Positivity 12 (2008), 691–709

1385-1292/040691-19, *published online* May 1, 2008
DOI 10.1007/s11117-008-2164-4

Positivity

Positive bases in spaces of polynomials

Bálint Farkas and Szilárd Gy. Révész

Abstract. For a nonempty compact set $\Omega \subseteq \mathbb{R}$ we determine the maximal possible dimension of a subspace $X \subseteq \mathcal{P}_m(\Omega)$ of polynomial functions over Ω with degree at most m which possesses a *positive basis*. The exact value of this maximum depends on topological features of Ω, and we will see that in many of the cases m can be achieved. Whereas only for low m or finite sets Ω it is possible that we have a subspace X with positive basis and with $\dim X = m+1$. Hence there is no Ω for which a positive basis exists in $\mathcal{P}_m$ for all $m \in \mathbb{N}$.

Mathematics Subject Classification (2000). Primary 41A17; Secondary 30E10, 41A44.

Keywords. Positive polynomials, positive bases, Bernstein-Lorentz representation, Bernstein-Lorentz degree.

1. Introduction

Consider the interval $I = [-1, 1]$ and the space $\mathcal{P}_n$ of algebraic polynomials with degree at most $n \in \mathbb{N}$ over I. In many problems, e.g., in approximation theory, it is desirable to represent a given polynomial $p \in \bigcup_n \mathcal{P}_n =: \mathcal{P}$ in terms of positive linear combinations of preliminary given, *positive* polynomials (positive means here and in the following that the polynomial is pointwise nonnegative). For example consider the Bernstein polynomials

$$e_{nk}(x) := (1-x)^k(1+x)^{n-k} \quad \text{for } k = 0, \dots, n,$$

which are evidently positive on $[-1, 1]$. The following is classical.

Proposition. *The system $E_n := \{e_{n0}, e_{n1}, \dots, e_{nn}\}$ is a basis of $\mathcal{P}_n$.*

The 2[nd] named author was supported in part by the Hungarian National Foundation for Scientific Research, Project #s T-049301, T-049693 and K-61908.
This work was accomplished during the 2[nd] author's stay in Paris under his Marie Curie fellowship, contract # MEIF-CT-2005-022927.

Proof. For the number of elements $\#E_n = n+1$ equals $\dim \mathcal{P}_n$, it suffices to show that E_n generates $\mathcal{P}_n$. For this purpose, we can represent the monomials x^k as

$$\begin{aligned} x^k &= \left(\tfrac{1}{2}[(1+x)-(1-x)]\right)^k \\ &= \left(\tfrac{1}{2}[(1+x)-(1-x)]\right)^k \left(\tfrac{1}{2}[(1+x)+(1-x)]\right)^{n-k}, \end{aligned}$$

which, when expanded, yields a desired representation in terms of e_{nk}. From this the assertion follows. $\square$

Now back to the original question. Does every positive element $p \in \mathcal{P}$ have a positive representation

$$p(x) = \sum_{k=0}^{N} a_k (1-x)^k (1+x)^{N-k} = \sum_{k=0}^{N} a_k e_{Nk}(x) \quad \text{with } a_0, a_1, \dots, a_N \geq 0 \quad (1)$$

for some $N \in \mathbb{N}$? The affirmative answer is due to Bernstein [2], see also [6, vol. II p. 83, Aufgabe 49].

Theorem (Bernstein)**.** *Every polynomial positive on* $[-1, 1]$ *has a positive representation* (1).

However, the precise truth is that the degree N of the representation (1) is in general not equal to the ordinary algebraic degree $n = \deg p$ of p. The minimal such N is called the *Lorentz-degree* $L(p)$ of p. For a positive polynomial p Bernstein's theorem is thus equivalent to $L(p) < \infty$. An example for the use of positive representations in approximation theory is the improvement of Bernstein, Schur and Markov type inequalities for polynomials, which can be deduced by replacing the ordinary degree by the Lorentz degree; for the details see [12] and the references therein.

Among the various known estimations of the Lorentz degree, the first, and probably the simplest, is the following.

Theorem (Lorentz [4]; see also [13])**.** *Assume that the positive polynomial* p *does not vanish in the unit disk. Then* $L(p) = \deg p$.

On the other hand, it is well-known that for $p \in \mathcal{P}_n$ positive the Lorentz degree can be arbitrarily large. In other words, $\operatorname{span}\{e_{N0}, e_{N1}, \dots, e_{NN}\}$ does not contain $\mathcal{P}_n$ for any $N \in \mathbb{N}$. Thus in particular E_N is not a positive basis for $\mathcal{P}_N$, where under a *positive basis* in an arbitrary ordered Banach space we understand the following.

Definition. *Let* X *be an ordered Banach space with positive cone* X_+*. A Schauder basis* $\{b_j\}_{j\in\mathbb{N}}$ *is called a* positive basis *of* X *if* $b_j \in X_+$ *for all* $j \in \mathbb{N}$*, and non-negative elements* $x \in X_+$ *have positive representation with respect to this basis, i.e.,*

$$X_+ = \left\{ x = \sum_{j=1}^{\infty} \lambda_j b_j : \lambda_j \geq 0 \text{ for each } j \in \mathbb{N} \right\} \quad \textit{holds.}$$

Geometric and order theoretic properties can be characterised by existence of positive bases in finite dimensional ordered Banach spaces. Suppose that $b_1, \dots, b_n$ is a positive basis in X. Then the positive cone X_+ is generating, i.e., we have $X = X_+ - X_+$. Further, the set $B = \operatorname{conv}\{b_1, \dots, b_n\}$, which is a base for the cone X_+, is a simplex and of course (linearly) compact. Thus by the Choquet-Kendall Theorem, see [5, Thm. 1.3.11.], we obtain that X is actually a vector lattice. Conversely, suppose that X is a vector lattice with a closed, generating cone X_+. It is easy to see that X_+ has a base, which by the Choquet-Kendall Theorem is a compact simplex. Actually, B is an $(n-1)$-dimensional simplex because X_+ is generating. Therefore the n extremal points $b_1, \dots, b_n$ of B form a positive basis. We see that, essentially as the consequence of the Choquet-Kendall Theorem, the following holds, see [9, Sec. 1.] (cf. also [5, p. 9 and 32]).

Theorem (Choquet-Kendall). *Let X be a finite-dimensional ordered Banach space with closed, generating cone X_+. Then X possesses a positive basis if and only if X is a vector lattice.*

Motivated by the fact that E_n is not a positive basis in $\mathcal{P}_n$, one may ask the question, whether $\mathcal{P}_n$ possesses a positive basis at all. In view of the above theorem, the following result of Polyrakis to this question is little surprising. Polyrakis proves actually far more general results, from which also the statement below follows easily (see Proposition 3.2, Corollary 3.3 and Example 4.2 of [9]).

Proposition 1 (Polyrakis). *Unless $n = 0, 1$, the space $\mathcal{P}_n$ does not have a positive basis at all.*

We are primarily interested in positive bases in subspaces Y of $\mathcal{P}_n[-1, 1]$, the polynomials considered on $[-1, 1]$, that is in the question whether the ordered subspace Y is actually a lattice subspace of $C[-1, 1]$ (see definition and explanations below).

Definition. *We say that Y is a* lattice-subspace *of the ordered Banach space X if Y is a subspace, and it is also a lattice with the induced ordering from X.*

For finite dimensional subspaces X of $C[-1, 1]$ which inherit ordering from $C[-1, 1]$, the existence of a positive basis in X is equivalent to the fact that X is actually a lattice-subspace of $C[-1, 1]$. Studying lattice-subspaces of $C[-1, 1]$ can be motivated by the following analogue of the Banach-Mazur Theorem proved by Polyrakis [8, Sec.4.]. He has shown that every separable Banach lattice is order-isomorphic to a lattice-subspace of $C[-1, 1]$.

As regards positive bases and lattice-subspaces, structure theory of lattice-subspaces in spaces of continuous functions, and geometric characterisations of existence of positive bases we refer the reader to [1, 8–11].

