COMPLEX ANALYSIS AND APPLICATIONS '85
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ON A THEOREM OF PHRAGMEN

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1. Investigating the distribution of primes E. Phragmen [2] proved the following function-theoretical result about the connection of the oscillation of a function and the singularities of its Dirichlet-Laplace transform.

<u>Theorem</u> (Phragmen). If $r(x)$ is a piece-wise continuous complex function with Dirichlet-Laplace transform $R(s)$, converging in a right halfplane, i.e.

$$R(s) = \int_1^\infty x^{-s} dr(x) \qquad (\sigma > A)$$

and if R can be meromorphically continued to $\sigma > a$ $(a < A)$, then for any singularity $\alpha + i\tau$ of R in $a < \sigma \leq A$ and for any $\varepsilon > 0$ we have

$$r(x) = \Omega(x^{\alpha-\varepsilon}).$$

In the important special case, when $r(x)$ is the remainder term of various forms of the prime number theorem, this relation was sharpened by E. Schmidt [8], and in case of the prime ideal theorem by E. Landau [4]. However these results were ineffective ones, and Littlewood's [5] question for explicit Ω-results was unanswered for more than two decades. The breakthrough was made by P. Turán [11], who could use his famous power-sum method for proving an explicit oscillatorical theorem on $\Delta(x) = \Psi(x) - x$. After this, the aim was to find the correct size of oscillation "caused by" a $\rho_0 = \beta_0 + i\gamma_0$ zero of $\zeta(s)$ (i.e. a pole of $-\frac{\zeta'}{\zeta}(s) - \frac{s}{s-1}$, the transform of $\Delta(x)$). This problem was settled by J. Pintz [6], who — roughly speaking — could bring down the ε from the exponent (see Corollary 2). Developing certain effective results in the theory of the distribution of the prime ideals, W. Staś and K. Wiertelak [10] began to investigate the possibility of proving general and explicit function-theoretical oscillation theorems. In the present paper we show, that the theoratically expectable largest oscillation can be proved for general function-classes too. As corollaries, we will get back the above mentioned result of J. Pintz, and corresponding results in algebraic number fields, due to the present author [7].

The author wishes to express his thank to J. Pintz for the stimulating consultations during his work.

2. We call a function (real or complex valued) piecewise continuous, if it is continuous in every finite interval except a finite number of jumps. For simplicity, we will suppose, that our functions are left continuous, and that any integral $\int_a^b$ means $\int_{a-0}^{b-0}$, when a or b is a jump of the corresponding integrand. For any function f, let $f_1=f$, $f_{\nu+1}(x) = f(f_\nu(x))$, i.e. f_ν is the ν-times iterated f function. We introduce for $z \in \mathbb{C}$

$$\operatorname{Log} z := \max\{\log|z|, 2\}.$$

Complex variables are written in the form $s = \sigma+it$ ($\sigma = \operatorname{Re} s$), and an integral on a vertical line with real part σ will be denoted by $\int_{(\sigma)}$.

We use the Landau o and O notations and the Vinogradov $\ll$ notation as well. $K_1, K_2, \ldots$ are positive parameters, fixed once for all, and $c_1, c_2, \ldots$ are explicitly calculable absolute constanst {(independent of any parameter)}.

<u>Definition</u>. We say, that $r \in \mathcal{R}$, if the following conditions hold:

I. $r(x): [1,\infty) \to \mathbf{C}$, $r(1) = 0$, r piece-wise continuous, and

$$|r(x)| < K_1 x \mathrm{Log}^{K_2} x .$$

II. The Dirichlet-Laplace transform of r,

$$R(s) = \int_1^\infty x^{-s} dr(x) = -\int_1^\infty r(x) dx^{-s} \qquad (\sigma > 1),$$

can be presented as the negative of the logarithmic derivative of a function F , regular for $\sigma \geq 0$ except for a possible pole at $s=1$:

