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Oscillation of Fourier transforms and Markov–Bernstein inequalities

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Abstract

Under certain conditions on an integrable function P having a real-valued Fourier transform $\hat{P}$ and such that $P(0)=0$, we obtain an estimate which describes the oscillation of $\hat{P}$ in $[-C\|P'\|_\infty/\|P\|_\infty, C\|P'\|_\infty/\|P\|_\infty]$, where C is an absolute constant, independent of P. Given $\lambda > 0$ and an integrable function ϕ with a non-negative Fourier transform, this estimate allows us to construct a finite linear combination P_λ of the translates $\phi(\cdot + k\lambda)$, $k \in \mathbf{Z}$, such that $\|P_\lambda'\|_\infty > c\|P_\lambda\|_\infty/\lambda$ with another absolute constant $c > 0$. In particular, our construction proves the sharpness of an inequality of Mhaskar for Gaussian networks.

Keywords: Oscillation of Fourier transform; Markov–Bernstein inequalities; Sums of translates; Gaussian networks

1. Introduction

The original Markov inequality states that $\|P'\|_{L^\infty(I)} \leqslant n^2\|P\|_{L^\infty(I)}$ for any algebraic polynomial P of degree n. Here, $I = [-1, 1]$. This inequality becomes an equality if P is the Chebyshev polynomial $P(x) = \cos nt$ where $x = \cos t$. The reader may find the details of this on [7, p. 40].

Upper estimates of the derivative norm by that of the function itself are usually termed Markov–Bernstein inequalities. There is an extensive literature on such inequalities, which play an important role in inverse theorems, where smoothness of a function is deduced from rates of convergence of polynomial approximations. For an excellent survey on Markov–Bernstein and related inequalities, the reader may consult the book [1] of Borwein and Erdélyi.

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By imposing additional assumptions on the zeros of the polynomials, one can obtain estimates which give lower bounds for the norm of a derivative in terms of the norm of the function. These results are usually termed inverse Markov–Bernstein inequalities or Turán type inequalities. For instance, Turán [10] proved that

$$\|P'\|_{L^\infty(I)} \geqslant \frac{\sqrt{n}}{6}\|P\|_{L^\infty(I)}$$

for any polynomial P of degree n, provided that all of its zeros lie in the interval $I = [-1, 1]$. We also refer the reader to an important paper of Eröd [5].

There has been an upsurge of interest in such estimates, with a number of recent results dealing with the topic [2,6,9,11]. For instance, in [11], Zhou showed that if $0 < r \leqslant q \leqslant \infty$ and $1 \geqslant 1/r - 1/q$, then

$$\|P'\|_{L^r(I)} \geqslant Cn^\alpha \|P\|_{L^q(I)}$$

for every polynomial P whose zeros lie in the interval I. Here, $\alpha = \frac{1}{2} - \frac{1}{2r} + \frac{1}{2q}$.

Our work is more closely related to results of Erdélyi and Nevai [4], who obtained

$$\lim_{n\to\infty} \frac{\|p_n'\|_X}{\|p_n\|_Y} = \infty$$

for sequences of polynomials p_n whose zeros satisfy certain conditions.

Markov–Bernstein inequalities have also been obtained for other classes of functions such as Gaussian networks. For instance, in [8], Mhaskar showed that for some constant c, $\|g'\|_p \leqslant cm\|g\|_p$ for any function g of the form

$$g(x) = \sum_{k=1}^{N} a_k \exp(-(x - x_k)^2), \quad x \in \mathbf{R},$$

where $|x_j - x_k| \geqslant 1/m$ for $j \neq k$, and $\log N = \mathcal{O}(m^2)$.

One of our goals in this note is to show that under certain conditions on an integrable function $P : \mathbf{R} \longrightarrow \mathbf{R}$ having a real-valued Fourier transform $\hat{P}$ with $P(0) = 0$,

$$r \geqslant C\frac{\|P'\|_\infty}{\|P\|_\infty} \implies \int_{-r}^{r} (\hat{P})_\pm \geqslant \frac{\sqrt{2\pi}}{4}\|P\|_\infty. \tag{1}$$

Here, $x_\pm = \max\{\pm x, 0\}$ and we can take $C = 8^3/\pi$. This estimate not only tells us that $\hat{P}$ will have a zero in the interval $[-r, r]$, but also provides an effective estimate on how it oscillates in the interval.