Actually Polyrakis's negative result, Proposition 1, is the starting point of this paper. We study the following problem. For $X \subseteq \mathcal{P}_m$ a subspace with positive basis, we are interested in how large the dimension $\dim X$ can be. As observed above the maximal possible dimension must be less than or equal to m, the exact answer will be given in Section 2.

We also consider the more general case when the polynomials are considered over a given nonempty compact subset $\Omega \subseteq \mathbb{R}$. For the determination of maximal dimension of a subspace of $\mathcal{P}_m$ with positive basis we have to take into account some topological-geometrical features of the set Ω. This will be carried out in Section 3.

We begin with some preparation and recall the necessary results from [9].

Subspaces of $C(\Omega)$ with positive bases

We denote by Ω a nonempty compact set in $\mathbb{R}$, and by $C(\Omega)$ the space of real valued, continuous functions defined on Ω. We consider $C(\Omega)$ (partially) ordered by the pointwise ordering. The notation $\mathcal{P}_n$ stands for the space of polynomials (polynomial functions) p with $\deg p \leq n$ restricted to the set Ω. The ordering of the polynomials is the one inherited from $C(\Omega)$. As Ω is fixed once for all, we will simple use the notation $\mathcal{P}_n$ instead of $\mathcal{P}_n(\Omega)$, as well $\mathcal{P}$ for $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n(\Omega)$.

Let $\{b_j\}$ be a (finite or infinite) sequence of $C(\Omega)$. Following [9] we can introduce the next notions. If t is a point of Ω and there exists $k \in \mathbb{N}$ such that $b_k(t) \neq 0$ and $b_j(t) = 0$ for each $j \neq k$, then we call that t a *k-node* (or simply a *node*) of the sequence $\{b_j\}$. If for each k there exists an k-node t_k of $\{b_j\}$, we shall say that $\{b_j\}$ is a *sequence* of $C(\Omega)$ *with nodes* and also that $\{t_j\}$ is a *sequence of nodes* of $\{b_j\}$. If $\{b_j\}$ is also a positive basis, then it is a *positive basis with nodes.*

The following results form the background for our paper (see Theorem 2.1 and Propositions 2.2 and 2.4 of [9] or Theorem 1 of [11]) and help to find node systems. Note that Polyrakis has proved these for arbitrary compact metric spaces Ω.

Theorem 2 (Polyrakis)**.** *Let X be a subspace of $C(\Omega)$ and $\{b_r\}$ be a sequence of X consisting of positive functions.*

i) If $\{b_r\}$ is a positive basis of X, then for each k there exists a sequence $\{\omega_{k\nu}\}$ of Ω such that for each $\nu \in \mathbb{N}$ we have

$$0 \leq \sum_{i=1, i\neq k}^{\nu} \frac{b_i(\omega_{k\nu})}{b_k(\omega_{k\nu})} < \frac{1}{\nu}.$$

Therefore $\lim_{\nu\to\infty} \frac{b_i(\omega_{k\nu})}{b_k(\omega_{k\nu})} = 0$ for each $i \neq k$. Moreover, there exists a sequence t_k of Ω where any t_k is a limit point of the sequence ω_{kn}, so that $b_m(t_k) = 0$ for any $m \neq k$.

*ii) If X is an n-*dimensional* subspace of $C(\Omega)$ and $b_1, b_2, \ldots, b_n \in C(\Omega)$ are positive functions such that for each $1 \leq k \leq n$ there exists a sequence $\{\omega_{k\nu}\}$ of Ω satisfying*

$$\lim_{\nu\to\infty} \frac{b_i(\omega_{k\nu})}{b_k(\omega_{k\nu})} = 0 \quad \textit{for each } i \neq k,$$

then $\{b_1, \ldots, b_n\}$ is a positive basis of X.

Let $a = \min \Omega$, $b = \max \Omega$. Unless Ω is finite, which leads to an equivalent question of considering the space $\mathbb{R}^K$, where $K := \#\Omega$ the number of elements in Ω, we have $\dot{\Omega} \neq \emptyset$ (where $\dot{H}$ stands for the set of limit points of the set H). Unless otherwise stated, this will be assumed from now on. For the finite case of $\mathbb{R}^K$, see for example [8] (see also Proposition 8). Suppose that X is a subspace of $\mathcal{P}_m$ with positive basis $\{b_1, b_2, \ldots, b_n\}$. For each $k = 1, 2, \ldots, n$ let $\omega_{k\nu} \longrightarrow t_k$ be a sequence in Ω with the properties formulated above in Theorem 2, that is

$$\lim_{\nu\to\infty} \frac{b_i(\omega_{k\nu})}{b_k(\omega_{k\nu})} = 0 \ \text{ for each } i \neq k. \tag{2}$$

Note that, since the functions b_i are polynomials, (2) entails that b_i vanishes in a strictly greater order at t_k than b_k for $i \neq k$. In particular we have $t_k \neq t_i$ for each $i \neq k$. We now can write that

$$b_i(t) = \prod_{k=1}^{n} (t - t_k)^{r_{ik}} u_i(t),$$

with some $u_i \in \mathcal{P}_m$, not vanishing at any of the points t_j, where r_{ik} is the order of t_k as root of the polynomial b_i. (If t_k is not a root of b_i, then $r_{ik} = 0$.) By the above we have $r_{ik} > r_{kk}$, whereas the latter may be 0.

Using the above facts, we are going to estimate the degrees of b_k from *below* in terms of n, which, conversely, will give us an *upper* estimate of n in terms of the degrees of b_k. Thus we can estimate the dimension of a subspace $X \subset \mathcal{P}_m$ with positive basis.

2. Positive bases in subspaces of polynomials on intervals

Assume now that $\Omega = [a, b]$ and that $X \subseteq \mathcal{P}_m$ is subspace with positive basis $b_1, b_2, \ldots, b_m$. By the consequence of Polyrakis's result we have $t_1, t_2, \ldots, t_n \in [a, b]$ such that

$$b_i(t) = \prod_{k=1}^{n} (t - t_k)^{r_{ik}} u_i(t), \tag{3}$$

where r_{ik} are *even* natural numbers whenever the corresponding t_k satisfies $a < t_k < b$, because otherwise b_k would change its sign at t_k.

As explained at the end of Section 1, we want to estimate the degrees of b_k from below. Let us do it for the sake of clarity in detail. We therefore distinguish three cases depending on whether $\{t_1, t_2, \ldots, t_n\} \cap \{a, b\}$ has $0, 1$ or 2 elements.

Consider first the case $a = t_1 < t_2 < \cdots < t_n = b$. Then for $k = 1, n$ we obtain the following from (3)

$$\deg b_1 = \deg u_1 + \sum_{i=1}^{n} r_{1i} \geq 0 + r_{11} + 2\sum_{i=2}^{n-1} r'_{1i} + r_{1n} \geq 0 + 2(n-2) + 1 = 2n - 3,$$

$$\deg b_n = \deg u_n + \sum_{i=1}^{n} r_{ni} \geq 0 + r_{n1} + 2\sum_{i=2}^{n-1} r'_{ni} + r_{nn} \geq 1 + 2(n-2) + 0 = 2n - 3.$$

For $1 < k < n$ we have

$$\deg b_k = \deg u_k + \sum_{i=1}^{n} r_{ki} \ge 0 + r_{k1} + 2\sum_{\substack{i=2\\ i\neq k}}^{n-1} r'_{ki} + r_{kk} + r_{kn} \ge 1 + 2(n-3) + 0 + 1 = 2n-4.$$

Where in all of these estimates, we used that $r_{kk} \ge 0$, $r_{ki} > 0$ for $k \neq i$ and $r_{ki} = 2r'_{ki} > 0$ if $1 < k < n$, which again follow from the considerations at the end of the Section 1

Similar arguments can be repeat in the case, when no t_l coincides with a or b, so all the exponents r_{ki} with $k \neq i$ must be even and positive. Thus we can conclude from (3)

$$\deg b_k = \deg u_k + \sum_{i=1}^{n} r_{ki} \ge 0 + 2\sum_{\substack{i=1\\ i\neq k}}^{n} r'_{ki} + r_{kk} \ge 2(n-1) + 0 = 2(n-2).$$

Finally, a very same argumentation can be executed also for the cases when the set $\{a,b\} \cap \{t_1, t_2, \ldots, t_n\}$ has exactly 1 element; in that case we find again $\max_{k=1,2,\ldots,n} \deg b_k \ge 2n-2$.

