



Linear polarization constants of Hilbert spaces[☆]

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Abstract

This paper has been motivated by previous work on estimating lower bounds for the norms of homogeneous polynomials which are products of linear forms. The purpose of this work is to investigate the so-called n th (linear) polarization constant $c_n(X)$ of a finite-dimensional Banach space X , and in particular of a Hilbert space. Note that $c_n(X)$ is an isometric invariant of the space. It has been proved by J. Arias-de-Reyna [Linear Algebra Appl. 285 (1998) 395–408] that if H is a complex Hilbert space of dimension at least n , then $c_n(H) = n^{n/2}$. The same value of $c_n(H)$ for real Hilbert spaces is only conjectured, but estimates were obtained in many cases. In particular, it is known that the n th (linear) polarization constant of a d -dimensional real or complex Hilbert space H is of the approximate order $d^{n/2}$, for n large enough, and also an integral form of the asymptotic quantity $c(H)$, that is the (linear) polarization constant of the Hilbert space H , where $\dim H = d$, was obtained together with an explicit form for real spaces. Here we present another proof, we find the explicit form even for complex spaces, and we elaborate further on the values of $c_n(H)$ and $c(H)$. In particular, we answer a question raised by J.C. García-Vázquez and R. Villa [Mathematika 46 (1999)

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315–322]. Also, we prove the conjectured $c_n(H) = n^{n/2}$ for real Hilbert spaces of dimension $n \leq 5$. A few further estimates have been also derived.

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1. Introduction and notation

Throughout this paper X will be a Banach space over \mathbb{K} , where \mathbb{K} is the real or complex field and X^* will denote the dual space. The closed unit ball and the unit sphere will be denoted by B_X and S_X , respectively. A function $P : X \rightarrow \mathbb{K}$ is a continuous n -homogeneous polynomial if there is a continuous symmetric n -linear form $L : X^n \rightarrow \mathbb{K}$ for which $P(x) = L(x, \dots, x)$, for all $x \in X$. In this case it is convenient to write $P = \hat{L}$. We define

$$\|P\| := \sup\{|P(x)| : x \in B_X\}.$$

For general background on polynomials, we refer to [8]. If f_1, f_2, \dots, f_n are bounded linear functionals on a Banach space X , then the product $(f_1 f_2 \cdots f_n)(x) := f_1(x) \times f_2(x) \cdots f_n(x)$ is a continuous n -homogeneous polynomial on X . R.A. Ryan and B. Turett have shown (see Theorem 9 in [23]), in their study of the strongly exposed points of the predual of the space of continuous 2-homogeneous polynomials, that there exists $C_n > 0$ such that

$$\|f_1\| \|f_2\| \cdots \|f_n\| \leq C_n \|f_1 f_2 \cdots f_n\|.$$

For complex Banach spaces, Benítez et al. [6] have obtained $C_n \leq n^n$, where the constant n^n is best possible. On the other hand, for real Banach spaces X , Ball's solution of the famous *plank problem* of Tarski, see [4], yields the same general result. However, the estimate " n^n " can be improved for specific spaces. Whence the following definition was introduced in [6].

Definition 1 (Benítez et al. [6]). The n th (linear) polarization constant of a normed space X is defined by

$$\begin{aligned} c_n(X) &:= \inf\{M > 0 : \|f_1\| \cdots \|f_n\| \leq M \cdot \|f_1 \cdots f_n\| \ (\forall f_1, \dots, f_n \in X^*)\} \\ &= 1 / \inf_{f_1, \dots, f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)|. \end{aligned} \quad (1)$$

Obviously $c_n(X)$ is a nondecreasing sequence. Its growth, as $n \rightarrow \infty$, is closely related to the structure of the space, hence the asymptotic value is of interest. In particular, a characteristic number is the *linear polarization constant* of the space X .

Definition 2 (Révész and Sarantopoulos [22]). The linear polarization constant of a normed space X is

$$c(X) := \lim_{n \rightarrow \infty} c_n(X)^{1/n}. \quad (2)$$

In fact, we should have put only $c(X) := \limsup_{n \rightarrow \infty} c_n(X)^{1/n}$ as a definition. However, it was proved recently that the limit does exist, see [22, Proposition 4]. Note that $c(X)$ can be infinity as well. More specifically, from [22, Theorem 12] we have.

Theorem A (Révész and Sarantopoulos [22]). *Let X be a real or complex normed space. Then $c(X) = \infty$ if and only if $\dim(X) = \infty$.*

In the special case where X is a Hilbert space, it is easy to see that

$$c_n(X) = \sup\{c_n(Y) : Y \text{ is a closed subspace of } X, \dim Y \leq n\}. \quad (3)$$

The Banach–Mazur distance $d(X, Y)$ between two isomorphic Banach spaces X and Y can be used in comparing the n th polarization constants of these spaces, see [6, Lemma 12]. Recall that the Banach–Mazur distance is defined as

$$d(X, Y) := \inf\{\|T\| \cdot \|T^{-1}\| : T : X \rightarrow Y \text{ is an isomorphism}\}.$$

Proposition A (Benítez et al. [6]). *If X and Y are isomorphic normed spaces, then*

$$c_n(X) \leq d^n(X, Y) c_n(Y). \quad (4)$$

It seems very likely that $c_n(X) \geq c_n(\ell_2^n)$ ($\forall n \leq \dim X$), but to our knowledge this is not known. However, for each $n \in \mathbb{N}$, Hilbert spaces have the smallest n th polarization constant among *infinite-dimensional* normed spaces.

Proposition B (Révész and Sarantopoulos [22]). *If X is an infinite-dimensional normed space, then*

$$c_n(X) \geq c_n(\ell_2^n), \quad \forall n \in \mathbb{N}. \quad (5)$$

Note that for any n -dimensional Banach space X we have

$$d(X, \ell_2^n) \leq \sqrt{n}, \quad (6)$$

which is due to F. John [13]. Thus, to determine $c_n(\mathbb{R}^n)$ is interesting not only in the context of Hilbert space theory. For example, a combination of (3), (4), (6), and (5), yields the following result.

Theorem B. *Let X be an infinite-dimensional normed space over \mathbb{K} and let H be the space ℓ_2 over \mathbb{K} . If $\mathbb{K}^n = \mathbb{R}^n$ or \mathbb{C}^n , that is the space ℓ_2^n over \mathbb{K} , then we have*

$$c_n(H) = c_n(\mathbb{K}^n) \leq c_n(X) \leq n^{n/2} c_n(\mathbb{K}^n) = n^{n/2} c_n(H) \quad (n \in \mathbb{N}). \quad (7)$$

In this work we focus our attention on the problem of estimating the linear polarization constants of Hilbert spaces. Although this is a classical topic, [15–17], there has been a flourishing activity on this field even in the last ten years, see [2,3,6,11,14,20,22] and just recently also [1,18,19,21].