For a fixed function ϕ, let

$$E_n(\lambda) := \left\{ \sum_{k=-n}^{n} b_k \phi(x + \lambda k) : b_k \in \mathbf{R},\ k = -n, \ldots, -1, 0, 1, \ldots, n \right\}. \tag{2}$$

The estimate in (1) allows us to construct $P_\lambda \in E_n(\lambda)$ for each $\lambda > 0$ and for sufficiently large positive integers n (depending on λ) such that $\|P_\lambda'\|_\infty > c\|P_\lambda\|_\infty/\lambda$ with some absolute constant $c > 0$. In particular, our construction proves the sharpness of the above-mentioned inequality of Mhaskar [8] for Gaussian networks.

2. Notations and preliminaries

For any integrable function f on the real line, we write for its Fourier transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-i\omega x}\, dx.$$

Given a real number x, its positive and negative parts are $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$, respectively.

We will write h for the Fejér kernel, that is,

$$h(x) := \frac{1}{\sqrt{2\pi}} \left(\frac{\sin x/2}{x/2} \right)^2.$$

Its Fourier transform is given by

$$\hat{h}(\omega) = \max\{1 - |\omega|, 0\}.$$

For the rest of the paper we fix an auxiliary function H. We could use any even, 2π-periodic, and twice continuously differentiable function $H : \mathbf{R} \longrightarrow \mathbf{R}$, not identically one, such that $H(x) = 1$ if $|x| \leqslant \pi/2$. The special constants and values in the following choice are not relevant, only some order is essential. Nevertheless, for definiteness and more explicit calculation, we take

$$H(x) = \begin{cases} 1 & \text{if } |x| \leqslant \pi/2, \\ \sin^2 x & \text{if } \pi/2 < |x| \leqslant \pi. \end{cases} \tag{3}$$

Then H has the Fourier cosine series expansion

$$H(x) = \sum_{k=0}^{\infty} a_k \cos kx,$$

where a_k are the Fourier cosine coefficients of H. Although precise values are not needed here, a calculation leads to $a_0 = 3/4$, $a_1 = 4/(3\pi)$, $a_2 = -1/4$ and

$$a_k = \frac{-4 \sin k\pi/2}{\pi k(k^2 - 4)} = \begin{cases} 0 & k \text{ even} \\ \dfrac{-4}{\pi k(k^2-4)} & k \equiv 1 \bmod 4 \\ \dfrac{4}{\pi k(k^2-4)} & k \equiv 3 \bmod 4. \end{cases} \quad \text{for } k \geqslant 3, \quad k \in \mathbf{N}. \tag{4}$$

It is immediate that $|a_k| \leqslant k^{-2}$ for all $k \in \mathbf{N}$; moreover, a direct calculation yields

$$\sum_{k=1}^{\infty} |a_k| = 1 + \frac{5}{3\pi} = 1.530516... < 1.6 \quad \text{and} \quad \sum_{k=1}^{\infty} a_k^2 = \frac{9}{8}. \tag{5}$$

3. Oscillation of Fourier transforms

Lemma 1. *Let $P : \mathbf{R} \longrightarrow \mathbf{R}$ be bounded, differentiable, and integrable, and assume that $\hat{P}$ is real-valued. Suppose $P(0) = 0$, and let*

$$r > \frac{8^3 \|P'\|_\infty}{\pi \|P\|_\infty}. \tag{6}$$

Then

$$\frac{4}{\sqrt{2\pi}}\int_{-r}^{r}(\hat{P})_{\pm}\geqslant\|P\|_{\infty}. \tag{7}$$

Proof of Lemma 1. There is nothing to prove if $\|P'\|_\infty=\infty$. Hence, we assume $\|P'\|_\infty<\infty$. Fix r satisfying (6) and define

$$f(x)=P\star h_r(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}P(x-t)h_r(t)\,dt,$$

where $h_r(t)=rh(rt)$. Since $(2\pi)^{-1/2}\int_{\mathbf{R}}h_r=1$, for any real number x,

$$f(x)-P(x)=S(x)+L(x), \tag{8}$$

where

$$S(x)=\frac{1}{\sqrt{2\pi}}\int_{|t|<\delta}(P(x-t)-P(x))h_r(t)\,dt,$$

$$L(x)=\frac{1}{\sqrt{2\pi}}\int_{|t|\geqslant\delta}(P(x-t)-P(x))h_r(t)\,dt$$

and $\delta>0$ is chosen such that $8\delta q=1$ with $q=\|P'\|_\infty/\|P\|_\infty$. Combining the straightforward estimates