Therefore we conclude the inequality

$$\max_{k=1,\ldots,n} \deg b_k \ge 2n-3 \tag{4}$$

for all possible cases with respect to $\#\{a,b\} \cap \{t_1, t_2, \ldots t_n\}$ being 0, 1 or 2. As $b_i \in \mathcal{P}_m$, the lower bound (4) for the degree has to match with the upper bound m of the degree. Therefore the maximal possible value for n is

$$n = \left[\frac{m+3}{2}\right].$$

An easy construction shows this to be sharp. Let $n = [\frac{m+3}{2}]$ and take n different points $a = t_1 < t_2 < \cdots < t_n = b$ of $\Omega = [a,b]$ and define

$$\begin{aligned} b_1(t) &= (t_n - t)\prod_{i=2}^{n-1}(t-t_i)^2, \qquad (t_1 = a) \\ b_k(t) &= (t-t_1)(t_n-t)\prod_{\substack{i=2\\ i\neq k}}^{n-1}(t-t_i)^2, \qquad (k = 2,\ldots,n-1) \\ b_n(t) &= (t-t_1)\prod_{i=2}^{n-1}(t-t_i)^2, \qquad (t_n = b) \end{aligned} \tag{5}$$

and take X to be the subspace of $\mathcal{P}_m$ generated by the b_k's. Then $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of X, because if we suppose that $p = \sum_{i=1}^{n} \lambda_i b_i$ we have that $p(t_i) = \lambda_i b_i(t_i)$, therefore p is positive if and only if $\lambda_i \ge 0$ for each i. So we have the following

Theorem 3. *The maximal possible dimension of a subspace X of $\mathcal{P}_m$ with a positive basis is $n = [\frac{m+3}{2}]$.*

Remark 4. *If $X \subseteq \mathcal{P}_m$ is a subspace of $\mathcal{P}_m$ with positive basis $\{b_1, b_2, \ldots, b_n\}$, where $n = [\frac{m+3}{2}]$, then there exist a node system $t_1, t_2, \ldots, t_n \in [a, b]$ with $t_i \neq t_j$ for each $i \neq j$ such that the basis elements are either of the form (5) with $a = t_1 < \cdots < t_n = b$, or one of the following forms: either with $a = t_1 < \cdots < t_n \leq b$*

$$b_1(t) = \prod_{k=2}^{n} (t - t_k)^2, \quad \text{and} \quad b_i(t) = (t - t_1) \prod_{\substack{k=2 \\ k \neq i}}^{n} (t - t_k)^2, \quad (i = 2, \ldots, n);$$

or with $a \leq t_1 < \cdots < t_n = b$

$$b_i(t) = (t_n - t) \prod_{\substack{k=1 \\ k \neq i}}^{n-1} (t - t_k)^2, \quad (1 \leq i \leq n-1) \quad \text{and} \quad b_n(t) = \prod_{k=1}^{n-1} (t - t_k)^2;$$

or, only in case of even m, with $a \leq t_1 < \cdots < t_n \leq b$

$$b_i(t) = \prod_{\substack{k=1 \\ k \neq i}}^{n} (t - t_k)^2, \quad (1 \leq i \leq n).$$

3. Positive bases of polynomials on compact sets of $\mathbb{R}$

In this section we suppose that Ω is a compact subset of $\mathbb{R}$. Our aim is to determine the minimum of possible maximal degrees of systems of basis functions b_j $(j = 1, \ldots, n)$ of some subspace $X \subset \mathcal{P}_m$ on Ω. The answer will be more delicate than in the case of an interval.

We will always write $a := \min \Omega$ and $b := \max \Omega$, i.e., $\operatorname{conv} \Omega = [a, b]$. We also need to introduce some auxiliary terminology.

Definition. *Let $t = (t_j)_{j=1}^n$, ordered naturally as $a \leq t_1 < t_2 < \ldots, t_n \leq b$, be a node system for the basis functions b_j, $(j = 1, \ldots, n)$ of the subspace X in $\mathcal{P}_m$. For technical reasons we take now $t_0 := -\infty$ and $t_{n+1} := +\infty$, and we define the* type *of the node sequence t by a sequence $\omega := (\omega_j)_{j=0}^n$ of 0's and 1's, of length $n+1$, so that $\omega_j = 0$ if $\Omega \cap (t_j, t_{j+1}) = \emptyset$, and $\omega_j = 1$ if $\Omega \cap (t_j, t_{j+1}) \neq \emptyset$. Therefore, there is $x_j \in \Omega \cap (t_j, t_{j+1})$ if and only if $\omega_j = 1$. The notation $x(t)$ or x_j will always refer to such a sequence from now on.*

As in (3) for a given system of nodes t we can write

$$b_j(x) = \prod_{i=1}^{n} (x - t_i)^{r_{ji}} u_j(x) \qquad (j = 1, \ldots, n),$$

with $r_{ji} \geq 1$ for all $j \neq i$ and u_j having no zeroes at points of t.

Our task is to determine, in terms of geometrical-topological features of the set Ω, the minimal m so that all b_j are contained in $\mathcal{P}_m$. This means that we consider all possible positive basis $b_1, b_2, \ldots, b_n$ and minimize the occurring maximal degree of basis elements. Obviously, it is equivalent to considering all basis $b_1, b_2, \ldots, b_n$ and then all nodes t for them, or conversely, all node systems t and then all basis systems $b_1, b_2, \ldots, b_n$ having these as nodes. Therefore, our goal is the determination of

$$\begin{aligned} d_n(\Omega) &:= \min\left\{\max_{j=1,\ldots,n} \deg b_j \ : \ b_1,\ldots b_n \text{ is a positive basis}\right\} \\ &= \min\left\{\max_{j=1,\ldots,n} \deg b_j \ : \ (b_j)_{j=1}^n \text{ is a p.b. with nodes } t\in\Omega^n\right\} \\ &= \min_{t\in\Omega^n}\min\left\{\max_{j=1,\ldots,n} \deg b_j \ : \ (b_j)_{j=1}^n \text{ is a p.b. with nodes } t\right\}. \end{aligned} \tag{6}$$

Before computing $d_n(\Omega)$, we consider the closely related, auxiliary question of finding

$$\min\left\{\deg b \ : \ 0\le b|_\Omega,\ b(x)=\prod_{i=1}^n (x-t_i)^{r_i}u(x),\ r_i\ge 1 \ \text{ for each } i\right\} \tag{7}$$

for a fixed system of nodes $t\subset\Omega$. Simplifying this we consider a modified problem, with a particular subset $\Omega' := t\cup x(t)$ replacing Ω in the nonnegativity condition:

$$\min\left\{\deg b \ : \ 0\le b|_{\Omega'},\ b(x)=\prod_{i=1}^n (x-t_i)^{r_i}u(x),\ r_i\ge 1 \ \text{ for each } i\right\}; \tag{8}$$

Or just as a function of ω:

$$\begin{aligned} \tau(\omega) := \min\{&\deg p:\ p\not\equiv 0,\ p(t_i)=0 \text{ for each } i, \\ &p(x_j)\ge 0 \text{ for all } j \text{ with } \omega_j = 1\}. \end{aligned} \tag{9}$$

Note that according to the requirement $p(t_i)=0$, we always must have $\tau(\omega)\ge n$. Although in (8) the quantity seems to depend on particular values of t_i and x_j, in the reworded formulation (9) we have already retreated in notation to the mere mentioning of ω. That the quantities in (8) and (9) are equal is an easy argument: here we spare the reader the details, as this will also follow from our considerations below, see Corollary 6. First we determine the value of $\tau(\omega)$ for a given $\omega=\omega(t)$. We need the following.