$$R(s) = -\frac{F'}{F}(s) \qquad (\sigma \geq 0) .$$

III. For $0 \leq \sigma \leq 4$

$$|F(s)(s-1)| \leq K_3(|t|+1)^{K_4} .$$

IV. $|F(2+it)| \geq K_5$.

<u>Remark</u>. By I and II, $F(s) \neq 0$ for $\sigma > 1$. Let us denote the order of the singularity of F at $s=1$ by $\varkappa$; i.e. $\varkappa=1$, if F has a pole, and $\varkappa=0$, if not. (It is not important, that in III we set $F(s)\cdot(s-1)$ — we could do everything with $F(s)\cdot(s-1)^{\varkappa}$ as well — but at present by III we have $\varkappa \leq 1$.) If r is real and monotone, and $\varkappa=1$, then by Wiener's Tauberian theorem [13] the relation

(2.1) $$r(x) \sim x \qquad (x \to \infty)$$

is equivalent to

(2.2) $$F(1+it) \neq 0 \qquad (t \in \mathbb{R}).$$

But, by an application of a classical theorem of Landau [3], it can be proved that for such an r we have $\varkappa=1$. Further, by the well-known method of De la Vallée Poussin, it can be proved, that (2.2) holds, so (2.1) follows for any real and monotone r. When r is not such, we donot have the corresponding asymptotical results, but nevertheless, we can prove the same oscillation. I.e. we have

Theorem. Let $r \in \mathcal{R}$, let $\varkappa$ be the order of the singularity of F at $s=1$, and

(2.3) $$d(x) := r(x) - \varkappa \cdot x \ .$$

Let $0 < \varepsilon < 1$ be arbitrary, $\rho_0 = \beta_0 + i\gamma_0$ a zero of F with $\beta_0 \geq \frac{1}{2}$, and

(2.4) $$Y \geq \max\{c_1, e^{|\rho_0|}, \exp(\frac{12}{\varepsilon^2}), K_1, K_4, e^{(1+K_2)^2}, K'\} \quad (K' := \log\frac{K_3}{K_5}) .$$

Then there exists an

(2.5) $$x \in [Y, Y^{c_2(K_4 \mathrm{Log}(|\rho_0|)+K')}] ,$$

for which

(2.6) $$|d(x)| > (1-\varepsilon)\frac{x^{\beta_0}}{|\rho_0|} .$$

3. Proposition 1. If $A \geq 1$, $0 < \alpha \leq 1$ and $x \geq 1$, we have

(3.1) $$\mathrm{Log}^A x \leq e^{\frac{A}{\alpha}+A^2} \cdot x^\alpha .$$

Proposition 2. If u and v are arbitrary complex numbers, we have

(3.2) $$\mathrm{Log}(u+v) \leq \mathrm{Log}(u) \cdot \mathrm{Log}(v) .$$

Proposition 3. For $B \geq \frac{1}{2}$

(3.3) $$\int_B^\infty e^{-x^2} dx < e^{-B^2} .$$

Proposition 4. For $a > 0$, b complex, $c > 0$ we have

(3.4) $$\frac{1}{2\pi i}\int_{(c)}^{\infty} e^{as^2+bs}ds = \frac{1}{2\sqrt{\pi a}}\exp(-\frac{b^2}{4a}) .$$

Lemma 1. (The continuous form of the power-sum theorem of J.W.S. Cassels [1].) For $h > 1$, $n \in \mathbb{N}$ and α_j complex $(j=1,\dots,n)$ we have

(3.5) $$\max_{h\le u\le(2n-1)h} |e^{-\alpha_1 u}\sum_{j=1}^{n} e^{\alpha_j u}| \ge 1 .$$

In the following we suppose that F satisfies III, IV and V, and collect general versions of well-known facts in the theory of the Riemann zeta function.