Ideally, one should look for the exact values of $c_n(\ell_2^d) = c_n(\mathbb{K}^d)$, for any $d, n \in \mathbb{N}$, which, in view of (3), reduces to $d \leq n \in \mathbb{N}$. In fact, this question is posed in [14], attributed to the referee of the paper. In this direction a remarkable success is Arias-de-Reyna's result.

Theorem C (Arias-de-Reyna [2]).

$$c_n(\mathbb{C}^n) = n^{n/2}.$$

An even more precise description was obtained recently by K. Ball [3]. Estimating $c_n(\mathbb{R}^n)$ seems to be a harder task. In particular, the proof of the stronger result due to K. Ball relies heavily on complex function theoretic tools which are not valid in the case of \mathbb{R}^n . Observe also that Arias-de-Reyna's entirely different technique, based on permanents, multilinear algebra and probability theory (Gaussian random variables in particular), strongly depends on the complex structure of \mathbb{C}^n . As Arias-de-Reyna has mentioned in [2], his Theorem C leads to an upper estimate $c_n(\mathbb{R}^n) \leq 2^n n^{n/2}$ even for the real case. This has been improved in [11, 14], until a more refined approach was worked out in [20], using the natural complexification of a real Hilbert space. This has led to the best estimate we are aware of.

Theorem D (Révész and Sarantopoulos [22]).

$$n^{n/2} \leq c_n(\mathbb{R}^n) \leq 2^{n/2-1} n^{n/2}.$$

In Section 4 we shall discuss the real approach in detail. This seems to be interesting even though the resulting exponential factor falls, unfortunately, only between $\sqrt{2}$ and 2, as this approach avoids any reference to the complex case. Let us mention here the following conjecture, mentioned already in [2, 6] and formulated also in [22].

Conjecture.

$$c_n(\mathbb{R}^n) = n^{n/2}. \quad (8)$$

In all, for a Hilbert space H of infinite dimension we only know that $c(H) = \infty$ and $\log c_n(H) \sim \frac{1}{2}n \log n$. On the other hand, if $\dim H = d$ is fixed then by Theorem 12 in [22] $c(H)$ must have a finite value as $n \rightarrow \infty$.

Trying to determine $c_n(\mathbb{K}^d)$ for arbitrary $d \leq n \in \mathbb{N}$, by natural extrapolation one might have thought that $c_n(\mathbb{K}^d) = d^{n/2}$. However, this fails by a recent result [1] which connects polarization constants to *Chebyshev constants*.

Definition 3. The n th *Chebyshev constant* of a subset $F \subseteq X$ in a metric space (X, ρ) is

$$M_n(F) := \inf_{y_1, \dots, y_n \in F} \sup_{y \in F} \rho(y, y_1) \cdot \dots \cdot \rho(y, y_n). \quad (9)$$

In particular, if X is a normed space and thus $\rho(x, y) = \|x - y\|$, then

$$M_n(F) := \inf_{y_1, \dots, y_n \in F} \sup_{y \in F} \|y - y_1\| \cdot \dots \cdot \|y - y_n\|. \quad (10)$$

Theorem E (Anagnostopoulos and Révész [1]). *We have*

$$c_n(\mathbb{R}^2) = 2^n / M_n(S^1) \quad \text{and} \quad c_n(\mathbb{C}^2) = 2^n / M_n(S^2), \quad (11)$$

where S^m denotes the unit sphere in \mathbb{R}^{m+1} , for any $m \in \mathbb{N}$.

This result is instructive for two reasons. First, since $M_n(S_1) = 2$ is well known and easy to derive, we immediately obtain not only $c_2(\mathbb{R}^2) = 2$, as we wanted, but also that $c_n(\mathbb{R}^2) > 2^{n/2}$ for all $n > 2$. Second, the constant $M_n(S^2)$ is not known and to find its exact value seems to be out of reach. In fact, this problem leads to nontrivial geometrical problems fitting into geometric discrepancy theory and known as “Whyte’s problem,” see [1,26]. The situation clearly suggests that to determine $c_n(\mathbb{K}^d)$ is, in general, even more hopeless. On the other hand, some estimates on the Chebyshev constants are known. The best known estimate is due to G. Wagner [25], which leads to

Corollary A (Anagnostopoulos and Révész [1]). *We have*

$$c(\mathbb{R}^2) = 2 \quad \text{and} \quad c(\mathbb{C}^2) = \sqrt{e}. \quad (12)$$

In the course of proof of Theorem 12 in [22], Révész and Sarantopoulos have shown that

$$c_n(\mathbb{K}^d) \leq e^{-nL(d, \mathbb{K})}. \quad (13)$$

Here and throughout the constant $L(d, \mathbb{K})$ is defined by

$$L(d, \mathbb{K}) := \int_{S_{\mathbb{K}^d}} \log | \langle x, s \rangle | d\sigma(x) \quad (< 0) \quad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}), \quad (14)$$

where $d\sigma(x)$ denotes the normalized surface Lebesgue measure of $S_{\mathbb{K}^d}$ and $s \in S_{\mathbb{K}^d}$ is arbitrary. Furthermore, it has been mentioned in [22, Remark 3], that by more elaborate analysis asymptotic sharpness of (13) can be shown. However, the authors of [22] were not aware of the fact that for real Hilbert spaces this has already been obtained by García-Vázquez and Villa [11].

Theorem F (García-Vázquez and Villa [11]). *We have $c(\mathbb{R}^d) = e^{-L(d, \mathbb{R})}$, with the constants $L(d, \mathbb{R})$ defined by (14).*

The outset of the present paper was to accomplish the work hinted in [22]. The real case being already known, the new part of our result is the following.

Theorem 1. *We have $c(\mathbb{C}^d) = e^{-L(d, \mathbb{C})}$, with the constants $L(d, \mathbb{C})$ defined by (14).*

We present our unified, different proof for Theorems F and 1 in Section 2. While [11] is based on a result for symmetric functions on compact metric spaces, our direct approach avoids use of anything like that, and is based on probabilistic as well as potential theoretic arguments. For completeness, we describe the full proof, although some overlapping occurs naturally at some points. In particular, we present the explicit exact formula of $L(d, \mathbb{K})$ in Lemmas 10 and 11, although the real case has already been computed in [11].