Definition. *A* block *of zeros or ones in ω is a maximal family of neighboring zeros respectively ones. A zero block is* inner*, if in ω there are ones before and after it. For any finite sequence $\omega\in\{0,1\}^{n+1}$ for some $n\in\mathbb{N}$ set*

$$\begin{aligned} N(\omega) &:= \#\{j \ : \ \omega_j = 1\}, \\ K(\omega) &:= \#\ \textit{inner blocks of 0 digits of odd length}, \\ \nu(\omega) &:= \prod_{i=0}^n (1-\omega_i) = \begin{cases} 1 & \textit{if } \omega_i = 0 \textit{ for each } i \\ 0 & \textit{if } \omega_i = 1 \textit{ for at least one } i. \end{cases} \end{aligned}$$

So, e.g., if $n=10$ and $\omega=(1,0,0,1,0,1,1,0,0,0,1)\in\{0,1\}^{11}$, then $N(\omega)=5$ and $K(\omega)=2$, and for $\omega=(0,1,0,0,1,0,1,1,0,0,0,1)\in\{0,1\}^{12}$ the same values of N and K occur, too. For both cases $\nu(\omega)=0$. Note that in view of the condition that K is the number of *inner* zero blocks of odd length, we must have $N(\omega) > K(\omega)$, unless both numbers are zero.

Lemma 5. *We have $\tau(\omega) = n-1+N(\omega)-K(\omega)+\nu(\omega)$. Moreover, there is always an extremal polynomial of minimal degree $\deg p = \tau(\omega)$, so that p is the product of linear factors vanishing at the nodes t_i $(i = 1, \ldots, n)$ only.*

Proof. **Step 1.** First of all, consider the case of $n = 1$, when there is only one node t_1, where p must vanish. That entails that for any polynomial p satisfying the conditions of (9), $\deg p \geq 1$ and $(x - t_1)|p(x)$, in the sense of polynomial division in $\mathbb{R}[x]$. There is the case $\omega = (0, 0)$, when $p(x) = x-t_1$ suffices; and it also suffices for $\omega = (0, 1)$, too, while for $\omega = (1, 0)$ the opposite, i.e. $-(x - t_1)$ will suffice. Finally, no linear function, vanishing at t_1, can remain positive on both sides of t_1, so $\tau(\omega)$ must be at least 2 if $\omega = (1, 1)$: and indeed, $(x - t_1)^2$ is appropriate. In all these cases $K(\omega) = 0$, and it is easy to check that the formula, stated by the lemma, holds true for $n = 1$: also, the given minimal degree polynomials are products of zero factors vanishing at the only node t_1.
Step 2. Let now $n = 2$. Now $\tau(\omega) \geq 2$ and $p_0(x) := (x - t_1)(x - t_2)|p(x)$. So we have $\tau(\omega) = 2$ for $\omega = (0, 0, 0)$, $\omega = (1, 0, 1)$, $\omega = (1, 0, 0)$ and $\omega = (0, 0, 1)$, as p_0 itself shows, and also for $\omega = (0, 1, 0)$, as shown by $-p_0$. On the other hand, with exactly two neighboring 1's in the sequence ω, $c \cdot p_0$ itself will not suffice, as it necessarily changes sign at t_1 and at t_2, and only there. So at two intermediate points x_j around any of these nodes the sign of $c \cdot p_0$ is always the opposite. On the other hand, $(x - t_1)(x - t_2)^2$ works for $\omega = (0, 1, 1)$, and symmetrically $(x - t_1)^2(t_2 - x)$ works for $\omega = (1, 1, 0)$, so for these two sequences $\tau(\omega) = 3$. At last, consider $\omega = (1, 1, 1)$. We claim that $\tau(\omega) > 3$, (and whence $\tau(\omega) = 4$, as shown by $(x-t_1)^2(x-t_2)^2$). But it is easy to see that $q(x) = c(x-\xi)(x-t_1)(x-t_2)$ cannot work for any choice of ξ and $c \neq 0$, since $c(x - \xi)$ can not have alternating signs at the points $x_0 < x_1 < x_2$. It follows $\tau((1, 1, 1)) = 4$. Again, it is easy to check that the asserted formula holds for $n = 2$, and of course the above given extremal polynomials are just products of factors of the form $(x-t_1)$ and $(x-t_2)$.
Step 3. The very last case is easily generalized to the cases when, for $n \in \mathbb{N}$ arbitrary, our sequence of ω is the identically 1 sequence, $\omega = \mathbf{1} = (1, 1, \ldots, 1) \in \{0, 1\}^{n+1}$. We assert that then $\tau(\mathbf{1}) = 2n$. The cases $n = 1, 2$ having already been obtained, our proof is by induction. Let $n > 2$ and assume the assertion for all $n' < n$.

Let $p(x)$ be a polynomial in the definition (9) of τ. We distinguish two cases: first assume that there exists some x_j with $p(x_j) = 0$. Then with any neighboring node – say, if $j \neq 0$, we can take t_j, otherwise t_1 for x_0 – we can consider the new polynomial $\widetilde{p}(x) := p(x)/((x - t_j)(x - x_j))$, or $\overline{p}(x) := p(x)/((x - t_j)(x - x_j))$ if $j = 0$. Since these cases are similar, we describe the case of $\widetilde{p}$ only. This polynomial vanishes for all t_i unless $i = j$, and is still nonnegative at all x_i's, save $i = j$, because all the other x_i's are outside the interval (x_j, t_j), and thus the denominator of $\widetilde{p}(x)$ remains positive there. At t_j and x_j we neither know, nor we are interested any more in the sign of $\widetilde{p}(x)$. But the above means that $\widetilde{p}(x)$ is a proper polynomial for $\widetilde{t} = (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$ and intermediate points $\widetilde{x} := x(\widetilde{t}) = (x_0, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots,)$ of length one less, and with $\omega = \mathbf{1} \in \{0,1\}^n$ now. So the induction hypothesis applies with $n' = n-1$, and $\deg \widetilde{p} \geq 2n'$, hence $\deg p \geq 2n$.

Consider now the second case, when for all $i = 0, \ldots, n$, we have $p(x_i) > 0$. Then in the open interval (x_i, x_{i+1}) p must have at least two zeroes, (when zeroes are counted with multiplicity), since it has at least one (at t_i), and only one simple zero does not allow for the strictly positive values at both ends of the interval. But that means that altogether p has at least $2n$ zeroes, proving again that $\deg p \geq 2n$, as claimed.

On the other hand, $\prod_i (x - t_i)^2$ is obviously a proper polynomial in the definition of $\tau(\omega)$ for all ω, (furthermore, it is the product of zero factors $(x - t_i)$, too) hence $\tau(\mathbf{1}) \leq 2n$, and thus $\tau(\mathbf{1}) = 2n$.

Step 4. The above argument is also a model for our next considerations: we argue by induction, reducing the number of digits (the length) of ω, whenever possible. Let $n \geq 3$, and assume that ω contains some zero digits, as $\omega = \mathbf{1}$ has already been settled. Our reduction steps will concentrate on the zero digits.

First, consider the case when ω has some zero at one of its ends, say at the start of it. Then for p any proper polynomial in (9), the polynomial $\widetilde{p}(x) := p(x)/(x - t_1)$ is a proper polynomial for the new system of nodes $\widetilde{t} := t \setminus \{t_1\}$ arising by removal of t_1 but keeping all the x_j's with $\omega_j = 1$ (and thus replacing ω by $\widetilde{\omega}$, the sequence with the first 0 digit of ω removed). Note that the length decreased by one, N and K remained unchanged, and the degree of p was decreased by 1. Furthermore, we have removed only a zero from ω, hence $\nu(\omega) = 1$ if and only if $\nu(\widetilde{\omega}) = 1$. So the induction hypothesis with $\widetilde{n} = n - 1$ and with the given system $\widetilde{t}$, $x(\widetilde{t})$ and $\widetilde{\omega}$ yields the assertion. Remark that with a minimal degree polynomial $\widetilde{p}$ for $\widetilde{t}$ and $\widetilde{\omega}$ is a product of zero factors $(x - t_i)$, thus the corresponding $p(x) := (x - t_1)\widetilde{p}(x)$ is also of the this type, as asserted.