The complex number $\rho = \beta+i\gamma$ represents a zero of F , $\sum_\rho$ is extended over all zeros satisfying the indicated conditions under the summation symbol, and for any $a > 0$ and T_1, T_2, T real

(3.6) $$N(a,T_1,T_2) := \sum_{\substack{\rho \\ \beta\ge a \\ T_1\le\gamma\le T_2}} 1 , \qquad N(a,T) := N(a,-T,T) .$$

Lemma 2. For $0 < a < 1$, T real

(3.7) $$N(a,T-1,T+1) \ll \frac{1}{a}(K_4 \mathrm{Log} T+K') .$$

Lemma 3. For $\ell > 1$, $0 < a < 1$ and any real T

(3.8) $$N(a,T-\ell,T+\ell) \ll \frac{\ell}{a}(K_4\mathrm{Log}(|T|+\ell)+K') ,$$

and

(3.9) $$N(a,T) \ll \frac{|T|+1}{a}(K_4\mathrm{Log}(|T|+\ell)+K') .$$

Lemma 4. For $0 < a$, $2a \le b < 1$ we have uniformly in $b \le \sigma \le 4$

$$(3.10)\qquad \frac{F'}{F}(s) = \sum_{|\rho - s| \le b-a} \frac{1}{s-\rho} - \frac{\varkappa}{s-1} + O\left(\frac{1}{a(b-a)}(K_4 \operatorname{Log} t + K')\right) .$$

Lemma 5. If $0 < b < \frac{1}{4}$, then there exists a poligon L, symmetric to the real axis, which consists of horizontal and vertical segments, lies in $b \le \sigma \le 2b$, has arc length $< 2T$ for that part belonging to $0 \le t \le T$, and for which

$$(3.11)\qquad \left|\frac{F'}{F}(s)\right| \ll \frac{1}{b^2}(K_4 \operatorname{Log} t + K')^2$$

whenever $s \in L$ or t is the imaginary part of any horizontal segment of L and $b \le \sigma \le 4$.

As for the proofs, Lemma 2 uses Jensen's inequality $2[\frac{1}{a}]$ times for $R=2$, $r = \sqrt{(2-a)^2+a^2}$, $s_k = 2+i(T+ka)$ $(k=\pm 1,\dots,\pm[\frac{1}{a}])$, Lemma 3 follows from Lemma 2, Lemma 4 uses the Borel-Caratheodory lemma with $z_0 = 2+it$, $R = 2-a$, $r = 2-b$, $G(z) = F(z)(z-1)^{\varkappa}$ and (3.8), and the construction of L can be done in view of Lemmas 1 and 4 by avoiding the zeros of F as much as possible. Compare [10].

4. Let us denote by D the Dirichlet-Laplace transform of d,

$$(4.1)\qquad D(s) = -\int_1^{\infty} d(x)\,dx^{-s} = \int_1^{\infty} x^{-s}\,dd(x) = R(s) - \frac{\varkappa s}{s-1} .$$

Let $k,m \ge 1$, and define

$$U = U(\rho_0) := \frac{1}{2\pi i}\int_{(2)} D(s+\rho_0)e^{ks^2+ms}ds =$$

$$(4.2)\qquad = \frac{1}{2\pi i}\int_{(2)} \left(-\int_1^{\infty} d(x)\frac{d}{dx}(x^{-s-\rho_0})dx\right)e^{ks^2+ms}ds =$$

$$(4.2) \qquad = -\int_1^{\infty} d(x)\frac{d}{dx}\{x^{-\rho_o}\frac{1}{2\sqrt{\pi k}}\exp(-\frac{(\log x-m)^2}{4k})\}\,dx =$$

$$= \frac{1}{2\sqrt{\pi k}}\int_1^{\infty}\frac{d(x)}{x}x^{-\rho_o}(\frac{\log x-m}{2k}+\rho_o)\exp(-\frac{(\log x-m)^2}{4k})\,dx ,$$

where we applied (3.4) and interchanged the order of the integrations and the derivation. The first and the last representations of U in (4.2) give the way to estimate U from above and from below, resp. Split up U to

$$(4.3) \qquad U_1 = \int_1^A , \quad U_2 = \int_A^B , \quad U_3 = \int_B^{\infty}$$

