Volume Dedicated to the Memory of Paul Turán, Springer, 1983, pp. 21–34.
[2] V.V. Arestov, E.E. Berdysheva, H. Berens, On pointwise Turán's problem for positive definite functions, East J. Approx. 9 (1) (2003) 31–42.
[3] R.P. Boas Jr., M. Kac, Inequalities for Fourier transforms of positive functions, Duke Math. J. 12 (1945) 189–206.
[4] C. Carathéodory, Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen, Rend. Circ. Mat. Palermo 32 (1911) 193–217.
[5] J. Egerváry, O. Szász, Einige Extremalprobleme in bereiche der trigonometrische Polynome, Math. Z. 27 (1928) 641–692.
[6] W. Ehm, T. Gneiting, D. Richards, Convolution roots of radial positive definite functions with compact support, Trans. Amer. Math. Soc. 356 (11) (2004) 4655–4685.
[7] L. Fejér, Über trigonometrische Polynome, J. Angew. Math. 146 (1915) 53–82.
[8] L. Fejér, Gesammelte Arbeiten I–II, Akadémiai Kiadó, Budapest, 1970.
[9] D.V. Gorbachev, A.S. Manoshina, The extremal Turán problem for periodic functions with small support. (Russian), in: Proceedings of the IV International Conference Modern Problems of Number Theory and its Applications, Tula, 2001. Chebyshevskiĭ Sb. 2, 2001, pp. 31–40.

[10] D.V. Gorbachev, A.S. Manoshina, The Turán extremal problem for periodic functions with small support and its applications. (English. Russian orginal), Math. Notes 76 (5) (2004) 640–652; translation from Mat. Zametki 76 (5) (2004) 688–700.
[11] G. Herglotz, Über Potenzreihen mit positivem reelen Teil im Einheitskreis, Leipz. Ber. 63 (1911) 501–511.
[12] V.I. Ivanov, A.V. Ivanov, Turan problems for periodic positive definite functions, Ann. Univ. Sci. Budapest. Sect. Comput. 33 (2010) 219–237.
[13] M.N. Kolountzakis, Sz.Gy. Révész, On pointwise estimates of positive definite functions with given support, Canad. J. Math. 58 (2) (2006) 401–418.
[14] M.N. Kolountzakis, Sz.Gy. Révész, Turán's extremal problem for positive definite functions on groups, J. Lond. Math. Soc. 74 (2006) 475–496.
[15] S.V. Konyagin, I. Sharplinski, Character Sums with Exponential Functions and their Applications, Cambridge University Press, Cambridge, 1999.
[16] M. Mathias, Über positive Fourier-Integrale, Math. Z. 16 (1923) 103–125. (German).
[17] Sz.Gy. Révész, Extremal problems and a duality phenomenon, in: A.G. Law, C.L. Wang (Eds.), Approximation, Optimization and Computing, Elsevier, 1990, pp. 279–281.
[18] Sz.Gy. Révész, A Fejér-type extremal problem, Acta Math. Hungar. 57 (3–4) (1991) 279–283.
[19] Sz.Gy. Révész, On Beurling's prime number theorem, Period. Math. Hungar. 28 (1994) 195–210.
[20] Sz.Gy. Révész, Minimization of maxima of nonnegative and positive definite cosine polynomials with prescribed first coefficients, Acta Sci. Math. (Szeged) 60 (1995) 589–608.
[21] Sz.Gy. Révész, The least possible value at zero of some nonnegative cosine polynomials and equivalent dual problems, J. Fourier Anal. Appl. (1995) 485–508. Kahane Special Issue.
[22] Sz.Gy. Révész, Turán's extremal problem on locally compact abelian groups, Anal. Math. 37 (1) (2011) 15–50.
[23] W. Rudin, Fourier Analysis on Groups, in: Interscience Tracts in Pure and Applied Mathematics, Vol. 12, Interscience Publishers (a division of John Wiley and Sons), New York, London, 1962, p. ix+285.
[24] I.Z. Ruzsa, Connections between the uniform distribution of a sequence and its differences, in: G. Halász (Ed.), Colloquia Mathematica Societatis János Bolyai, Vol. 34, North Holland, 1981, pp. 1419–1443.
[25] G. Szegő, Coeffizientenabschatzungen bei ebenen und räumlichen harmonischen Entwicklungen, Math. Ann. 96 (1926–1927) 601–632.
[26] G. Szegő, Orthogonal Polynomials, Vol. XXXIII, fourth ed., AMS Colloquium Publications, Providence, Rhode Island, 1975, p. 432.
[27] A. Weil, L'intégration dans les Groupes Topologiques et Ses Applications, Gauthiers-Villars, Paris, 1938.