Step 5. Second, consider the case when there are neighboring zeroes anywhere in the sequence ω. Let us pick up the first such pair, say $\omega_j = 0 = \omega_{j+1}$ with j minimal having this property. Then define $\widehat{\omega}$ by deleting these two zeroes from the sequence ω. The characteristics ν, N and K do not change, while the length decreases by 2. Let p be a polynomial satisfying the conditions in (9), and put now $\widehat{p}(x) := p(x)/((x - t_j)(x - t_{j+1}))$. Let now $\widehat{t} = (t_1, \ldots, t_{j-1}, t_{j+2}, \ldots, t_n)$ (of length $n-2$). Then $\widehat{p}$ is appropriate for $\widehat{t}, \widehat{\omega}, \widehat{x}$ in the sense of (9) if and only if p is appropriate for t, ω, x in the same sense. Conversely, starting out from a polynomial $\widehat{p}$, proper for $\widehat{t}$ etc., we find that $p(x) := \widehat{p}(x) \cdot (x - t_j)(x - t_{j+1})$ will satisfy all conditions for t etc. This shows that $\tau(\widehat{\omega}) = \tau(\omega) - 2$, as the minimal degree polynomials correspond. That the minimal degree polynomials are the product of zeros factors $(x - t_i)$ is again apparent.

Step 6. Third, it remains to consider the case when there are no ending zeroes and neither there are double zeroes in ω. That is, we then must have a sequence of 1's with a few, say k, isolated interior zeroes inside; let the set of indices with $\omega_j = 0$

be denoted by z. Then removing $\{t_j \ : \ j \in z\}$ from t, there remains a set of $n-k$ nodes $\overline{t} := \{t_i \ : \ 1 \le i \le n, \ \omega_i = 1\}$, strictly interlacing with the set $x(t)$. Clearly, $\tau(\omega) \ge \overline{\tau}(\omega)$ with

$$\overline{\tau}(\omega) := \min\{\deg \overline{p} \ : \ \overline{p} \not\equiv 0, \ \overline{p}(x_j) \ge 0 \ (\omega_j = 1), p(t_i) = 0 \ (i \notin z)\},$$

since in $\overline{\tau}(\omega)$ we have just relaxed a few conditions in the definition (9). Now we have a new system of $\overline{t}, \overline{x}, \overline{\omega} = \mathbf{1} \in \{0,1\}^{n-k+1}$, for which a minimal degree polynomial is known, according to the above settled case of $\omega = \mathbf{1}$. We thus have $\overline{\tau}(\omega) = \tau(\overline{\omega}) = 2(n-k)$, hence in particular $\tau(\omega) \ge 2(n-k)$, and an extremal polynomial for $\overline{\tau}(\omega)$ is

$$\overline{p}(x) := \prod_{t_i \in \overline{t}} (x - t_i)^2.$$

However, if $\omega_j = 0$, then $j \in z$ and, on the other hand, $j+1 \notin z$. Since $\omega_j = 0$, in the interval (t_j, t_{j+1}) there is no condition on the sign of p in (9), and outside of $[t_j, t_{j+1}]$ the function $(x-t_j)/(x-t_{j+1})$ is strictly positive. Therefore, together with $\overline{p}$, also the polynomial

$$p(x) := \overline{p}(x) \prod_{\omega_j = 0} \frac{x - t_j}{x - t_{j+1}} = \prod_{\omega_j=0} (x-t_j)(x-t_{j+1}) \prod_{\omega_{i-1}\cdot\omega_i = 1} (x-t_i)^2$$

satisfies all sign conditions of (9); observe, that $p(t_i) = 0$ now for all $i = 1, \ldots, n$. It shows that $\tau(\omega) \le 2(n-k)$, hence $\tau(\omega) = 2(n-k)$.

It remains to observe that also in this case the asserted formula holds true, since now $\nu(\omega) = 0$, $N(\omega) = n-k+1$ and $K(\omega) = k$; moreover, p is again a product of zero factors with the nodes as roots. □

Corollary 6. *Let $\Omega \subset \mathbb{R}$ be arbitrary, $t_1, t_2, \ldots, t_n \in \Omega$ a node sequence, and ω be the type of $t = (t_1, t_2, \ldots, t_n)$ with respect to Ω. Then the minimal degree of polynomials, as defined in (7) and (8), are both equal to $\tau(\omega)$ in (9).*

Proof. Clearly, the restriction $0 \le b|_\Omega$ is more stringent than $0 \le b|_{\Omega'}$, which is equivalent to the condition in the definition of $\tau(\omega)$ in (9). But by Lemma 5, at least one τ-extremal polynomial p can be taken as product of linear factors $(x - t_i)$, which does not vanish, hence does not change sign, but only at nodes of t. Therefore, p must be strictly positive on the whole of $\cup_{\omega_j=1}(t_j, t_{j+1})$. Putting $\Omega^\star := \cup_{\omega_j=1}(t_j, t_{j+1}) \cup t$, we are led to $p|_{\Omega^\star} \ge 0$, so in view of $\Omega \subset \Omega^\star$, also $p|_\Omega \ge 0$. Whence (7) cannot exceed $\tau(\omega)$, and the quantities (7) and (8) are equal. □

To give the relationship between the values of $\tau(\omega)$ and $d_n(\Omega)$ the following considerations are appropriate. For a given node-system we define

$$t^{(j)} := (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$$

and then

$$\omega^{(j)} := \omega(t^{(j)}) = (\omega_0, \ldots, \omega_{j-2}, 1, \omega_{j+1} \ldots, \omega_n) \in \{0,1\}^n,$$

since in the interval (t_{j-1}, t_{j+1}) there is always an element of Ω, namely at least t_j. This means that in ω we replace the pair t_{j-1}, t_j by a single digit 1 thus resulting in $\omega^{(j)}$.

Given a node system t (or equivalently ω), first we have to determine the maximal degree of the corresponding basis elements $b_1, b_2, \ldots, b_n$, and take the minimum with respect to the choice of t. Also we have the restriction that b_i $(i \neq j)$ vanishes at t_j in a strictly larger order than b_j (see the end of Section 1). However, by the above we know that the minimal degree system of b_j's can be constructed with zeroes only in points of $t^{(j)}$. This means that by assuming strictly positive values at t_j, we can forget about the extra assumption on the order of vanishing. That is we first have to determine

$$\sigma(\omega) := \max_{j=1,\ldots,n} \tau(\omega^{(j)}),$$

which then will be the lowest possible maximal degree of the b_j's, given the set Ω and the node system $t \subset \Omega$.

It is obvious that $d_n(\Omega) = \min_{t\subset\Omega,\#t=n} \sigma(\omega)$ holds; see the formulation in (6).

Before determining $d_n(\Omega)$ we first compute $\sigma(\omega)$ for any 0-1 sequence ω.

Lemma 7. *For any $n \in \mathbb{N}$ and $\omega \in \{0,1\}^{n+1}$ the following formula holds:*

$$\sigma(\omega) = \begin{cases} n-1 = \tau(\omega)-1 & \text{if} \quad \omega = \mathbf{0}, \\ 2n-2 = \tau(\omega)-2 & \text{if} \quad \omega = \mathbf{1}, \\ 2n-2 = \tau(\omega)-1 & \text{if} \quad \omega = (0,1,\ldots,1) \text{ or } (1,\ldots,1,0), \\ 2n-3 = \tau(\omega)-1 & \text{if} \quad \omega = (0,1,\ldots,1,0), \\ \tau(\omega) & \text{otherwise.} \end{cases} \tag{10}$$

Proof. The reader will have no difficulty in checking the first four exceptional cases, hence we can skip the details.