$$|S(x)|\leqslant\delta\|P'\|_\infty=\frac{\|P\|_\infty}{8}\quad\text{and}\quad|L(x)|\leqslant\frac{8\|P\|_\infty}{\pi r\delta}<\frac{\|P\|_\infty}{8},$$

with (8), we obtain for any real number x,

$$|f(x)-P(x)|<\|P\|_\infty/4. \tag{9}$$

Since f and $\hat{f}$ are both integrable, the inversion formula for the Fourier transform shows that

$$\sqrt{2\pi}\|f\|_\infty\leqslant\int_{\mathbf{R}}|\hat{f}|=\int_{\mathbf{R}}\left(\hat{f}+2(\hat{f})_-\right)=\sqrt{2\pi}f(0)+2\int_{\mathbf{R}}(\hat{f})_-.$$

Applying (9) with $x=0$ and noting that $P(0)=0$, we conclude that

$$\|f\|_\infty\leqslant\frac{1}{4}\|P\|_\infty+\frac{2}{\sqrt{2\pi}}\int_{\mathbf{R}}(\hat{f})_-.$$

Making use once more of (9) and the last inequality gives

$$\|P\|_\infty\leqslant\|f\|_\infty+\frac{1}{4}\|P\|_\infty\leqslant\frac{1}{2}\|P\|_\infty+\frac{2}{\sqrt{2\pi}}\int_{\mathbf{R}}(\hat{f})_-$$

and therefore

$$\|P\|_\infty\leqslant\frac{4}{\sqrt{2\pi}}\int_{\mathbf{R}}(\hat{f})_-.$$

Finally, we observe that $\hat{f}(\omega) = \hat{P}(\omega)\hat{h}(\omega/r)$, $0 \leqslant \hat{h} \leqslant 1$ and $\hat{h} = 0$ outside $[-1, 1]$. These imply that $(\hat{f})_- = 0$ outside $[-r, r]$ and $(\hat{f})_- \leqslant (\hat{P})_-$. Therefore,

$$\|P\|_\infty \leqslant \frac{4}{\sqrt{2\pi}} \int_{-r}^{r} (\hat{P})_-.$$

A similar argument leads to the same inequality for $(\hat{P})_+$. □

4. Construction of sums of translates with large oscillation

Theorem 1. *Let $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ be an even, continuous, integrable function such that $\phi(0) = 1$. In addition, suppose that its Fourier transform $\hat{\phi}$ is non-negative, integrable and analytic on* **R**. *Given $\lambda > 0$, there exist a positive integer n and $P \in E_n(\lambda)$, with $E_n(\lambda)$ defined in* (2), *such that*

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \geqslant \frac{C}{\lambda}.$$

Here, we could take $C = \pi^2/2^{10}$.

Proof. For each positive integer n and for each real number x, we define

$$P_n(x) = 2A_n\phi(x) + \sum_{k=1}^{n} a_k(\phi(x + \lambda k) + \phi(x - \lambda k)) \tag{10}$$

and also

$$P_\infty(x) := \lim_{n\to\infty} P_n(x) = 2A_\infty\phi(x) + \sum_{k=1}^{\infty} a_k(\phi(x + \lambda k) + \phi(x - \lambda k)), \tag{11}$$

where the coefficients a_k are the Fourier cosine coefficients of H in (4), and

$$A_n := A_n(\lambda) := -\sum_{k=1}^{n} a_k\phi(\lambda k), \quad A_\infty := A_\infty(\lambda) := -\sum_{k=1}^{\infty} a_k\phi(\lambda k). \tag{12}$$

We start with showing that P_∞ is not identically zero. □

Lemma 2. *Under the assumptions of Theorem* 1, *we have $\|P_\infty\|_\infty > 0$.*

Proof of Lemma 2. For each $\omega \in \mathbf{R}$ and $n \in \mathbf{N}$ we define

$$T_n(\omega) := \sum_{k=1}^{n} a_k(\cos(k\lambda\omega) - \phi(\lambda k)) \tag{13}$$

and also

$$T_\infty(\omega) := \lim_{n\to\infty} T_n(\omega) = \sum_{k=1}^{\infty} a_k(\cos(k\lambda\omega) - \phi(\lambda k)). \tag{14}$$

Thus, $\hat{P}_\infty(\omega) = \hat{\phi}(\omega)2T_\infty(\omega) = 2\hat{\phi}(\omega)(H(\lambda\omega) - F(\lambda))$, where

$$F(\lambda) := \sum_{k=0}^{\infty} a_k \phi(\lambda k) = a_0 - A_\infty(\lambda) \tag{15}$$

is a uniformly convergent sum of bounded functions of λ. By the Fourier inversion formula, $P_\infty \equiv 0$ if and only if $\hat{P}_\infty \equiv 0$. Thus, it suffices to show that for any given $\lambda > 0$, $\hat{\phi}(\omega)(H(\lambda\omega) - F(\lambda))$ does not vanish identically.

