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PERIODIC DECOMPOSITIONS OF FUNCTIONS

1. Let $f:\mathbb{R}\to\mathbb{R}$ be a real function and let $a_1,\dots,a_n$ be given real numbers. We say that $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-decomposition of f if f_i is periodic mod a_i for every $i=1,\dots,n$. If $\mathcal{F}$ is a class of real functions and each f_i belongs to $\mathcal{F}$ then we say that $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-decomposition in $\mathcal{F}$.

Let Δ_a denote the difference operator, that is let

$$\Delta_a f(x)=f(x+a)-f(x) \qquad (x\in\mathbb{R};\ f:\mathbb{R}\to\mathbb{R}).$$

If $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-decomposition then $\Delta_{a_i} f_i=0$ for every i and, as the operators Δ_{a_i} commute, we obtain

$$(1) \qquad \Delta_{a_1}\dots\Delta_{a_n} f = 0.$$

A class $\mathcal{F}$ of real functions is said to have the decomposition property (d.pr.) if, for every $f\in\mathcal{F}$ and $a_1,\dots,a_n\in\mathbb{R}$, (1) implies that f has an $(a_1,\dots,a_n)$-decomposition in $\mathcal{F}$. Neither the class of all real functions, nor $C(\mathbb{R})$, the class of all continuous functions defined on $\mathbb{R}$ has the d.pr. Indeed, if f is the identity function $f(x)=x$ then $\Delta_a\Delta_b f=0$ for every $a,b\in\mathbb{R}$. On the other hand, if, say, $a=b$ then f does not have an (a,b)-decomposition since f is not periodic.

The following result shows that $BC(\mathbb{R})$, the class of all bounded and continuous functions has the d.pr.

THEOREM 1. *Let $a_1,\dots,a_n$ be real numbers and $f\in BC(\mathbb{R})$. Then f has an $(a_1,\dots,a_n)$-decomposition in $C(\mathbb{R})$ if and only if (1) holds.*

A special case of the theorem above, namely when $n=2$ and a_1/a_2 is irrational, was proved by M. Wierdl in [6].

2. By the <u>norm</u> of the decomposition $f=f_1+\dots+f_n$ we mean $\max_{1\le i\le n} \|f_i\|_\infty$, where $\|g\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}$. We denote by C_n the greatest lower bound of all positive numbers C with the property that whenever $f \in BC(\mathbb{R})$ satisfies (1) then f has a continuous $(a_1,\dots,a_n)$-decomposition of norm $\leqq C\|f\|_\infty$.

THEOREM 2. *For every* $n \geqq 2$ *we have* $C_n \leqq 2^{n-2}$.

In certain cases better estimates can be proved.

THEOREM 3. *Suppose that* $f \in BC(\mathbb{R})$ *satisfies* (1), *where* $a_1,\dots,a_n$ *are pairwise incommensurable. Then* f *has a continuous* $(a_1,\dots,a_n)$*-decomposition with norm not exceeding* $(2-\frac{1}{n})\|f\|_\infty$.

Probably neither of the bounds 2^{n-2} and $2-\frac{1}{n}$ is sharp; the problem of finding the best constants in these theorems proves to be surprisingly difficult. The next two theorems give sharp estimates in some special cases.

THEOREM 4. *Suppose that* $f \in BC(\mathbb{R})$ *satisfies* (1), *where either* $n=2$ *or* $n=3$ *and* a_1,a_2,a_3 *are pairwise incommensurable. Then* f *has a continuous* $(a_1,\dots,a_n)$*-decomposition with norm not exceeding* $\|f\|_\infty$.

THEOREM 5. *Suppose that* $1/a_1,\dots,1/a_n$ *are linearly independent over the rationals. If* $f \in BC(\mathbb{R})$ *satisfies* (1) *then*

(i) $$\sup f = \sum_{i=1}^{n} \sup f_i \quad , \quad \inf f = \sum_{i=1}^{n} \inf f_i$$

hold for every continuous $(a_1,\dots,a_n)$*-decomposition of* f;

(ii) *there is a continuous* $(a_1,\dots,a_n)$*-decomposition* $f=f_1+\dots+f_n$ *such that* $\|f\|_\infty = \sum_{i=1}^{n} \|f_i\|_\infty$.