Assume now that ω is not of the four exceptional case. Note that since now $\omega \neq \mathbf{0}$, we necessarily have $\nu(\omega) = \nu(\omega^{(j)}) = 0$ for all $j = 1, \ldots, n$, hence by Lemma 5, for these ω we have $\tau(\omega) = n - 1 + N(\omega) - K(\omega)$ and $\tau(\omega^{(j)}) = n-2+N(\omega^{(j)})-K(\omega^{(j)})$, in view of the decrease of the length of the 0-1 sequence. So actually have to keep track the change of N and K when the $\omega^{(j)}$ are formed from the given non-exceptional ω.

Observe that $K(\omega^{(j)}) \geq K(\omega) - 1$, since no two different (odd length) inner zero blocks can be influenced by the operation of substituting ω_{j-1} and ω_j by one single digit of 1. Moreover, this decrease in the number of odd length inner blocks can arise only if at one end of such a block, the ending combination of 0 and 1 (or, 1 and 0) is replaced by 1. If any other, totally inner segment of an (odd) inner zero block is replaced by 1, then the number of zeros remaining from the block remains odd, hence there will be at least one remaining heir of the original odd block, again counted in $K(\omega^{(j)})$, and thus not allowing K to decrease. But if a pair of a 0 and 1 is substituted by 1, then N remains unchanged, so in all, in case K decreases, the difference of $N-K$ can increase only by 1 (at most). Similarly, if N increases, that requires that a pair of zeros is substituted by 1, so N increases by 1, but then K can not decrease, as seen above. (It can increase, though, but that is out of our concern now.) So, again, the difference $N-K$ can increase at most by 1. Summing up, we have, apart from the four extra cases, $(N-K)(\omega^{(j)}) \leq (N-K)(\omega)+1$.

We now show that for some particular index j this unit increase of $N-K$ really occurs. If $K(\omega) \geq 1$, then we just take an ending pair of 0 and 1, as analyzed above, and replacing it by 1 will do the stated increase.

Finally, if there are no inner odd length zero blocks, then there must be a pair of zeros somewhere. Indeed, either there is an even inner block of zeros (in this case we find neighboring zeros), or there is no inner block of zeros at all. But in the latter case there must be at least two neighboring zeros at beginning or at the end of ω, since otherwise we would be in the exceptional cases, which was excluded.

Also, $\omega \neq \mathbf{0}$, so there must be some 1's in ω. So, let us consider an occurrence of a double zero in the sequence ω, which is neighboring a digit of 1. Now replace this two double zeros by 1. Thus N increases by 1, while no new inner block is generated moreover all odd inner zero blocks remain odd inner zero blocks. Therefore we than have $N-K$ increasing by 1.

In all we find that apart from the four exceptional cases, $(N-K)(\omega^{(j)}) \leq (N-K)(\omega)+1$ for all $j=1,\dots,n$, and for some particular choice of j this inequality becomes an equality: that is, according to the discussion above, we obtain $\max_j \tau(\omega^{(j)}) = \tau(\omega)$, as claimed. $\square$

For low values of n and for finite Ω the value of $d_n(\Omega)$ is trivial to determine. These cases are included in the next proposition for the sake of completeness only.

Proposition 8. *If $n=1$, then a one dimensional positive basis of the constant function space $\mathcal{P}_0$ exists, for arbitrary choice of $t=\{t_1\}$. If $n=2$, then $d_n(\Omega)=1$ for any (compact) Ω having at least two elements, with the choice of $t=\{t_1,t_2\}=\{a,b\}$. If Ω is finite, then $\dim C(\Omega)=\#\Omega$, so if $\#\Omega<n$, then there is no nodes and basis systems of n elements. If $\#\Omega=n$, then $t=\Omega$ is the only choice, and $d_n(\Omega)=n-1$. For all other cases $d_n(\Omega)>n-1$.*

Proof. Apart from the trivial cases, it suffices to show that for $\#\Omega>n\geq 3$ we necessarily have $d_n(\Omega)>n-1$. Indeed, for any choice of t, there remains a point $x\in\Omega\setminus t$, hence the type ω of t with respect to Ω can not be $\mathbf{0}$. We can now combine Lemmas 5 and 7: from the latter, we are free of the first exceptional case, and the next three give all larger values of $\sigma(\omega)$ (at least for $n\geq 3$).

In the last case of (10) we have $\sigma(\omega)=\tau(\omega)$, and from Lemma 5 that $\tau(\omega)=n-1+N(\omega)-K(\omega)$, since now $\nu(\omega)=0$. It remains to see that $N(\omega)>K(\omega)$, which is clear if $K(\omega)>0$, and also if $K(\omega)=0$ and $N(\omega)>0$, while $N(\omega)=K(\omega)=0$ is excluded (for it leads to $\omega=\mathbf{0}$). This concludes the proof. $\square$

Corollary 9. *If $\#\Omega > n$, then the n-dimensional polynomial space $\mathcal{P}_{n-1}$ does not have a positive basis in $C(\Omega)$: in particular, if Ω is infinite, then no polynomial spaces $\mathcal{P}_m$ can have a positive basis in $C(\Omega)$.*

We consider now the case of Ω being still finite but possessing sufficiently many points.

Proposition 10. *If $3 \leq n \leq \#\Omega < \infty$, then we have*

$$d_n(\Omega) = \begin{cases} n & \text{if } n < \#\Omega \\ n-1 & \text{if } n = \#\Omega. \end{cases}$$

Proof. Proposition 8 gives the case of $n = \#\Omega$ and also $d_n(\Omega) \geq n$ for $n < \#\Omega$.

For the converse inequality let us take as t_j, for $j = 1, \ldots, n$, the first n points of Ω, according to increasing order. Then $\omega(t) = (0, 0, 0, \ldots, 0, 1)$, as $b > t_n$ is a point of Ω. We have $\sigma(\Omega) = n - 1 + N(\omega) - K(\omega) = n - 1 + 1 - 0 = n$. So $d_n(\Omega) \leq n$, the proof is complete. □

We have settled the case of a compact interval in Section 2, and the above Proposition 8 and 10 contains the otherwise easy case of a finite Ω. Therefore, we now consider the remaining cases of a compact, non-convex Ω with infinitely many elements. This is actually the point where the topological-geometrical features of Ω enter the picture. Since $\mathbb{R}\backslash\Omega$ is open, it is a union of disjoint open intervals: more precisely, $\mathbb{R}\backslash\Omega = (-\infty, a) \cup (b, \infty) \cup (\cup_{k\in\mathcal{K}} I_k)$, where $\mathcal{K}$ is a finite or countably infinite index set which is nonempty. We will use this decomposition of $\mathbb{R}\backslash\Omega$ as a canonical representation in the following. Note that the intervals I_k are disjoint and inside (a, b), but they might have common endpoints. We will denote the endpoints of I_k as α_k and β_k, so $I_k = (\alpha_k, \beta_k)$.

Let $\mathcal{L} \subseteq \mathcal{K}$. We say that the family $\{I_\ell \,:\, \ell \in \mathcal{L}\}$ of intervals is a *free family of holes* if $\overline{I_\ell} \cap \overline{I_{\ell'}} = \emptyset$, for any $\ell \neq \ell'$ with $\ell, \ell' \in \mathcal{L}$.

Let $\dot{\Omega}$ denote the set of limit points of Ω. There might be some points in $\Omega\backslash \operatorname{conv}\dot{\Omega}$. They are all isolated points of Ω, and we call these *eccentric* points of Ω. Further, we define

$$\lambda(\Omega) := \max\{\#\mathcal{L} \,:\, \mathcal{L} \subset \mathcal{K} \text{ is a free family of holes}\}.$$

Note that obviously $\lambda(\Omega) = \infty$ if and only if Ω has infinitely many components. This for example the case, if there are infinitely many eccentric points. Whereas if there are only finitely many of them, then we can introduce the notations Θ_ℓ, Θ_r for the parity of their number on left respectively on the right hand side of Ω.

We now describe $d_n(\Omega)$ in terms of these topological characteristics.

