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Open Problem

Norm of Extension from a Circle to a Triangle

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This problem I posed first at a conference in 2005 at an approximation theory conference in Bommerholz, Germany. In 2007 it was also incorporated into the collection of open problems [1], but this is the first time it gets also printed.

In recent years we have seen a number of quite good estimates on derivatives of multivariate polynomials P under condition of controlling the maximum norm of P on say a convex, or a symmetric convex body of $\mathbb{R}^N$. For details we refer to [4] and to our survey [9] in this very volume. The problem, if the otherwise converging estimates are really sharp, seem to be the next question to answer. The following simple-looking question is related to lower estimations, that is, sharpness questions of the Bernstein problem.

Problem. *Let $\Delta \subset \mathbb{R}^2$ be any triangle, with its inscribed circle denoted by $\mathcal{C}$. Determine (at least asymptotically, when $n \to \infty$)*

$$M_n(\Delta) := \sup_{\substack{P \in \mathcal{P}_n \\ \|P|_{\mathcal{C}}\|=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} : Q|_{\mathcal{C}} = P|_{\mathcal{C}},\ Q \in \mathcal{P}_n \right\}.$$

Equivalently, determine (at least asymptotically)

$$M_n(\Delta) = \sup_{\substack{T \in \mathcal{T}_n \\ \|T\|_{\mathbb{T}}=1}} \left\{ \inf \|Q(x,y)\|_{C(\Delta)} : Q(\cos t, \sin t) = T(t) \right\}.$$

Clearly, knowing the minimax type quantity $M_n(\Delta)$, we can then determine, by suitable affine transformations, the same quantities for any pair of triangles and inscribed ellipses $\mathcal{E}$: we just have to consider the affine transformation which takes $\mathcal{E}$ to a circle.

The strongest possible hypothesis would be $M_n(\Delta) = 1 + o(1)$, when $n \to \infty$, for all triangles. However, not even the question if $M_n(\Delta) \sim M_n(\Delta_0)$ $(n \to \infty)$, if Δ_0 is say the standard triangle, seems to be simple. It may well be, in particular when these quantities do not converge to 1, that they are indeed different for different triangles. A warning sign may be the following. Naidenov found [6] – also using computer search – several counterexamples to a conjecture of mine with Sarantopoulos. The conjecture was to say that gradients of polynomials may be subject to an estimate with the so-called generalized Minkowski functional in place of $G_K(x,y)$ below. Now what happened is that the counterexamples showed varying degree of failure, with constants from something like one percent in case of Δ_0 to "rather large" (say 10–20%) deficiencies when Δ is a rather elongated, flat triangle. As may be seen from what follows, this phenomenon may suggest a problem with the above extension constants.

As already noted, my interest in the question comes from the multivariate Bernstein problem, that is, estimates from above the directional derivative of a polynomial P, say of norm 1 on a convex body on $K \subset \mathbb{R}^d$, at a point $x \in K^o$ and in a direction y. The known estimates have the form

$$|D_y P(x)| \le \deg P \sqrt{\|P\|^2_{C(K)} - P(x)^2}\, G_K(x,y),$$

where this $G_K(x,y)$ are constants only depending on the geometry, i.e. the body K and the points x, y, but independent from P. That is, the estimation separates the effects of geometry and analysis, giving the degree and the so-called "Bernstein-Szegő factor" (the square root term) as the result of the "analysis inputs", plus another factor, which is a purely geometry-related quantity.

In fact, we have basically two types of quantities for $G_K(x,y)$, one being the semiderivative $(V_K)'_y(x) := \lim_{t\to 0+} V_K(x+ity)/t$ of the Siciak-Zaharjuta extremal function, and the other the reciprocal of the (tangentially) best inscribed ellipse constant $E_K(x,y)$. For details see [9]. Now these quantities are rarely known precisely – a nice exception being when K is a simplex, see [5] – but one of the astonishing recent findings was that *they are equal* in case of any convex body K, interior point x and directional vector y [4]. This of course strengthened the expectation that these estimates then may as well be "the right ones", that is, sharp. In fact, in the form of the Siciak-Zaharjuta extremal function semiderivative this was already conjectured by Baran [2].