Remark 1. As the direct calculation of $L(2, \mathbb{R})$ and $L(2, \mathbb{C})$ shows, Corollary A is in fact a special case of Theorem 1 for $d = 2$. However, the connection of $c_n(\mathbb{K}^d)$ and $M_n(S^m)$, which is special to the geometry of \mathbb{K}^2 only, does not extend to arbitrary d .

Still, we can provide some information concerning a few low-dimensional cases which seem to support our conjecture.

The next statement has already been claimed (for $n = 2, 3, 4$), cf. [6, Lemma 10], but no proof has been published yet.

Theorem 2. *We have $c_n(\mathbb{R}^n) = n^{n/2}$ for $n = 2, 3, 4$, and 5. Therefore, for $n \leq \min\{d, 5\}$ we also have $c_n(\mathbb{R}^d) = n^{n/2}$.*

Further relations and estimates on the values of the polarization constants will be discussed in Section 4.

2. Common proof of Theorems F and 1

By the Riesz representation theorem the polynomial $P_n(x) := f_1(x) \cdots f_n(x)$, where $f_k \in S_{H^*}$, $1 \leq k \leq n$, can be written in the form

$$P_n(x) = \langle x, y_1 \rangle \cdot \langle x, y_2 \rangle \cdots \langle x, y_n \rangle, \quad (15)$$

where $y_j \in S_H^* = S_H$.

For any system of points $\{y_j\}_{j=1}^n \subset S_H$ let us consider the discrete measure

$$\mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{y_j}, \quad (16)$$

where δ_y stands for the Dirac measure concentrated at y . Clearly $0 \leq \mu_n$ and $\|\mu_n\| = 1$. We consider $(S^{d-1}, \mathcal{B}, \sigma)$ as a probability space, where \mathcal{B} is the sigma algebra of Borel measurable sets. When thought of as the probability space, we denote S^{d-1} as Ω . Now our basic random variable ξ from Ω to S^{d-1} is a random selection of a point $x \in S^{d-1}$ according to the uniform distribution σ , that is $P(\xi \in A) = \sigma(A)$ for sets $A \in \mathcal{B}$. Moreover, we consider now totally independent copies ξ_k for $k \in \mathbb{N}$ of the above $\xi = \xi_1$. On the product probability space or *sample space* $\Omega_\infty := \prod_{k=1}^\infty \Omega_k$, where Ω_k are identical copies of Ω , a random event or *sample* is a sequence $\omega = (y_k)_{k=1}^\infty$. First we prove the following.

Lemma 3. *For any given continuous function $f \in C(S^{d-1})$ we have*

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\sigma \quad \text{with } \mu_n = \mu_n(\omega) \text{ for a.a. } \omega \in \Omega_\infty. \quad (17)$$

Here for $\omega = (y_k)_{k=1}^\infty$ $\mu_n(\omega)$ is defined in (16) with the points $(y_k)_{k=1}^n$.

Proof. With ξ as above, we define $\zeta := f(\xi) = \int_S f d\delta_\xi$. Then ζ is a real valued random variable on Ω . From the normalizing conditions we have that the expectation (mean) is $m := E(\zeta) = \int_\Omega f(\xi(\omega)) dP(\omega) = \int_S f d\sigma$. Consider now the totally independent copies $\zeta_n := f(\xi_n)$ of the random variable $\zeta = f(\xi)$ for all $n \in \mathbb{N}$. So we have a sequence (ζ_n) of independent, identically distributed random variables with common mean m . The assertion is that the average of these variables tend to the mean almost surely, i.e., we have $\sigma_n := \frac{1}{n}(\zeta_1 + \cdots + \zeta_n) \rightarrow m$ for $n \rightarrow \infty$ a.s.

For these random variables the *strong law of large numbers* applies, cf. Theorem of [10, X. 7, p. 244]. That is, for any given $\epsilon, \delta > 0$ we have $P(|\sigma_n - m| \leq \epsilon) > 1 - \delta$ when $n \geq N := N_0(\delta, \epsilon)$. This formulation of the law from [10, X. 7. p. 243] is equivalent to state that for a.a. $\omega \in \Omega_\infty$ we have $\sigma_n(\omega) \rightarrow m$, cf. footnote 12 on p. 243 of [10]. (This comprises a straightforward application of the Borel Cantelli lemma in the product space Ω_∞ .) Whence the lemma obtains. \square

We will need the following, seemingly plausible assertion, which requires a proof.

Lemma 4. *There exist appropriate choice of points $\{y_1^n, \dots, y_n^n\} \subset S^{d-1}$ so that with the corresponding sequence of measures (μ_n) of the form (16) we have $\mu_n \xrightarrow{\text{weak}^*} \sigma$ ($n \rightarrow \infty$), where σ denotes the normalized surface measure of S^{d-1} .*

Proof. Let $\mathcal{F} := \{f_j : j \in \mathbb{N}\}$ be a countable, dense subset of $\mathcal{C}(S^{d-1})$, and let us fix first any function $f := f_j \in \mathcal{F}$. In view of the above Lemma 3, there is a subset $\Omega^j \subset \Omega_\infty$ with $\mathbb{P}(\Omega^j) = 1$ (where \mathbb{P} is probability measure on the product space Ω_∞), so that (17) holds for all $\omega \in \Omega^j$.

It follows that for almost all sequences $\omega \in \Omega_\infty$ we have (17) simultaneously for *all* functions $f_j \in \mathcal{F}$. Now if $g \in \mathcal{C}(S^{d-1})$, ϵ is any given positive number, and $\|g - f_j\| < \epsilon$, then for a “good” $\omega \in \Omega_\infty$ we have by (17)

$$-\epsilon \leq \liminf_{n \rightarrow \infty} \int g d\mu_n - \int f_j d\sigma \leq \limsup_{n \rightarrow \infty} \int g d\mu_n - \int f_j d\sigma \leq \epsilon, \quad (18)$$

that is $\limsup_{n \rightarrow \infty} \int g d\mu_n - \liminf_{n \rightarrow \infty} \int g d\mu_n \leq 2\epsilon$, and the limit exists: moreover, (18) implies that then the limit is equal to $\int g d\sigma$, as asserted. \square

Remark 2. During the period when we were looking for a concise proof or direct reference for this Lemma 4, J. Diestel [7] gave us the suggestion of using W. Maak’s proof for the existence of a Haar measure (see, e.g., [9]), while V. Totik [24] suggested use of the Krein–Milman Theorem. Both ideas lead to alternative proofs of the lemma. However, we still do not know any direct reference which would cover the statement as it is.