$$(A = e^{m-\mu} , \quad B = e^{m+\mu} , \quad 0 < \mu \le \frac{3m}{4}).$$

For any $0 < \alpha \le \frac{1}{2}$ we have by (3.1)

$$(4.4) \qquad m^{\frac{K_2+1}{2}} \ll e^{(1+K_2)^2+\frac{\frac{K_2+1}{2}}{\alpha}} e^{\alpha m} ,$$

and so by I, using (3.3) we have for any $m \ge |\rho_o|$, $m \ge \mu+4k$, $1 \le k \le \frac{\mu}{4}$

$$|U_1| \le \frac{1}{2\sqrt{\pi k}}\int_1^A (K_1 \mathrm{Log}^{K_2}x+\varkappa)x^{-\beta_o}(\frac{m-\log x}{2k}+|\rho_o|)\exp(-\frac{(\log x-m)^2}{4k})\,dx$$

$$\le \frac{K_1}{\sqrt{k}}m^{K_2}(\frac{m}{2k}+m)\int_1^{e^{m-\mu}} e^{-\frac{(\log x-m)^2}{4k}-\beta_o \log x}\,dx \ll$$

$$\ll K_1 e^{(1+K_2)^2+\frac{\frac{K_2+1}{2}}{\alpha}} e^{\alpha m}\int_{-m/2\sqrt{k}}^{-\mu/2\sqrt{k}} e^{-u^2-\beta_o(2\sqrt{k}u+m)} e^{2\sqrt{k}u+m}\,du \le$$

$$(4.5) \qquad \le K_1 e^{(1+K_2)^2+\frac{\frac{K_2+1}{2}}{\alpha}} e^{m(1-\beta_o+\alpha)+k(1-\beta_o)^2}\int_{\mu/2\sqrt{k}-\sqrt{k}(1-\beta_o)}^{\infty} e^{-v^2}\,dv$$

$$<< M_1 \exp(m(1-\beta_o+\alpha)+\mu(1-\beta_o)-\frac{\mu^2}{4k})$$

$$(M_1 = M_1(\alpha) = K_1 e^{(1+K_2)^2+\frac{K_2+1}{\alpha}}) .$$

Similarly for $m > |\rho_o|$, $m > \mu+4k$, $1 \le k \le \frac{\mu}{4}$, and M_1 defined as in (4.5),

$$|U_3| \le \frac{M_1}{\sqrt{k}} \int_B^\infty \exp\{(\alpha-\beta_o)\log x - \frac{(\log x-m)^2}{4k}\} dx , \tag{4.6}$$

since $m < \log B \le \log x$. By the same substitution $u = \frac{\log x-m}{2\sqrt{k}}$

$$|U_3| \le M_1 \int_{\mu/2\sqrt{k}}^\infty e^{-u^2-(\beta_o-\alpha)(2\sqrt{k}u+m)+2\sqrt{k}u+m} du =$$

$$= M_1 e^{(1+\alpha-\beta_o)m+k(1+\alpha-\beta_o)^2} \int_{\mu/2\sqrt{k}-\sqrt{k}(1+\alpha-\beta_o)}^\infty e^{-v^2} dv << \tag{4.7}$$

$$<< M_1 \exp((1+\alpha-\beta_o)m+\mu(1+\alpha-\beta_o) - \frac{\mu^2}{4k}) .$$

Now, if for a constant C we suppose

$$|d(x)| < Cx^{\beta_o} \qquad x \in [A,B] , \tag{4.8}$$

then for U_2 we get

$$|U_2| \le \frac{1}{2\sqrt{\pi k}} \int_A^B C \left|\frac{\log x-m}{2k}+\rho_o\right| \exp(-\frac{(\log x-m)^2}{4k}) \frac{dx}{x} =$$