Note that for any $\lambda > 0$, $H(\lambda\omega) \neq 1$ for $\omega \in \mathcal{I} = (\frac{\pi}{2\lambda}, \frac{3\pi}{2\lambda}) + (2\pi/\lambda)\mathbf{Z}$ while $H(\lambda\omega) = 1$ for $\omega \in \mathcal{J} = [-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}] + (2\pi/\lambda)\mathbf{Z}$. Therefore, if $F(\lambda) = 1$, then $F(\lambda) \neq H(\lambda\omega)$ for $\omega \in \mathcal{I}$, while if $F(\lambda) \neq 1$, then $F(\lambda) \neq H(\lambda\omega)$ for $\omega \in \mathcal{J}$. In any case, $H(\lambda\omega) - F(\lambda) \neq 0$ for ω in a union of non-empty open intervals. If $\hat{P}_\infty \equiv 0$, then $\hat{\phi}$ would have to be zero on these intervals, which is impossible since $\hat{\phi}$ is assumed to be analytic on $\mathbf{R}$. This completes the proof of Lemma 2. □

To finish the proof of the theorem it suffices to show the next assertion.

Lemma 3. *If a positive integer n is chosen such that $20\sum_{k>n} |a_k| < \|P_\infty\|_\infty$, then*

$$\frac{\|P_n'\|_\infty}{\|P_n\|_\infty} \geqslant \frac{\pi^2}{2^{10}\lambda}.$$

Proof of Lemma 3. Recall $\hat{P}_n(\omega) = 2\hat{\phi}(\omega)T_n(\omega)$ with T_n in (13). We also define $\Delta_n(\omega) = T_n(\omega) - H(\lambda\omega) + F(\lambda)$ for $\omega \in \mathbf{R}$, with $F(\lambda)$ in (15).

Meanwhile, in view of the assumptions that $\hat{\phi} \geqslant 0$ and $\hat{\phi} \in L^1$, the inversion formula for the Fourier transform shows that $\|\phi\|_\infty = \phi(0) = 1$. With this in mind, we obtain for every positive integer n

$$\|\Delta_n\|_\infty \leqslant 2\sum_{k>n} |a_k| \quad \text{and} \quad \|P_\infty - P_n\|_\infty \leqslant 4\sum_{k>n} |a_k|. \tag{16}$$

Suppose $0 < r \leqslant \pi/(2\lambda)$. Then $H(\lambda\omega) = 1$ for $|\omega| \leqslant r$. Therefore, if $F(\lambda) \geqslant 1$, then

$$\int_{-r}^{r} (\hat{P}_n)_+ = 2\int_{-r}^{r} \hat{\phi}(1 - F(\lambda) + \Delta_n)_+ \leqslant 4\sqrt{2\pi}\sum_{k>n} |a_k|.$$

Here, we have again made use of the conditions $\hat{\phi} \geqslant 0$ and $\phi(0) = 1$. Similarly, if $F(\lambda) < 1$, we also obtain

$$\int_{-r}^{r} (\hat{P}_n)_- \leqslant 4\sqrt{2\pi}\sum_{k>n} |a_k|.$$

Thus, we have shown that if $0 < r \leqslant \pi/(2\lambda)$, then for each positive integer n,

$$\min\left(\int_{-r}^{r} (\hat{P}_n)_-, \int_{-r}^{r} (\hat{P}_n)_+\right) \leqslant 4\sqrt{2\pi}\sum_{k>n} |a_k|. \tag{17}$$

On the other hand, Lemma 1 together with the second inequality in (16) asserts that if $r > (8^3/\pi)\|P_n'\|_\infty/\|P_n\|_\infty$, then

$$\frac{4}{\sqrt{2\pi}} \min\left(\int_{-r}^{r} (\hat{P}_n)_-, \int_{-r}^{r} (\hat{P}_n)_+\right) \geqslant \|P_n\|_\infty \geqslant \|P_\infty\|_\infty - 4\sum_{k>n} |a_k|. \tag{18}$$