Consider now the polynomials (15) determined by the points $\{y_1^n, \dots, y_n^n\} \subset S_H$ defined in Lemma 4. Clearly $c_n(H) \geq 1/\|P_n\|$. Hence by definition (1), for a lower estimate of $c_n(H)$ we need an upper estimate of $\|P_n\|$. Since $\dim H = d < \infty$, S_H is compact, and $\|P_n\| = |P_n(z_n)|$ for some $z_n \in S_H$. Thus, by the compactness of S_H , z_n has a convergent subsequence. Without loss of generality we can suppose $z_n \rightarrow z$.

Now let $\epsilon > 0$ be fixed and consider the function $\phi(x) := \log |\langle x, s \rangle|$ for $x \in S_H$ and $s = (1, 0, \dots, 0) \in S_H$. Clearly $-\infty \leq \phi \leq 0$ on S_H , and ϕ is $d\sigma$ integrable. Since ϕ is upper semi-continuous on the compact set S_H , there exists a sequence $\phi_m \in C(S_H)$ of continuous functions, which converges to ϕ monotonically. In fact, it suffices to take

$\phi_m := \max\{\phi, -m\}$. Thus since $\phi \in L^1(\sigma)$ we also have $\phi_m \in L^1(\sigma)$ and by the monotone convergence theorem

$$\int_{S_H} \phi_m(x) d\sigma(x) \rightarrow \int_{S_H} \phi(x) d\sigma(x) \quad \text{as } m \rightarrow \infty. \quad (19)$$

So with ε given, choose $m_0(\varepsilon)$ so that

$$\int_{S_H} \phi_m(x) d\sigma(x) < \int_{S_H} \phi(x) d\sigma(x) + \varepsilon \quad (m > m_0). \quad (20)$$

For the following, m and thus $\phi_m \in C(S_H)$, will be fixed. Note that by definition (14) and that of $\phi(x)$ above we have

$$\int_{S_H} \phi d\sigma = L(d, \mathbb{K}). \quad (21)$$

So now we can write

$$\begin{aligned} \|P_n(x)\|^{1/n} &= |P_n(z_n)|^{1/n} = \exp\left\{\frac{1}{n} \sum_{j=1}^n \log|\langle z_n, y_j^n \rangle|\right\} \\ &= \exp\left\{\int_{S_H} \log|\langle z_n, y \rangle| d\mu_n(y)\right\} \\ &\leq \exp\left\{\int_{S_H} \max\{\log|\langle z_n, y \rangle|, -m\} d\mu_n(y)\right\} \\ &= \exp\left\{\int_{S_H} \Psi_n(y) d\mu_n(y)\right\} \quad (\text{with } \Psi_n(y) := \max\{\log|\langle z_n, y \rangle|, -m\}). \end{aligned} \quad (22)$$

Take now $\Psi := \max\{\log|\langle z, y \rangle|, -m\}$. Since the function $\alpha(u) := \max\{\log|u|, -m\}$ belongs to $C([-1, 1])$, and $\beta_n(y) := \langle z_n, y \rangle$ converges uniformly to $\beta(y) := \langle z, y \rangle$, it is easy to see that the functions Ψ_n are uniformly convergent to Ψ , as $\Psi = \alpha \circ \beta$ and $\Psi_n = \alpha \circ \beta_n$.

Hence as $\mu_n \xrightarrow{\text{weak}^*} \sigma$ we get

$$\int_{S_H} \Psi_n d\mu_n \rightarrow \int_{S_H} \Psi d\sigma \quad \text{as } n \rightarrow \infty. \quad (23)$$

Taking now $n_0 = n_0(\varepsilon, m)$ sufficiently large we obtain for $n > n_0$

$$\int_{S_H} \Psi_n(y) d\mu_n(y) < \int_{S_H} \Psi(y) d\sigma(y) + \varepsilon = \int_{S_H} \phi_m d\sigma + \varepsilon, \quad (24)$$

by the symmetry of $\Psi(y) = \max\{\log|\langle z, y \rangle|, -m\}$ and $\phi_m(y) = \max\{\log|\langle s, y \rangle|, -m\}$ with respect to z or s . Collecting (20), (23), (24), and (21) now gives

$$\|P_n\|^{1/n} \leq \exp\{L(d, \mathbb{K}) + 2\varepsilon\} \quad (n \geq n_0). \quad (25)$$

As noted above, (25) now leads to

$$c_n(H)^{1/n} \geq \exp\{-L(d, \mathbb{K}) - 2\varepsilon\} \quad (n \geq n_0),$$

and thus by taking limits we obtain

$$c(H) \geq \exp\{-L(d, \mathbb{K}) - 2\varepsilon\}. \quad (26)$$

Since (26) holds for all $\varepsilon > 0$, the lower estimate $c(H) \geq e^{-L(d, \mathbb{K})}$ is established.

Now let $y_1, \dots, y_n \in S_H$. By compactness, we can even choose these points to be extremal, that is, take $1/c_n(H) = \max_{x \in S_H} \prod_{m=1}^n |\langle y_m, x \rangle|$. Similarly to (22), we are dealing with the quantity

$$\frac{1}{c_n(H)} = \max_{x \in S_H} \prod_{m=1}^n |\langle y_m, x \rangle| = \exp\left(\max_{x \in S_H} \sum_{m=1}^n \log |\langle y_m, x \rangle|\right) =: \exp\left(\max_{x \in S_H} \varphi(x)\right).$$

Note that $\varphi(x)$ is bounded from above by 0 and is upper semicontinuous, hence it really has a maximum over the compact set S_H . To estimate the maximum value M of the function $\varphi(x)$, we use averaging over the unit sphere. That is, we integrate with respect to the normalized surface measure $d\sigma$ of S_H . Then we are led to

$$M := \max_{x \in S_H} \varphi(x) \geq \int_{S_H} \varphi(x) d\sigma(x) = \int_{S_H} \sum_{m=1}^n \log |\langle y_m, x \rangle| d\sigma = nL(d, \mathbb{K}), \quad (27)$$

since, independently of the particular value of the unit vectors y_m , we have

$$\int_{S_H} \log |\langle y_m, x \rangle| d\sigma = L(d, \mathbb{K}) \quad (m = 1, \dots, n).$$

Collecting the facts above leads to

$$c_n(H) = e^{-M} \leq e^{-nL(d, \mathbb{K})}, \quad (28)$$

which concludes the proof of Theorems F and 1.

García-Vázquez and Villa point out on [11, p. 319] that

$$\inf_{n \in \mathbb{N}} \frac{1}{c_n(\mathbb{R}^d)^{1/n}} = e^{L(d, \mathbb{R})}, \quad (29)$$

and expose the question whether this infimum is actually a minimum. Analyzing the above proof, we can answer this question to the negative both for real and complex spaces.