$$= \frac{C}{\sqrt{\pi}} \int_{-\mu/2\sqrt{k}}^{\mu/2\sqrt{k}} \left|\frac{v}{\sqrt{k}}+\rho_o\right| e^{-v^2} dv \le \frac{2C}{\sqrt{\pi}} \int_0^\infty (\frac{v}{\sqrt{k}} e^{-v^2} + |\rho_o| e^{-v^2}) dv = \tag{4.9}$$

$$= (|\rho_o| + \frac{1}{\sqrt{\pi k}})C .$$

From (4.5), (4.7) and (4.8) for $m > |\rho_o|$, $k \geq 1$ and for any $0 < \alpha < 1$, $4k < \mu \leq \frac{3}{4}m$ we infer

$$(4.10) \qquad |U| \leq (|\rho_o| + \frac{1}{\sqrt{\pi k}})C + O(M_1 \exp(m(1-\beta_o+\alpha)+\mu(1+\alpha-\beta_o) - \frac{\mu^2}{4k})).$$

Now, all conditions on the parameters are satisfied with the choice of

$$(4.11) \qquad \alpha = \frac{1}{2}, \quad \mu = 12k, \quad m = 16k, \quad k \geq \log Y,$$

and we conclude from (4.10) in view of (2.4) and $\beta_o \geq \frac{1}{2}$, that

$$(4.12) \qquad |U| \leq C(|\rho_o| + \frac{\varepsilon}{6}) + O(M_1 Y^{-8}) \leq C|\rho_o|(1+\frac{\varepsilon}{3}) + \frac{\varepsilon}{6}.$$

5. For the estimation from below, we transform the way of integration to the broken line $L-\rho_o$ (which is justified by Lemma 5, $e^{ks^2+ms} = O(e^{-t^2})$ and III), and get a sum of residues, to be large for a proper choice of k according to Lemma 1.

Namely, from the left-hand side of U in (4.2) we get

$$(5.1) \qquad U = \frac{1}{2\pi i} \int_{L-\rho_o} D(s+\rho_o) e^{ks^2+ms} ds + \sum_\rho{}' e^{k(\rho-\rho_o)^2+m(\rho-\rho_o)},$$

where $\sum'$ indicates the restricted sum over the zeros right from L. With the notation

$$(5.2) \qquad M_2 = M_2(\rho_o) := K_4 \operatorname{Log}|\rho_o| + K'$$

we get from Lemma 5 and (3.2) for the integral in (5.1)

$$(5.3) \qquad \left|\int_L\right| \ll \frac{M_2^2}{b^2} e^{(2b-\beta_o)m+(\beta_o-b)^2 k} \int_0^\infty \operatorname{Log}^2 t e^{-kt^2} \ll \frac{M_2^2 e^{2bm+k-\beta_o m}}{\sqrt{k}\, b^2}$$

and from Lemma 2 and (3.2) for any $\ell \geq 1$ that

$$| \sum_{\substack{\rho \\ |\gamma-\gamma_o|>\ell}}{}' e^{k(\rho-\rho_o)^2+m(\rho-\rho_o)} | \leq \sum_{j=o}^{\infty} e^{(1-\beta_o)m+(1-\beta_o)^2k-(j+\ell)^2k} .$$

(5.4) $$\cdot\{N(b,\gamma_o+\ell+j,\gamma_o+\ell+j+1)+N(b,\gamma_o-\ell-j,\gamma_o-\ell-j-1)\} \ll$$

$$\frac{M_2}{b} e^{(1-\beta_o)m+(1-\beta_o)^2k} \sum_{j=\ell}^{\infty} e^{-j^2k} \mathrm{Log} j \ll \frac{M_2 \mathrm{Log}\ell}{b} e^{2(1-\beta_o)m-\ell^2k} .$$

As for the remaining summands, their number n is at most $N(b,\gamma_o-\ell,\gamma_o+\ell)$, so by Lemma 3

(5.5) $$n \ll \frac{M_2\ell \,\mathrm{Log}\ell}{b} .$$

Applying Lemma 1 for the remaining finite sum we obtain a

(5.6) $$k \in [\log Y, 2n\cdot\log Y]$$

with

(5.7) $$| \sum_{\substack{\rho \\ |\gamma-\gamma_o|\leq\ell}} \{e^{(\rho-\rho_o)^2+\frac{m}{k}(\rho-\rho_o)}\}^k | \geq 1 .$$