Combining (17) and (18) we conclude that if $(8^3/\pi)\|P_n'\|_\infty/\|P_n\|_\infty < \pi/(2\lambda)$, then $\|P_\infty\|_\infty - 4\sum_{k>n} |a_k| \leqslant 16\sum_{k>n} |a_k|$ and therefore $\|P_\infty\|_\infty \leqslant 20\sum_{k>n} |a_k|$. This proves the lemma which gives the conclusion of the theorem. □

5. Application to Gaussian networks

Our goal in this section is to prove sharpness of an inequality of Mhaskar (mentioned in the introduction of this paper) for Gaussian networks. We shall apply Theorem 1 (in particular, Lemma 1 in the proof) with $\phi(x) = \exp(-x^2)$. In this section $E_n(\lambda)$ is defined according to (2) with our above given Gaussian ϕ.

The following theorem is the main result of this section.

Theorem 2. *Let $n \in \mathbf{N}$ and $\lambda \in (0, 1)$ satisfy*

$$n > N_0 := C_0\lambda \exp\left(\frac{\pi^2}{2\lambda^2}\right) \quad \left(C_0 := \frac{1280}{3\pi}\right). \tag{19}$$

Then there exists $P \in E_n(\lambda)$, such that

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \geqslant \frac{\pi^2}{2^{10}\lambda}.$$

Remark. Note $\log N_0 = O(1/\lambda^2)$, in complete agreement with the above mentioned result of Mhaskar. Thus, the result proves sharpness of the result in [8] for an arithmetic progression of shifts $x_k := \lambda k$ with separation $1/m = \lambda$.

We retain the function H from (3) and its Fourier coefficients a_k in (4) also in this section. With these Fourier coefficients a_k and for each $\lambda > 0$ and $x \in \mathbf{R}$, $P_\infty(\lambda, x)$ will again be as in (11) with $A_\infty(\lambda)$ defined in (12). However, in contrast to the proof of Theorem 1, λ is no longer fixed.

As we are dealing with the Gaussian function $\phi(x) := \exp(-x^2)$, a number of properties are immediate.

First of all, the fact that ϕ is even and decreasing on $[0, \infty)$ implies that for each $\lambda > 0$ and for any real number x,

$$\sum_{k\in\mathbf{Z}} \phi(k\lambda - x) \leqslant 1 + \frac{1}{\lambda}\int_{\mathbf{R}} \phi = 1 + \frac{\sqrt{\pi}}{\lambda}. \tag{20}$$

Indeed, all values of $\phi(k\lambda - x)$ can be replaced by the $\int$ over the interval of length λ from $k\lambda - x$ towards 0, except perhaps the function value at the (single, if $x \neq \pm\lambda/2$) point which is closest to 0 (and thus is estimated by 1).

Also, the Fourier transform of ϕ is given by $\hat{\phi}(\omega) = (1/\sqrt{2})\exp(-\omega^2/4)$. Keeping only the term with maximal absolute value, we easily obtain

$$\sum_{l=-\infty}^{\infty} \left|\hat{\phi}\left(\frac{\omega + 2\pi l}{\lambda}\right)\right|^2 \geqslant \hat{\phi}^2\left(\frac{\pi}{\lambda}\right) \quad (\forall \omega \in \mathbf{R}). \tag{21}$$

Lemma 4. *For the function* (11) *we have*

$$|P_\infty(\lambda, x)| \leqslant \frac{24}{1 + x^2} \quad (x \in \mathbf{R}), \tag{22}$$

uniformly for all $\lambda \in (0, 1)$.

Proof of Lemma 4. Using $\phi(\lambda k) \leqslant 1$ and (5) we obtain

$$|A_\infty(\lambda)| \leqslant \sum_{k=1}^{\infty} |a_k| \leqslant 1.6.$$

As $\max_{\mathbf{R}}(1 + x^2)\phi(x) = \max_{[0,\infty)}(1 + t)e^{-t} = 1$, we get

$$|2A_\infty(\lambda)\phi(x)| \leqslant \frac{3.2}{1 + x^2}. \tag{23}$$

It follows that we indeed have

$$|P_\infty(\lambda, x)| \leqslant \frac{3.2}{1 + x^2} + \sum_{k \in \mathbf{Z}\setminus 0} |a_k|\phi(x - \lambda k), \tag{24}$$

where $a_k = a_{-k}$ if $k < 0$. As $\|\phi\|_\infty = \phi(0) = 1$, in case $|x| \leqslant 2$ this immediately leads to $|P_\infty(\lambda, x)| \leqslant 3.2/(1 + x^2) + 3.2 < 20/(1 + x^2)$, hence (22).