Proposition 5. For all $n \in \mathbb{N}$ we have $c_n(\mathbb{K}^d) < e^{-nL(d, \mathbb{K})}$.

Proof. It suffices to see that in the upper estimate of $c_n(H)$ the averaging over S_H with respect to $d\sigma$ leads to strict inequality, that is, $M > nL(d, \mathbb{K})$ in (27). As $\varphi(x)$ is integrable with respect to $d\sigma$, for equality in (27) we should have $\varphi(x) = M$ for σ -a.e. $x \in S_H$. However, considering any point y_m , we find that in the “equatorial strip” $\{x \in S_H: |\langle x, y_m \rangle| \leq \delta\}$ the inequality $\varphi(x) \leq \log \delta$ holds. For small $\delta > 0$ the strip is still a positive measure subset of S_H with values strictly less than M , hence the average differs from the maximum value M and the inequality is strict. \square

3. Proof of Theorem 2

We start with a lemma which was communicated to Y. Sarantopoulos by A.M. Tonge.

Lemma 6. *Let H be a real Hilbert space and let $a_k \in S_H$, $k = 1, \dots, n$, be arbitrary unit vectors. Suppose that for some unit vector $\xi \in S_H$ and with some $\delta > 0$ we have*

$$\max_{\|x\|=1, \|x-\xi\|\leq\delta} |\langle a_1, x \rangle| \cdots |\langle a_n, x \rangle| = |\langle a_1, \xi \rangle| \cdots |\langle a_n, \xi \rangle|, \quad (30)$$

that is, ξ is a local (conditional) maximum on S_H . Then we have

$$n\xi = \frac{a_1}{\langle a_1, \xi \rangle} + \cdots + \frac{a_n}{\langle a_n, \xi \rangle}. \quad (31)$$

Proof. [22] mentions a variational proof: we will give a more transparent geometric argument. A slight advantage of this approach is that it makes no use of linear independence of the vectors a_k . So let us write

$$P(x) := \prod_{k=1}^n \langle a_k, x \rangle.$$

Any sequence of vectors a_1, \dots, a_n can be supposed, for the present purposes, as lying in \mathbb{R}^n . Clearly, if ξ is a local maximum we must have $P(\xi) \neq 0$ and $\xi \in S_H$. So we also suppose, as we may, $H = \mathbb{R}^n$, and we consider the vector

$$v := \frac{1}{P(\xi)} \cdot \text{grad } P(\xi) = \sum_{k=1}^n \frac{a_k}{\langle a_k, \xi \rangle}. \quad (32)$$

If $u \in \mathbb{R}^n$ is any vector, then we have

$$P(\xi + tu) = P(\xi) + P(\xi)\langle v, tu \rangle + o(t^2) \quad (t \rightarrow 0). \quad (33)$$

Clearly for all vector with $\langle u, \xi \rangle < 0$ and t sufficiently small but positive $\|\xi + tu\| < 1$, and P being homogeneous, by condition of maximality $|P(\xi + tu)| < |P(\xi)|$. Thus (33) leads to $\langle v, tu \rangle + o(t^2) \leq 0$, i.e., $\langle v, u \rangle \leq 0$ whenever $\langle u, \xi \rangle < 0$. It follows that v can only be an outer normal of the unit sphere at ξ , that is, $v = \lambda\xi$ with $\lambda \geq 0$. Substituting this into (32), and taking inner product with ξ we obtain

$$\lambda = \langle \lambda\xi, \xi \rangle = \langle v, \xi \rangle = \sum_{k=1}^n \left\langle \frac{a_k}{\langle a_k, \xi \rangle}, \xi \right\rangle = n. \quad (34)$$

Hence $v = \lambda\xi = n\xi$ and (32) entails (30). \square

Proof of Theorem 2. Note that the case $n = 2$ is already contained in Theorem E, for $M(S^1) = 2$ is obvious. Let now $n = 2, 3, 4$ or 5 , and let the linear functionals be fixed as $a_1, \dots, a_n \in S_{\mathbb{R}^n}$. Consider the Gram matrix

$$A := (\langle a_i, a_j \rangle)_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}. \quad (35)$$

By an appropriate change of signs $\epsilon_i = \pm 1$ of the vectors a_i , which does not change norm of P , we want to achieve that the row (and thus the column) sums of the entries of A

are all add up to at least 1. To get this, select signs ϵ_i to maximize $\|\sum_{i=1}^n \epsilon_i a_i\|_2$. Write $a := \sum_{i=1}^n \epsilon_i a_i$ for this (or, any) maximal vector. If $1 \leq j \leq n$ is an arbitrary index, put $b := -2\epsilon_j a_j + a$: then $\|b\|_2 \leq \|a\|_2$ by assumption. On the other hand, by the parallelogram law $\|-\epsilon_j a_j + a\|_2^2 + \|\epsilon_j a_j\|_2^2 = 1/2(\|a\|_2^2 + \|b\|_2^2) \leq \|a\|_2^2$, that is, $\langle \epsilon_j a_j, a - \epsilon_j a_j \rangle$ is nonnegative. Obviously this implies $\langle a, \epsilon_j a_j \rangle \geq 1$, as needed. So without loss of generality we can assume

$$\begin{aligned} y_1 &:= \langle a_1, a_1 \rangle + \langle a_1, a_2 \rangle + \cdots + \langle a_1, a_n \rangle \geq 1, \\ y_2 &:= \langle a_2, a_1 \rangle + \langle a_2, a_2 \rangle + \cdots + \langle a_2, a_n \rangle \geq 1, \\ &\vdots \\ y_n &:= \langle a_n, a_1 \rangle + \langle a_n, a_2 \rangle + \cdots + \langle a_n, a_n \rangle \geq 1. \end{aligned} \quad (36)$$

Let us note that the above argument is the special case $r_j = 1$ ($j = 1, \dots, n$) of a more general version, known as Bang's lemma, see [5]. For this special case the argument occurs several places, see, e.g., [17] or [14, Lemma 2.4.1(i)]. Now let us consider the mean vector

$$x := \frac{a}{\|a\|_2} = \frac{a_1 + \cdots + a_n}{\|a_1 + \cdots + a_n\|_2}. \quad (37)$$

The theorem will be proved once we show the following lemma. \square

Lemma 7. *Let $n \leq 5$. Suppose that the signs of the unit vectors a_i ($i = 1, \dots, n$) are chosen so that (36) holds. Then the mean vector (37) satisfies $|P(x)| \geq n^{-n/2}$.*

Proof. By definition and (36), we have $1 \leq y_i \leq n$ ($i = 1, \dots, n$). The assertion is equivalent to state that the inequality

$$y_1^2 y_2^2 \cdots y_n^2 \geq \left(\frac{y_1 + y_2 + \cdots + y_n}{n} \right)^n \quad (38)$$

holds true for all the possible vectors $y := (y_1, y_2, \dots, y_n)$, which arise from Gram matrices (35) of unit vector systems satisfying (36). However, it is rather difficult to describe the exact set of the arising vectors y , so we settle with the following. \square

Lemma 8. *Let $n \leq 5$. Then (38) holds true for all $y \in [1, n]^n$.*

Proof. First we remark that $n^2 \geq (2 - \frac{1}{n})^n$ for $n = 2, 3, 4$, and 5, while it is false for $n > 5$. However, $n^2 \geq (2 - \frac{1}{n})^n$ is just the special case of (38) when $y = (1, \dots, 1, n)$, whence in general (38) fails at $y = (1, \dots, 1, n)$ for $n > 5$.