Now for this special k, choose

(5.8) $$\ell=5 , \quad b = \frac{1}{16}$$

and by (4.11), (5.5), (5.6) and (5.8) we conclude that the interval (2.5) contains $[A,B]$, defined in (4.3). Moreover, by (2.4), (4.11) and (5.8), we get from (5.1), (5.3), (5.4) and (5.7) that

(5.9) $$|U| \geq 1 - \frac{c_3M_2^2}{\sqrt{k}} e^{-5k} - c_4M_2e^{-9k} \geq 1 - \frac{\varepsilon}{2} .$$

Comparing (4.12) with (5.9), we see that

$$C \geq \frac{1-\frac{2}{3}\varepsilon}{|\rho_o|(1+\frac{\varepsilon}{3})} > \frac{1-\varepsilon}{\rho_o}, \tag{5.10}$$

which proves the existence of an $x \in [A,B]$ satisfying (2.6).

6. Number theoretical applications. Let K be an algebraic number field, n the degree and Δ the discriminant of it. The remainder of the prime ideal theorem is defined by

$$\Delta_K(x) := \Psi_K(x)-x := \sum_{m<x} \Lambda_K(m)-x$$

where

$$\Lambda_K(m) = \sum_{\substack{\mathfrak{P},k \\ N\mathfrak{P}^k=m}} \log N\mathfrak{P},$$

where $\mathfrak{P}$ runs over the prime ideals of K, $k \in \mathbb{N}$, and $N\mathfrak{P}$ is the norm of the ideal $\mathfrak{P}$. For $\zeta_K(s)$, the Dedekind zeta function of K, we have

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} \frac{\Lambda_K(m)}{m^s}.$$

With these notations, it follows (see e.g. [9])

Lemma 6. $\Psi_K \in \mathcal{R}$, and we have for $n \geq 2$

$$K_1 = \frac{n}{\log 2}, \quad K_2=2, \quad \varkappa=1, \quad K_3 = c_5^n|\Delta|^{3/2}, \quad K_4 = \frac{3}{2}n+2, \quad K_5=(\frac{6}{\pi^2})^n.$$

For the most important case $K=\mathbb{Q}$, we have

$$K_1 = 3, \quad K_2 = 0, \quad K_3 = c_6, \quad K_4 = \frac{3}{2}, \quad K_5 = \frac{6}{\pi^2}.$$

Corollary 1 ([7]). If $0 < \varepsilon < 1$, $\rho_o = \beta_o+i\gamma_o$, $\zeta_K(\rho_o) = 0$, and

$$Y > \max\{e^{|\rho_o|}, e^{12/\varepsilon^2}, c_7 n, c_8 \log|\Delta|\}$$

then there exists an

$$x \in [Y, Y^{c_9 n \mathrm{Log}|\rho_o| + c_{10}\log|\Delta|}]$$

for which

$$|\Delta_K(x)| > \frac{(1-\varepsilon)x^{\beta_o}}{|\rho_o|} .$$

Corollary 2 (J. Pintz [6]). If $0 < \varepsilon < 1$, $\rho_o = \beta_o + i\gamma_o$, $\zeta(\rho_o) = 0$, and

$$Y > \max\{c_{11}, e^{|\rho_o|}, e^{12/\varepsilon^2}\}$$

then there exists an

$$x \in [Y, Y^{c_{12}\log|\rho_o|}]$$

for which

$$|\Delta(x)| > (1-\varepsilon)\frac{x^{\beta_o}}{|\rho_o|} .$$

Remark. In the above investigations we defined $\mathcal{R}$ as a class of complex valued functions. But, if r is real-valued, we can prove more either regarding the value of oscillation, or concerning the sign of r (i.e. $\Omega_\pm$ results). To these questions we shall return later.

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