Proposition 11. *If $\lambda(\Omega) = \infty$, then we have*

$$d_n(\Omega) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ n & \text{if } 2 < n. \end{cases}$$

Proof. As before, Proposition 8 settles the cases $n = 1, 2$ and gives $d_n(\Omega) \geq n$ for $n \geq 2$, so again it suffices to show $d_n(\Omega) \leq n$ for $n > 2$.

By assumption we can take n free holes, such that none of their endpoints is a or b. Arrange them according to the increasing order, take every second of them starting with the first (altogether $[n/2]$ free holes), and consider their endpoints as node system t. That is, if the intervals are $I_1, \dots, I_n$, with $I_j = (\alpha_j, \beta_j)$, $(j = 1, \dots, n)$, then consider the point set $\{\alpha_1, \beta_1, \dots, \alpha_{2k-1}, \beta_{2k-1}, \dots\}$ of $2[n/2]$ elements, and, in case n is odd, add the point b to form t. For the corresponding type we have

$$\begin{aligned} \omega(t) &= (1,0,1,0,1,\dots,1,0,1,0) \quad \text{if } n \text{ is odd,} \\ \text{or} \quad \omega(t) &= (1,0,1,0,1,\dots,1,0,1,0,1) \quad \text{if } n \text{ is even.} \end{aligned}$$

From this we see $K(\omega) = N(\omega) - 1$, hence $d_n(\Omega) \leq \sigma(\omega) = n - 1 + N(\omega) - K(\omega) = n$. By Proposition 8 the equality $d_n(\Omega) = n$ follows. □

Theorem 12. *Let $3 \leq n$ and let Ω be an infinite compact subset of $\mathbb{R}$. Then we have*

$$d_n(\Omega) = \begin{cases} n & \textit{if} \quad \lambda(\Omega) > \frac{n}{2}, \\ n & \textit{if } \lambda(\Omega) = \left[\frac{n}{2}\right], \ n \textit{ is even}, \\ n + \Theta_\ell \Theta_r & \textit{if } \lambda(\Omega) = \left[\frac{n}{2}\right], \ n \textit{ is odd}, \\ 2(n - 1 - \lambda(\Omega)) + \Theta_\ell + \Theta_r & \textit{if} \quad \frac{n}{2} - 1 \geq \lambda(\Omega) \geq 1, \\ 2n - 3 & \textit{if} \quad \lambda(\Omega) = 0. \end{cases} \tag{11}$$

Proof. The case $\lambda(\Omega) = 0$ (the case of the interval) and $\lambda(\Omega) = \infty$ is known by Section 2 and Proposition 11. So we can assume $\lambda(\Omega)$ to be positive and finite. In this case Ω is the union of finitely many closed (possibly degenerate) intervals, and the interior of Ω is not empty. We will use Lemma 5 and Lemma 7 tacitly.
Step 1. First we consider the case $\lambda(\Omega) > \frac{n}{2} - 1$. The case of even $n = 2k$ is easy. Indeed, consider k free holes $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$, ordered increasingly, and their endpoints as nodes in t. Then $\omega = \omega(t) = (\omega_0, 0, \omega_2, 0, \omega_4, 0, \dots, \omega_{2k-2}, 0, \omega_{2k})$. From which $K(\Omega) = N(\omega) - 1$ and hence $\sigma(\omega) = n - 1 + N(\omega) - K(\omega) = n$ is easy to see. Proposition 8 then gives $d_n(\Omega) = n$.
Step 2. Let $n = 2k + 1$. If $\lambda(\Omega) > k$ then we can take similarly to the above $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ free holes, such that none of the endpoints coincides with a (or b). In this case $t = \{a, \alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$ (or with b instead of a) will be a proper node system for which (in case $\alpha_1 \neq a \in t$) we have

$$\omega = \omega(t) = (0, \omega_1, 0, \omega_3, 0, \dots, \omega_{n-2}, 0, \omega_n).$$

So we have $K(\omega) = N(\omega) - 1$, hence $\sigma(\omega) = n - 1 + N(\omega) - K(\omega) = n$. As above $d_n(\Omega) = n$ follows.
Step 3. For the case $n = 2k + 1$, $\lambda(\Omega) = k$ consider a *maximal* family of free holes with k elements $\{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$, and index them according to the increasing order. If $\Theta_\ell = 0$ we can shift, if necessary, the holes having endpoints among the left eccentric points, in such a way that $\alpha_1 \neq a$. We can do the analogous

procedure for the eccentric points on the right of Ω (with the condition $\beta_k \neq b$). Still we will have a free family of holes having exactly k members, which we will use in the following. The set of the endpoints of these intervals is t'. If $a \notin t'$ (or $b \notin t'$) we can add it to t', $t := t' \cup \{a\}$, say. Such steps we *cannot* accomplish if and only if $\Theta_\ell \Theta_r = 1$. In this latter case we take any point which lies in the interior of Ω and add it to t'. In both of these cases we have a node system of n elements. We can easily determine $\omega = \omega(t)$

$$\omega = (0, \omega_1, 0, \omega_3, \ldots, \omega_{n-2}, 0, \omega_n), \qquad \text{if } \Theta_\ell \Theta_r = 0, \tag{a}$$

$$\omega = (\omega_0, 0, \omega_2, 0, \ldots, \omega_{2j}, 0, 1, 1, 0, \omega_{2j+5}, 0, \ldots, 0, \omega_n), \qquad \text{if } \Theta_\ell \Theta_r = 1. \tag{b}$$

From this we can compute the value of $\sigma(\omega)$. In the case (a) we have $K(\omega) = N(\omega) - 1$, hence $\sigma(\omega) = n - 1 + N(\omega) - K(\omega) = n$. While in case (b) we obtain $K(\omega) = N(\omega) - 2$, so $\sigma(\omega) \leq n - 1 + N(\omega) - K(\omega) = n + 1$. Summing up $d_n(\Omega) \leq n + \Theta_\ell \Theta_r$. We leave the lower estimate to the end of this proof. Actually the case $\Theta_\ell \Theta_r = 0$ is already done, since Proposition 8 furnishes $d_n(\Omega) \geq n$ hence $d_n(\Omega) = n$.

Step 4. Now we turn to the case $\lambda \leq \frac{n}{2} - 1$. The arguments are very similar to the above. Take a family of free holes with $k = \lambda(\Omega)$ elements $\{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ and index them according to the increasing order. Repeat the shifting procedure for the holes involving eccentric points as above (if possible).

If $\Theta_\ell = 0$ we can shift, if necessary, the holes having endpoints among the left eccentric points, in such a way that $\alpha_1 \neq a$. We can do the analogous procedure for the eccentric points on the right of Ω (with the condition $\beta_k \neq b$). Consider the $k' = 2k + 2 - \Theta_\ell - \Theta_r$ points $t' := \{a, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k, b\}$. Note that, for example, $a = \alpha_1$ if $\Theta_\ell = 1$. Because Ω is infinite and there are only finitely many holes in Ω, there is an interval I completely belonging to Ω. Take now $n - k'$ many points in the interior of I, this complements t' to a node system t of n elements.