3. Among the (not necessarily bounded) continuous functions satisfying (1) are the polynomials of degree less than n. This observation leads to the following problem: which functions f can be written in the form $f=p+f_1+\dots+f_n$ where p is a polynomial of degree $< n$ and $\Delta_{a_i} f_i=0$ $(i=1,\dots,n)$. We call such a representation an $(a_1,\dots,a_n)$-*quasi-decomposition* of f.

If $f \in C(\mathbb{R})$ has a continuous $(a_1,\dots,a_n)$-quasi-decomposition then (1) must hold. However, it was shown by I.Z. Ruzsa and M. Szegedy that (1) is not sufficient for the existence of such a decomposition. We can give the exact condition in terms of the n-th modulus of continuity of f:

$$\delta_n(f) = \sup_{h \in \mathbb{R}} \|\Delta_h^n f\|_\infty = \sup\left\{\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j} f(x+jh)\right| : x,h \in \mathbb{R}\right\}.$$

THEOREM 6. *A function* $f \in C(\mathbb{R})$ *has an* $(a_1,\dots,a_n)$-*quasi-decomposition in* $C(\mathbb{R})$ *if and only if* (1) *and* $\delta_n(f) < \infty$ *hold simultaneously*.

As a simple application of this condition, we obtain

THEOREM 7. *A function* f *has an* $(a_1,\dots,a_n)$-*quasi-decomposition in* $C(\mathbb{R})$ *with a linear* p *if and only if* (1) *holds and* f *is uniformly continuous*.

4. Let S be a non-empty set and let T be a map of S into itself. A function $g:S \to \mathbb{R}$ is said to be T-periodic, if $g \circ T=g$ or, equivalently, if $\Delta_T g=0$, where $\Delta_T g = g - g\circ T$. Now let $T_1,\dots,T_n$ be maps of S into itself and let $f=f_1+\dots+f_n$ where f_i is T_i-periodic for every $i=1,\dots,n$. If the maps T_i commute, i.e. $T_i \circ T_j = T_j \circ T_i$ hold for every i,j, then the operators Δ_{T_i} also commute and we have

(2) $$\Delta_{T_1}\dots\Delta_{T_n} f = 0.$$

Let $\mathcal{F}$ be a class of real valued functions defined on S. We say that $\mathfrak{F}$ has the decomposition property (d.pr.) with respect to the maps (w.r.t.) $T_1,\dots,T_n$ if for every $f\in\mathfrak{F}$, condition (2) implies that there exists a $(T_1,\dots,T_n)$-decomposition of f in $\mathfrak{F}$, i.e. $f=f_1+\dots+f_n$, where $f_i\in\mathfrak{F}$ and $\Delta_{T_i}f_i=0$ $(i=1,\dots,n)$.

Suppose that the class $\mathfrak{F}$ is closed under linear operations and let T be a map of S into itself. Then $Af = f - f\circ T$ $(f\in\mathfrak{F})$ defines a linear operator on $\mathfrak{F}$ such that Ker A consists of all T-periodic functions from $\mathfrak{F}$. This observation together with the next theorem show that some Banach spaces of functions possess the d.pr. w.r.t. "reasonable" mappings.

THEOREM 8. Let X be a linear space over $\mathbb{R}$, $\|\cdot\|$ be a norm on X and τ be a vector topology on X such that $\{x\in X : \|x\|\leq 1\}$ is τ-compact, and if $x_k\in X$ $(k=1,2,\dots)$ and $\|x_k\|\to 0$ then $x_k\to 0$ in τ.