So these Bernstein-type estimates are conjecturally best possible, at least when the degrees are not restricted, but we consider all polynomials of all degrees. To arrive at this, one approach would be to show that the estimates in the course of proofs are sharp. So let us have a closer look at the method of the inscribed ellipses, which yields $G_K(x,y) = 1/E_K(x,y)$. Here we consider an *inscribed ellipse* $\mathcal{E} \subset K$, and estimate the derivative by considering $T := P|_{\mathcal{E}}$, which then has a derivative along the curve. This is then used to estimate $|D_y P(x)|$. For getting the best estimate, we choose the inscribed ellipse $\mathcal{E}$ (in a certain well-specified sense) maximal.

So now we are to see that once restricting to $\mathcal{E}$ or $\mathcal{C}$, we do not loose anything. In the course of proof we always estimate sharply, except when the yield of the trigonometrical Bernstein inequality, which is of the form $n\sqrt{\|T\|^2_{C(\mathcal{E})} - T^2(t_0)}$, where $T = P|_{\mathcal{E}}$ is estimated by $n\sqrt{\|P\|^2_{C(K)} - P^2(x)}$. That is, in the Bernstein-Szegő factor we substituted $\|P\|_{C(K)}$ for $\|T\|_{C(\mathcal{E})}$. Now this is put in the focus by the above extension problem, at least when K is a triangle. But, although the question seems to be rather particular, as for the choice of $K = \Delta$, note that it is already shown that sharpness of the above Bernstein type inequalities for this particular case already entail sharpness for all convex bodies of dimension 2, see the closing remark of [4]. One may then pose the analogous question to Δ being a simplex and the inscribed ellipse $\mathcal{E}$, or circle $\mathcal{C}$ being maximal in the appropriate sense.

Of course, it may well happen that for some polynomials P or T the extension increases the norm, while for others it does not. So if $M(\Delta)$ is large, it still may happen that in the case when the trigonometrical Bernstein inequality is sharp – when $T(t) = \cos(n(t - t_0))$ – then the extension has small norm. That also means that the question in its general form requires more, than is necessary for the affirmative answer in question of sharpness of the currently known Bernstein type inequalities.

Let us note one more related thing, which, however well-known to some, seems to cause surprise to others. That observation is that if we now denote by D the disk, encircled by the circle $\mathcal{C}$, then defining $M(D)$ as the corresponding extension quantity to D, we always have $M(D) = 1$. So extending a polynomial *into* $\mathcal{C}$ does not increase its norm at all. This comes from the fact that we always have some *harmonic* polynomial extensions, which then satisfy the maximum principle and thus $\max_{\mathcal{C}} |Q| = \max_D |Q|$. This fact is hard to look up in the literature, so D. Burns at al. describes an elegant proof – which they attribute to D. Khavinson – on [3, page 101].

The argument runs as follows. Fix $\mathcal{C}$ to be the unit circle together with a polynomial $P \in \mathcal{P}_n = \mathcal{P}_n(\mathbb{R}^2)$ to be extended, and consider the mapping $T : p \to \Delta(pq)$, Δ being the Laplace operator, and $q(x, y) := (1 - x^2 - y^2)$. This mapping is now clearly a linear mapping from $\mathcal{P}_m \to \mathcal{P}_m$, for any $m \in \mathbb{N}$, and it is *injective*; for if $T(p) = 0$, then pq satisfies the Laplace equation, i.e. harmonic, but as it vanishes on the boundary $\mathcal{C}$ (for there $q(x, y) \equiv 0$), by the maximum principle the harmonic function pq vanishes everywhere and is thus also $p \equiv 0$. But as $\mathcal{P}_m$ is a finite dimensional vector space, $\ker T = 0$ means that T is also *surjective*. We take now $m = n - 2$, and $R := \Delta P \in \mathcal{P}_{n-2}$. Because T is surjective, there is $r \in \mathcal{P}_{n-2}$ such that $Tr = R$, that is, $\Delta(qr) = \Delta P$. Clearly $Q := P - qr$ is then the right polynomial to pick, for $\Delta Q \equiv 0$ and $Q|_{\mathcal{C}} = P|_{\mathcal{C}}$.

For another discussion of extensions, and harmonic extensions in particular, see also [7], where the rather similar question of finding sharp norm estimates for extensions from $\mathcal{C}$ to a concentric circle $\mathcal{C}_r$ of radius r is solved. (This work also settles the above existence question of a harmonic extension, even if in a more involved way.) I would like to thank this reference to Professor V. V. Arestov.

I would say that the minimax problem of determining $M_n(\Delta)$ is certainly of some degree of difficulty and of independent interest, too.

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