Because the right-hand side of (24) is even, it remains to take $x > 2$.

Now let $\mathcal{A}$ be the set of all non-zero integers k such that $|x - \lambda k| < x/2$. Observe that for $k \in \mathcal{A}$, $\lambda|k| \geqslant x/2$ and thus $|k| \geqslant x/(2\lambda)$, which gives by $|a_k| \leqslant 1/k^2$, also $|a_k| \leqslant 4\lambda^2/x^2 \leqslant 5\lambda^2/(1 + x^2)$ for $x > 2$. Therefore, taking into account also (20) and $x > 2$, we are led to

$$\sum_{k \in \mathcal{A}} |a_k|\phi(x - \lambda k) \leqslant \frac{5\lambda^2}{1 + x^2}\left(1 + \frac{\sqrt{\pi}}{\lambda}\right) = \frac{5\lambda^2 + 5\sqrt{\pi}\lambda}{1 + x^2}. \tag{25}$$

On the other hand, in view of (5) and

$$\max_{[2,\infty)} (1 + x^2)\phi(x/2) = \max_{[4,\infty)} (1 + t)e^{-t/4} = 5/e,$$

we have

$$\sum_{k \notin \mathcal{A}} |a_k|\phi(x - \lambda k) \leqslant \phi\left(\frac{x}{2}\right) 2\sum_{k=1}^{\infty} |a_k| \leqslant \frac{10}{e(1 + x^2)} \sum_{k=1}^{\infty} |a_k| < \frac{6}{1 + x^2}. \tag{26}$$

Recalling $0 < \lambda < 1$ a combination of (24), (25) and (26) gives the result of the lemma. □

We shall also make use of an explicit lower bound for the L^2-norm of $P_\infty(\lambda, \cdot)$ in terms of the l^2-norm of its coefficients. Actually, a more general phenomenon can be observed here.

Lemma 5. *Let $\lambda > 0$ be fixed and $c_k \in \mathbf{C}$ ($k \in \mathbf{Z}$) be arbitrary coefficients satisfying $\sum_{k\in\mathbf{Z}} |c_k|^2 < \infty$, i.e., $(c_k) \in \ell_2(\mathbf{Z})$. Consider the function $f(\lambda, x) := \sum_{k=-\infty}^{\infty} c_k \phi(x - \lambda k)$. We then have*

$$\| f(\lambda, \cdot)\|_2^2 \geqslant \mu(\lambda) \sum_{k=-\infty}^{\infty} |c_k|^2, \tag{27}$$

where

$$\mu(\lambda) := \frac{2\pi}{\lambda} \inf_{\omega\in\mathbf{R}} \sum_{l\in\mathbf{Z}} \left| \hat{\phi}\left(\frac{\omega + 2\pi l}{\lambda}\right)\right|^2. \tag{28}$$

Proof of Lemma 5. First of all, for a fixed $\lambda > 0$, the series defining $f := f(\lambda, \cdot)$ converges in $L^2(\mathbf{R})$. To see this, we consider its sequence $f_n(\lambda, x) = \sum_{|k|\leqslant n} c_k \phi(x - \lambda k)$ of partial sums. The Fourier transform of $f_n := f_n(\lambda, \cdot)$ is given by $\hat{f}_n(\lambda, t) = \hat{\phi}(t) \sum_{|k|\leqslant n} c_k e^{-ik\lambda t}$. Applying Plancherel's theorem to $\| f_n(\lambda, \cdot) - f_m(\lambda, \cdot)\|_2^2$ and writing the resulting integral as a sum of integrals over the intervals $[2\pi\lambda^{-1} l, 2\pi\lambda^{-1}(l + 1)],\ l \in \mathbf{Z}$, we obtain

$$\| f_n(\lambda, \cdot) - f_m(\lambda, \cdot)\|_2^2 = \frac{1}{\lambda} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} \left| \hat{\phi}\left(\frac{\omega + 2\pi l}{\lambda}\right) \sum_{m<|k|\leqslant n} c_k e^{ik\omega}\right|^2 d\omega$$