Let $A_1,\dots,A_n$ be commuting, τ-continuous linear maps of X into itself such that

$$\|A_i - I\|\leqq 1 \quad (i=1,\dots,n).$$

Then $\mathrm{Ker}(A_1\dots A_n)$, as a linear subspace of X, is spanned by the null spaces $\mathrm{Ker}\,A_i$ $(i=1,\dots,n)$.

The conditions of this theorem are satisfied if X is a reflexive Banach space with τ being the weak topology. It can be shown that the assertion of the theorem does not hold for every Banach space X and for every system of commuting linear operators A_i satisfying $\|A_i - I\|\leqq 1$.

Applying this theorem it can be proved that the $L^p(S)$ classes for $1\leqq p<\infty$ possess the d.pr. w.r.t. commuting measurable maps which do not decrease measure, and in σ-finite spaces $L^\infty(S)$ has the d.pr. w.r.t. commuting measurable maps which do not map sets of positive measure into sets of measure zero. Also, the class of all bounded functions defined on S has the d.pr. w.r.t. every commuting system of maps.

As for classes of real functions, we have the following immediate corollary.

THEOREM 9. *Let $\mathcal{F}$ be a translation-invariant normed space of $\mathbb{R}\to\mathbb{R}$ functions. Suppose that there is a translation-invariant vector topology $\mathcal{T}$ on $\mathcal{F}$ such that $\{f\in\mathcal{F} : \|f\|\leqq 1\}$ is $\mathcal{T}$-compact, and whenever $f_n\in\mathcal{F}$ and $\|f_n\|\to 0$ then $f_n\to 0$ in $\mathcal{T}$. Then $\mathcal{F}$ has the d.pr. (w.r.t. translations).*

Making use of this condition, one can prove that each of the following classes has the d.pr.

$$\text{b-BV}^1 = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded and } \sup_x V(f;[x,x+1])<\infty\}$$

$$\text{b-Lip} = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded and Lipschitz}\}$$

$$\text{b-Lip}^k = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded, } f^{(k-1)} \text{ exists everywhere and is Lipschitz}\}.$$

We remark that the d.pr. of the class $BC(\mathbb{R})$ does not follow from Theorem 9. It was proved by V. Totik, that there does not exist a vector topology on $BC(\mathbb{R})$ satisfying the conditions of Theorem 9.

5. We conclude with the following problem:

Is every bounded, continuous solution of a homogeneous difference equation

$$(3)\qquad \sum_{i=1}^{n} c_i f(x+a_i) = 0$$

necessarily uniformly continuous?

(We remark that if we replace (3) by the more general convolution equation $\mu * f=0$ then the answer is negative; see [3]. We also point out the connection of this problem with the investigations of S. Bochner and others concerning continuous solutions of difference equations; see [2].)

If the answer to this problem is affirmative, it provides a simple proof of our Theorem 1. We note first that (1) is a homogeneous difference equation. Now, if $f \in BC(\mathbb{R})$ is uniformly continuous and satisfies (1) then an elementary construction gives a continuous $(a_1,...,a_n)$-decomposition of f via the Arzela-Ascoli lemma. Another approach is the following. Any solution of (3) is mean-periodic, and any bounded and uniformly continuous mean periodic function is uniformly almost periodic (see [3], p.43). Then we also can find an $(a_1,...,a_n)$-decomposition of f using the Fourier series of f (see [1]).

REFERENCES

[1] A.S. Besicovitch: Almost periodic functions. Dover, 1954.

[2] S. Bochner, Über gewisse Differential- und allgemeine Gleichungen, deren Lösungen fastperiodisch sind, Math. Ann. 103(1930),588-597.

[3] J.-P. Kahane: Lectures on mean periodic functions. Tata Institute, Bombay, 1959.

[4] M. Laczkovich and Sz.Gy. Révész, Periodic decompositions of continuous functions, submitted.

[5] M. Laczkovich and Sz.Gy. Révész, Decompositions into periodic functions belonging to a given Banach space, submitted.

[6] M. Wierdl, Continuous functions that can be represented as the sum of finitely many periodic functions, Mat. Lapok 32(1981-84),107-113 (in Hungarian)