So let $n \leq 5$ and let us exploit the fact that (38) holds when $y = (1, \dots, 1, n)$. First let us consider the variable values $y(t) := (1, \dots, 1, t)$ in the interval $1 \leq t \leq n$. For these special values the left-hand side of (38) is t^2 and the right-hand side is $(\frac{n-1+t}{n})^n$, hence (38) is equivalent to $q(t) \geq 0$ with $q(t) := 2 \log t - n \log(\frac{n-1+t}{n})$. By the above we have $q(n) \geq 0$, while $q(1) = 0$, hence it suffices to show that $q(t)$ is, in fact, a concave function on $[1, n]$. This follows from computing

$$q''(t) = \frac{-2}{t^2} + \frac{n}{(n-1+t)^2} = \frac{(n-2)t^2 - 4(n-1)t - 2(n-1)^2}{t^2(n-1+t)^2} < 0,$$

the last inequality being valid between the two roots $t_1^{(n)}, t_2^{(n)}$ of the quadratic polynomial in the numerator. (Here, again, one has to use the restricted range of n when calculating $[1, n] \subset [t_1^{(n)}, t_2^{(n)}]$.)

Let now m be the number of indices of coordinates y_j with $1 < y_j \leq n$. When $m = 0$, (38) degenerates to $1 = 1$, and when $m = 1$, we obtain (38) from the above consideration for $y(t)$. So we argue by induction. Let now $1 \leq m < n$, suppose that (38) holds for the values when at most m of the variables differ from 1, and let us prove (38) for the vector $y = (1, \dots, 1, y_k, y_{k+1}, \dots, y_n)$, where $k := n - m$. First let us apply the inductive hypothesis for $y = (1, \dots, 1, y_{k+1}, \dots, y_n)$ to get

$$y_{k+1}^2 \cdots y_n^2 \geq \left(\frac{k + y_{k+1} + \cdots + y_n}{n} \right)^n. \quad (39)$$

Now put $t := 1 + \frac{n(y_k - 1)}{k + y_{k+1} + \cdots + y_n}$. Then obviously $1 \leq t \leq y_k$, hence by the $m = 1$ case of $y(t)$ we get

$$y_k^2 \geq t^2 \geq \left(1 + \frac{t - 1}{n} \right)^n = \left(1 + \frac{y_k - 1}{k + y_{k+1} + \cdots + y_n} \right)^n. \quad (40)$$

Multiplying together Eqs. (39) and (40) gives (38). \square

The above approach was crude in two respects. First, the conjecture can be valid even if the mean vector (37) fails to satisfy $|P(x)| \geq n^{-n/2}$. In fact, this is necessary only for the vector ξ in (31), when also the 2^n possible choices of signs of a_k can be varied. In fact, for all system of signs the corresponding ξ is a local maxima of $|P(x)|$, but there is no guarantee, that among all 2^n local extremal values just the one used here will be the global maximum place. Still, in the orthogonal case all mean vectors will coincide with the extremal points, and the approach seem to work. So one can ask the following, even more daring question than the above conjecture.

Question. Is it true, that for any n and unit vectors a_i ($i = 1, \dots, n$), with suitably chosen signs of the unit vectors a_i ($i = 1, \dots, n$), the mean value vector (37) satisfies $|P(a)| \geq n^{-n/2}$?

Second, it is rather optimistic for one to seek the validity of (38) over the whole $[1, n]^n$. We have seen that this fails for $n > 5$ at $y(n) = (1, \dots, 1, n)$, however, note that this $y(n)$ can never be exhibited as a vector arising from the Gram matrix (35). Indeed, if $y_n = n$, then $\langle a_j, a_n \rangle = 1$ for all j , and thus all a_j are equal and $y_k = n$ for all k . It remains to future work to exploit the intrinsic geometric features hidden in the restriction on the admissible values of y .

4. Some further calculations

First we aim at finding the constants $L(d, \mathbb{K})$ defined by (14). For this, we need the exact value of some definite integrals. To express parity conditions we define

$$\varepsilon(k) = \begin{cases} 1 & \text{if } k \text{ odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases} \quad (41)$$

Recall that the semifactorials $m!!$ are defined as $m!! := \prod_{k=1}^m k^{\varepsilon(m-k+1)}$ and the logarithmic factorial function ψ is defined by the series

$$\psi(z) = -C + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right) \quad (z \in \mathbb{C}),$$

where $C := \lim_{n \rightarrow \infty} (\sum_{m=1}^n 1/m - \log n) = 0.5772 \dots$ is Euler's constant.

For our purposes we shall make use of the following two integrals which can be found in [12], see formulas 4.241/5, p. 533, 4.246, p. 534, and 4.253/1, p. 535:

$$\begin{aligned} \int_0^1 \sqrt{(1-x^2)^{2n-1}} \log x \, dx &= -\frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [\psi(n+1) + C + \ln 4] \\ &= -\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{4} \left[2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right] \end{aligned} \quad (42)$$

and

$$\int_0^1 x^{\mu-1} (1-x^r)^{\nu-1} \log x \, dx = \frac{1}{r^2} B\left(\frac{\mu}{r}, \nu\right) \left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right) \right], \quad (43)$$

where B is the beta function, $\operatorname{Re} \mu > 0$, $\operatorname{Re} \nu > 0$ and $r > 0$.

To calculate the integral $\int_0^1 (1-x^2)^{n/2} \log x \, dx$, if n is odd we just apply (42). If n is even, then we have to apply (43) with $\mu = 1$, $\nu = (n+2)/2$ and $r = 2$. As for the integral $\int_0^1 (1-x^2)^n x \log x \, dx$, we apply (43) with $\mu = 2$, $\nu = n+1$ and $r = 2$. After some easy calculations we eventually derive the following formulas.