for $m < n$. The rapid decay of $\hat{\phi}$ assures the finiteness of

$$M(\lambda) := \frac{2\pi}{\lambda} \sup_{\omega\in\mathbf{R}} \sum_{l=-\infty}^{\infty} \left| \hat{\phi}\left(\frac{\omega + 2\pi l}{\lambda}\right)\right|^2$$

and therefore by Parseval's theorem,

$$\| f_n(\lambda, \cdot) - f_m(\lambda, \cdot)\|_2^2 \leqslant M(\lambda) \sum_{m<|k|\leqslant n} |c_k|^2 \longrightarrow 0$$

as $n > m \longrightarrow \infty$. This proves convergence in L^2 of the series defining $f(\lambda, \cdot)$.

A similar argument furnishes the conclusion of the lemma except that we take the infimum $\mu(\lambda)$ (as defined in (28)), instead of the supremum $M(\lambda)$ above. □

Proof of Theorem 2. First of all, we estimate $\| P_\infty(\lambda, \cdot)\|_\infty$ from below by $\| P_\infty(\lambda, \cdot)\|_2$. In view of Lemma 4 we have

$$|P_\infty(\lambda, x)| \leqslant \frac{C}{|x|} \quad \text{(with } C = 12) \tag{29}$$

for all real numbers $x \neq 0$ and for each $\lambda > 0$.

Now let the parameter σ be chosen so that

$$\sigma := \frac{C}{\| P_\infty(\lambda, \cdot)\|_\infty}.$$

Note that P_∞ does not vanish identically, hence $\sigma > 0$. We write $\|P_\infty(\lambda, \cdot)\|_2^2$ as a sum of integrals over $[-\sigma, \sigma]$ and over $\mathbf{R} \setminus [-\sigma, \sigma]$. Estimating trivially in $[-\sigma, \sigma]$ and applying (29) to the second integral yields

$$\|P_\infty(\lambda, \cdot)\|_2^2 \leqslant 2\sigma \|P_\infty(\lambda, \cdot)\|_\infty^2 + 2C^2\sigma^{-1}.$$

Thus, a short calculation with the chosen value of σ gives, for each $\lambda > 0$,

$$\|P_\infty(\lambda, \cdot)\|_2^2 \leqslant 4C\|P_\infty(\lambda, \cdot)\|_\infty. \tag{30}$$

To evaluate $\|P_\infty(\lambda, \cdot)\|_2$, we note $P_\infty(\lambda, x) = \sum_{k\in\mathbf{Z}} \alpha_k \phi(x - k\lambda)$, where $\alpha_k = a_{|k|}$ if $k \neq 0$, and $\alpha_0 = 2A_\infty(\lambda)$, with $A_\infty(\lambda)$ defined in (12). For this function we clearly have $\sum_{k\in\mathbf{Z}} |\alpha_k|^2 \geqslant 2 \sum_{k=1}^{\infty} |a_k|^2 = 9/4$ in view of (5).

Meanwhile, we consider the function $\mu(\lambda)$ defined in (28). Recalling (21) and the explicit form of $\widehat{\phi}$ provides for each $\lambda > 0$ the estimate

$$\mu(\lambda) \geqslant \frac{\pi}{\lambda} \exp\left(-\frac{\pi^2}{2\lambda^2}\right).$$

Combining this with Lemma 5 and (30) we obtain

$$\|P_\infty(\lambda, \cdot)\|_\infty \geqslant \frac{\pi}{4C\lambda} \exp\left(-\frac{\pi^2}{2\lambda^2}\right) \sum_{k=-\infty}^{\infty} |\alpha_k|^2 = \frac{9\pi}{16C\lambda} \exp\left(-\frac{\pi^2}{2\lambda^2}\right). \tag{31}$$

Now recalling $|a_k| \leqslant 1/k^2$ we obtain $\sum_{k>n} |a_k| < 1/n$ for each positive integer n. Recalling also $C = 12$, this and (31) yields that whenever (19) holds, then

$$20 \sum_{k>n} |a_k| < 20/n < 20/N_0 = \frac{3\pi}{64\lambda} \exp\left(-\frac{\pi^2}{2\lambda^2}\right) < \|P_\infty(\lambda, \cdot)\|_\infty.$$

Therefore, an application of Lemma 3 concludes the proof of Theorem 2. □

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