The digits of ω are relatively well-known: first comes some block $\omega^{(\ell)}$ then a block of $n - k' + 1$ many 1's and finally a block $\omega^{(r)}$

$$\omega = (\omega^{(\ell)}, \underbrace{1, 1, \ldots, 1}_{n-k'+1 \text{ ones}}, \omega^{(r)}).$$

We observe only $\omega^{(\ell)}$ in detail; the analogous reasoning applies also for $\omega^{(r)}$. If $\Theta_\ell = 1$, then

$$\omega^{(\ell)} = (0, 0, \omega_2, 0, \ldots, \omega_{2j}, 0),$$

whereas if $\Theta_\ell = 0$, we have

$$\omega^{(\ell)} = (0, \omega_1, 0, \omega_3, 0, \ldots, \omega_{2j-1}, 0).$$

From this form we see that each 0 block in $\omega^{(r)}$ is either an inner block of zeros and in this case it is odd, or it is an ending block and only then it might be of even length. Among $\omega_2, \ldots, \omega_{2j}$ (or $\omega_1, \ldots, \omega_{2j-1}$) there are L digits of 1. Denoting the number of 1's in $\omega^{(r)}$ by R and reasoning analogously as above we obtain

$N = n - k' + 1 + L + R$, and $K = L + R$, hence $\sigma(\omega) = n - 1 + N - K = 2n - k'$ and

$$d_n(\Omega) \leq 2n - 2 - 2\lambda(\Omega) + \Theta_\ell + \Theta_r.$$

Step 5. Now we turn to the lower estimate of $d_n(\Omega)$ in general. Let us now start with an optimal t of type $\omega = \omega(t)$, i.e., with $d_n(\Omega) = \sigma(\omega)$. Denote $N := N(\omega)$ and $K := K(\omega)$. Denote by L and R the number of zeros at the left respectively the right side of ω. Suppose that there are M inner blocks of zeros with respective lengths μ_i, $i = 1\ldots, M$. So $n + 1 - N - (L + R) = \sum_{i=1}^{M} \mu_i$. The number of odd blocks is the number of odd μ_i, so $K = \sum_{i=1}^{M} ([(\mu_i + 1)/2] - [\mu_i/2])$. Moreover, to any zero-block of length, say, μ, there corresponds a family of $[(\mu + 1)/2]$ free intervals. The union of any number of such families, belonging to different zero-blocks, still remains a free family of intervals. Therefore there exists a free family of holes with $k := \sum_{i=1}^{M}[(\mu_i + 1)/2]$ members. So now we can write

$$\begin{aligned} 2k &= 2\sum_{i=1}^{M}\left[\frac{\mu_i+1}{2}\right] = \sum_{i=1}^{M}\left(\left[\frac{\mu_i+1}{2}\right] + \left[\frac{\mu_i}{2}\right]\right) + K \\ &= \sum_{i=1}^{M}\mu_i + K = n + 1 - N - (L+R) + K = 2n - (L+R) - \sigma(\omega). \end{aligned}$$

using again formula (10) and Lemma 5. Observe that m zeros at the left (or right) side of ω result in additional family of $[m/2]$ free holes. Thus we can write

$$\begin{aligned} 2\lambda(\Omega) &\geq 2k + 2\left[\frac{L}{2}\right] + 2\left[\frac{R}{2}\right] = 2n - (L+R) + 2\left[\frac{L}{2}\right] + 2\left[\frac{R}{2}\right] - \sigma(\omega) \\ &= 2n - 2 + \Theta_\ell + \Theta_r - \sigma(\omega). \end{aligned}$$

By optimality of t we have $\sigma(\omega) = d_n(\Omega)$. Thus $d_n(\Omega) \geq 2n-2-2\lambda(\Omega)+\Theta_\ell+\Theta_r$, hence the missing part of (11) follows.

Step 6. Only the case of odd n and $\lambda(\Omega) = [n/2]$ remains to be proved. We know the upper estimate $d_n(\Omega) \leq n + \Theta_\ell\Theta_r$ from Step 3 and the lower estimate $d_n(\Omega) \geq 2n - 2 - 2\lambda(\Omega) + \Theta_\ell + \Theta_r$ from Step 5. If $\Theta_\ell\Theta_r = 1$ these reduce to $d_n(\Omega) \leq n + 1$ and $d_n(\Omega) \geq 2n - 2\lambda(\Omega) = n + 1$, hence the assertion. Otherwise we obtain $d_n(\Omega) = n$ by Proposition 8. □

Note that $\lambda(\Omega)$ can be determined in a rather mechanical way through topological features. A way to do so is to consider all boundary points – of which there can be infinitely many if and only if $\lambda(\Omega) = \infty$ and in this case we are done. Otherwise, the finitely many boundary points determine finitely many intervals, and we can form a finite sequence of 0's and 1's according to the whether the intervals contain a point of Ω or not. If $\mu_1, \mu_2, \ldots, \mu_k$ are lengths of the inner zero-blocks, then we have $\lambda(\Omega) = \sum_{i=1}^{k}\left[\frac{\mu_i+1}{2}\right]$.

For the subspaces $X \subseteq \mathcal{P}_m$ with positive basis and of maximal dimension we obtain directly from Theorem 12 the following.

Theorem 13. *Let Ω be an infinite compact subset of $\mathbb{R}$, $X \subseteq \mathcal{P}_m(\Omega)$ be a subspace with positive basis and, as such, of maximal possible dimension. Then we have*

$$\dim X = \begin{cases} m & \text{if } \quad \frac{m}{2} \leq \lambda(\Omega), \\ m - \Theta_\ell \Theta_r & \text{if } \frac{m-1}{2} = \lambda(\Omega), \\ \left[\frac{m-\Theta_\ell-\Theta_r}{2}\right] + 1 + \lambda(\Omega) & \text{if } \quad \frac{m-2}{2} \geq \lambda(\Omega) \geq 1, \\ \left[\frac{m+3}{2}\right] & \text{if } \quad \lambda(\Omega) = 0. \end{cases}$$

Also in higher dimensions there are known geometric conditions that imply the existence a positive basis in a given finite dimensional subspace $X \subseteq C(\Omega)$, cf., e.g., [9]. In connection to our investigation we ask the following

Question. *Consider $\Omega \subset \mathbb{R}^d$ a compact, convex set – or, a general compact set. What is the maximal dimension of subspaces of $\mathcal{P}_n(\Omega)$ with positive basis?*

4. Acknowledgement

The authors are indebted to the anonymous referee for the helpful comments and suggestions.

References

[1] Y.A. Abramovich, C.D. Aliprantis, I.A., Polyrakis, *Lattice subspaces of positive projections*, Proc. R. Ir. Acad., **14A**(2), (1994), 237–253.

[2] S.N. Bernstein, Sur la repréèsentation des polynomes positifs, (originally published in 1915), In: *"Collected works of S. N. Bernstein"*, **1**, Constructive theory of functions (1905–1930), Academy of Sciences USSR, 1952, 251–252.

[3] T. Erdélyi, *Estimates for the Lorentz degree of polynomials*, J. Approx. Theory, **68** (1991), 187–198.

[4] G. Lorentz, *The degree of approximation by polynomials with positive coefficients*, Math. Annalen, **151** (1963), 239–251.

[5] A.L. Peressini, *Ordered Topological Vector Spaces*, Harper's Series in Modern Mathemtics, Harper & Row, New York (1967).

[6] Gy. Pólya, G. Szegő, *Aufgaben und Lehrsätze aus der Analysis, vol. II*, Die Grundlehren der matehmatischen Wissenschaften in Einzeldarstellungen, Band XX, Verlag der Julius Springer, (1925).

[7] I. Polyrakis, *Schauder bases in locally solid lattice Banach spaces*, Math. Proc. Camb. Phil. Soc. **101** (1987), 91–105.

[8] I. Polyrakis, *Lattice subspaces of $C[0,1]$ and positive bases*, J. Math. Anal. Appl., **184** (1994), 1–18.

[9] I. Polyrakis, *Finite dimensional lattice subspaces of $C(\Omega)$ and curves of $\mathbb{R}^n$*, Trans. Am. Math. Soc. **348/7** (1996), 2793–2809.

[10] I. Polyrakis, *Minimal lattice subspaces*, Trans. Am. Math. Soc., **351/10** (1999), 4183–4203.

[11] I. Polyrakis, *Lattice-subspaces and positive bases in function spaces*, Positivity, **7**(4)(2003), 267–284.

[12] Sz. Gy. Révész, *Schur type inequalities for polynomials with no zeros in the unit disk*, J. Ineq. Appl. (2008, to appear).

[13] T. Scheick, *Inequalities for derivatives of polynomials of special type*, J. Aprox. Theory, **6** (1972), 354–358.

Bálint Farkas
Technische Universität Darmstadt
Fachbereich Mathematik, AG4
Schloßgartenstraße 7, D-64289
Darmstadt, Germany
e-mail: `farkas@mathematik.tu-darmstadt.de`

Szilárd Gy. Révész
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O.B. 127, 1364 Hungary
e-mail: `revesz@renyi.hu`

Received 22 August 2007; accepted 6 January 2008