Lemma 9. For any $n \in \mathbb{N}$ we have

$$\int_0^1 (1-x^2)^{n/2} \log x \, dx = \begin{cases} -\frac{n!!}{(n+1)!!} \sum_{m=0}^{n/2} \frac{1}{2m+1} & \text{if } n \text{ is even,} \\ -\frac{n!!}{(n+1)!!} \frac{\pi}{2} \left(\sum_{m=1}^{(n+1)/2} \frac{1}{2m} + \log 2 \right) & \text{if } n \text{ is odd,} \end{cases} \quad (44)$$

and

$$\int_0^1 (1-x^2)^n x \log x \, dx = -\frac{1}{4(n+1)} \sum_{k=1}^{n+1} \frac{1}{k}. \quad (45)$$

Here, as usual, empty sums are considered 0.

In the following σ_m stands for the surface area of the unit sphere S^{m-1} in \mathbb{R}^m , that is,

$$\sigma_m = \begin{cases} \frac{\pi^{m/2} m}{(m/2)!} = \frac{2^{m/2} \pi^{m/2} m}{m!!} & \text{for } m \text{ even,} \\ \frac{2^{(m+1)/2} \pi^{(m-1)/2} m}{m!!} & \text{for } m \text{ odd.} \end{cases} \quad (46)$$

Let us consider first the real case $\mathbb{K} = \mathbb{R}$.

Lemma 10 (García-Vázquez and Villa [11]). *Treating empty sums as 0, for any $d \geq 2$ we have*

$$-L(d, \mathbb{R}) = \begin{cases} \sum_{m=1}^{(d-2)/2} \frac{1}{2m} + \log 2 & \text{if } d \text{ is even,} \\ \sum_{m=0}^{(d-3)/2} \frac{1}{2m+1} & \text{if } d \text{ is odd.} \end{cases} \quad (47)$$

Proof. Standard calculus gives

$$L(2, \mathbb{R}) = \frac{1}{2\pi} \int_0^{2\pi} \log |\cos \phi| d\phi = -\log 2,$$

as expected. For $d \geq 3$ we use polar coordinates to obtain with $S_+ := S \cap \{x: x_1 > 0\}$ and $d\sigma = \frac{1}{\sigma_d} dA$, with A the surface area measure on S ,

$$\begin{aligned} L(d, \mathbb{R}) &= \frac{2}{\sigma_d} \int_{S_+} \log x_1 dA(x) = \frac{2}{\sigma_d} \int_0^1 \log x_1 \cdot \frac{1}{\sqrt{1-x_1^2}} \sigma_{d-1} \left(\sqrt{1-x_1^2} \right)^{d-2} dx_1 \\ &= \frac{2\sigma_{d-1}}{\sigma_d} \int_0^1 (1-u^2)^{(d-3)/2} \log u du. \end{aligned} \quad (48)$$

Substituting (44) for $n = d - 3$ and using the values (46) in (48) yields Lemma 10. \square

Lemma 11. *With empty sums treated 0 as above, we have*

$$-L(d, \mathbb{C}) = \frac{1}{2} \sum_{k=1}^{d-1} \frac{1}{k} \quad (\forall d \in \mathbb{N}). \quad (49)$$

Proof. Observe that $z = (z_1, \dots, z_d) \in S_{\mathbb{C}^d}$ if and only if $z \sim (x_1, y_1, \dots, x_d, y_d) \in S_{\mathbb{R}^{2d}}$. Hence the surface area is σ_{2d} . Moreover, for $z \in S_{\mathbb{C}^d}$ we have $|\langle z, e \rangle| = |z_1 \cdot 1| = \sqrt{x_1^2 + y_1^2} = r_1$. Using again the directional cosine $\sqrt{1-r_1^2}$, the length of the circle $|z_1| = r_1$, and the $2d-3$ measure of the set $\{z \in S: z_1 = r_1 e^{i\theta}\}$ (which is $\sigma_{2d-2} \cdot (1-r_1^2)^{(2d-3)/2}$), we obtain

$$\begin{aligned} L(d, \mathbb{C}) &= \int_{S_{\mathbb{C}^d}} \log |\langle z, e \rangle| d\sigma(z) = \frac{1}{\sigma_{2d}} \int_0^1 \log r_1 \cdot 2\pi r_1 \cdot \sigma_{2d-2} (1-r_1^2)^{d-3/2} \frac{dr_1}{\sqrt{1-r_1^2}} \\ &= \frac{2\pi\sigma_{2d-2}}{\sigma_{2d}} \int_0^1 (1-r_1^2)^{d-2} \cdot r_1 \log r_1 dr_1 \\ &= (2d-2) \int_0^1 (1-x^2)^{d-2} x \log x dx. \end{aligned}$$

Thus substituting the value (45) for $n = d - 2$ leads to (49). \square

Remark 3. The asymptotic estimate of Theorems F and 1 give back the upper estimate (28), which is equivalent to the estimate already obtained in [22] (cf. the proof of Theorem 12 there), and [11] in the real case.

Indeed, if a_j ($j = 1, \dots, d$) are $c_d(\mathbb{K}^d)$ extremal vectors, then for any $m \in \mathbb{N}$ we have

$$\begin{aligned} \frac{1}{c_d(\mathbb{K}^d)} &= \left\| \prod_{j=1}^d \langle x, a_j \rangle \right\|_{S_{\mathbb{K}^d}} = \left(\left\| \prod_{j=1}^d \langle x, a_j \rangle \right\|_{S_{\mathbb{K}^d}}^m \right)^{1/m} \\ &\geq \left(\min \left\{ \left\| \prod_{k=1}^{md} \langle x, b_k \rangle \right\|_{S_{\mathbb{K}^d}} : b_k \in S_{\mathbb{K}^d} \ (k = 1, \dots, md) \right\} \right)^m \\ &= c_{dm}(\mathbb{K}^d)^{-1/m}. \end{aligned} \quad (50)$$

Here the right-hand side tends to $c(\mathbb{K}^d)^{-d}$ when $m \rightarrow \infty$, so by Theorems F and 1 (28) obtains.

Now let us define

$$q(d) := q(d, \mathbb{K}) := -L(d, \mathbb{K}) - \frac{1}{2} \log d \quad (d \in \mathbb{N}). \quad (51)$$

With C denoting Euler's constant, it is not difficult to see that the exact formulae of Lemmas 10 and 11 imply

Proposition 12. For the constant $q(d, \mathbb{K})$ defined in (51), we have

$$\lim_{d \rightarrow \infty} q(d, \mathbb{R}) = \frac{1}{2} \log 2 + \frac{C}{2} \quad \text{and} \quad \lim_{d \rightarrow \infty} q(d, \mathbb{C}) = \frac{C}{2}. \quad (52)$$

The real case was already given in [11] and in [14].

Remark 4. Note the somewhat surprising difference between the real and complex cases. Its meaning is the following. When estimating $c_n(\mathbb{R}^d)$ by natural complexification (as, e.g., in [22, Corollary 11]), we get $c_n(\mathbb{R}^d) \leq 2^{n/2} c_n(\mathbb{C}^d)$, what involves a factor of $2^{n/2}$. Now Proposition 12 means that this estimate is roughly best possible when n is large while d is kept fixed, or at least relatively small. In contrast, the above Conjecture is just expressing that in case $n = d$ the factor $2^{n/2}$, arising from estimating via the natural complexification, would be fully unnecessary.

In view of Theorems F and 1, $-L(d, \mathbb{K})$ has to be increasing. But we even have that also $q(d)$ is increasing.

Lemma 13. The quantity $q(d) := q(d, \mathbb{K})$, defined in (51), is strictly increasing in function of $d \in \mathbb{N}$.

Proof. Since $\log b - \log a = \int_a^b du/u > (b-a)/b$ for $0 < a < b$, we easily get this for $\mathbb{K} = \mathbb{C}$ from (49), whereas for the real case $q(d+2, \mathbb{R}) > q(d, \mathbb{R})$ follows the same way.

However, to see the full real case $q(d, \mathbb{R}) > q(d-1, \mathbb{R})$, we need a more refined estimation. To start with, use the well-known Leibniz series of $\log 2$ to get

$$\sum_{m=0}^k \frac{1}{2m+1} - \sum_{m=1}^k \frac{1}{2m} - \log 2 = \sum_{j=2k+2}^{\infty} \frac{(-1)^j}{j} \quad (53)$$

and that

$$\sum_{m=1}^k \frac{1}{2m} + \log 2 - \sum_{m=0}^{k-1} \frac{1}{2m+1} = - \sum_{j=2k+1}^{\infty} \frac{(-1)^j}{j}. \quad (54)$$

After elementary reformulations and applying geometric series expansion we get

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{n+\ell} = \sum_{\ell=0}^{\infty} \frac{1}{(n+2\ell)(n+2\ell+1)} = \sum_{\ell=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(n+2\ell+1)^{m+1}}. \quad (55)$$

Utilizing monotonicity of u^{-m} , change of order of summation in (55) leads to

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{n+\ell} &> \sum_{m=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{2} \int_{n+2\ell+1}^{n+2\ell+3} \frac{1}{u^{m+1}} du = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m(n+1)^m} \\ &= -\frac{1}{2} \log \left(1 - \frac{1}{n+1} \right), \end{aligned} \quad (56)$$

where Taylor series expansion of $\log(1-x)$ at $x = (n+1)^{-1}$ is applied in the last step. Comparing (47), (51), (53), and (54) leads to

$$q(d) - q(d-1) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{d-1+\ell} + \frac{1}{2} \log(d-1) - \frac{1}{2} \log d \quad (57)$$

both for odd $d = 2k+3$ and for even $d = 2k+2$. To conclude the proof it only remains to combine (57) and (56) (with $n = d-1$). \square

So q not only converges, but converges monotonically. Whence we also have

Corollary 14. Let C be Euler's constant and put $Q = \sqrt{2e^C} = 1.8874 \dots$. Then we have

$$c_n(\mathbb{R}^d) \leq e^{nq(d, \mathbb{R})} d^{n/2} \leq Q^n d^{n/2} \quad (n, d \in \mathbb{N}) \quad (58)$$

and

$$c_n(\mathbb{C}^d) \leq e^{nq(d, \mathbb{C})} d^{n/2} \leq (e^C d)^{n/2} \quad (n, d \in \mathbb{N}). \quad (59)$$

In particular, for the infinite-dimensional real Hilbert space $H := \ell_2(\mathbb{R})$ we have

$$n^{n/2} \leq c_n(H) = c_n(\mathbb{R}^n) \leq e^{nq(n, \mathbb{R})} n^{n/2} \leq Q^n n^{n/2} \quad (n \in \mathbb{N}). \quad (60)$$

The real case (58) and (60) were already covered by [11,14]. Observe that (60) is an independent, but slightly weaker estimate than Theorem D. However, here we did not use

the somewhat involved Theorem C of Arias-de-Reyna and subsequent complexification, as in [22]. For complex Hilbert spaces Corollary 14 does not provide any new information for the “diagonal case” $n = d$, but the general case estimate is best possible when n is large compared to d . The value of the asymptotic constant $\sqrt{e^C} = 1.334\dots$ is remarkably smaller than in the real case.

As noted above, choosing any orthonormal vector system the lower estimate $c_n(\mathbb{R}^n) \geq n^{n/2}$ is obvious. Below we utilize the same simple idea in a slightly more complex context.

Proposition 15. *For any $n, m \in \mathbb{N}$ we have*

$$c_{nm}(\mathbb{R}^{nm}) \geq n^{nm/2} c_m^n(\mathbb{R}^m). \quad (61)$$

Proof. Represent \mathbb{R}^{nm} as the orthogonal direct product of n copies H_j ($j = 1, \dots, n$) of \mathbb{R}^m , and take a_1, \dots, a_m to be a c_m -extremal unit vector set of \mathbb{R}^m . Let the vector system $b_{j,k}$ ($j = 1, \dots, n, k = 1, \dots, m$) be defined so that $b_{j,k}$ is the vector in H_j equivalent to a_k in \mathbb{R}^m .

Because of orthogonality, any vector splits as the orthogonal sum of its components in the factors H_j , that is, $x = x_1 + \dots + x_n$ with $x_j \in H_j$ ($j = 1, \dots, n$). Consequently, if $P(x) = \prod_{j=1}^n \prod_{k=1}^m \langle x, b_{j,k} \rangle$, then with the unit vectors $y_j := x_j / |x_j|$ ($j = 1, \dots, n$) we obtain

$$|P(x)| = \prod_{j=1}^n \prod_{k=1}^m (|x_j| |\langle y_j, b_{j,k} \rangle|) \leq \prod_{j=1}^n \left(|x_j|^m \frac{1}{c_m(\mathbb{R}^m)} \right) \leq n^{-mn/2} \frac{1}{c_m^n(\mathbb{R}^m)}, \quad (62)$$

since $\prod_{j=1}^n |x_j| \leq n^{-n/2}$ by the orthogonal case, and extremality of $b_{j,k}$ in H_j is assured by construction. (It is easy to see that (62) is sharp.) The assertion is immediate. \square

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