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Turán type reverse Markov inequalities for compact convex sets ☆

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Abstract

For a compact convex set $K \subset \mathbb{C}$ the well-known general Markov inequality holds asserting that a polynomial p of degree n must have $\|p'\| \leqslant c(K)n^2 \|p\|$. On the other hand for polynomials in general, $\|p'\|$ can be arbitrarily small as compared to $\|p\|$.

The situation changes when we assume that the polynomials in question have all their zeroes in the convex set K. This was first investigated by Turán, who showed the lower bounds $\|p'\| \geqslant (n/2) \|p\|$ for the unit disk D and $\|p'\| \geqslant c\sqrt{n} \|p\|$ for the unit interval $I := [-1, 1]$. Although partial results provided general lower estimates of order $\sqrt{n}$, as well as certain classes of domains with lower bounds of order n, it was not clear what order of magnitude the general convex domains may admit here.

Here we show that for all bounded and convex domains K with nonempty interior and polynomials p with all their zeroes lying in K $\|p'\| \geqslant c(K)n \|p\|$ holds true, while $\|p'\| \leqslant C(K)n \|p\|$ occurs for any K. Actually, we determine $c(K)$ and $C(K)$ within a factor of absolute numerical constant.

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0. Introduction

On the complex plane polynomials of degree n admit a Markov inequality [1] $\|p'\|_K \leqslant c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \geqslant c_K n^A \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials must belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n^{(0)}(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n^{(0)}(K)} M(p) \quad \text{with} \quad M := M(p) := \frac{\|p'\|}{\|p\|}. \tag{1}$$

Theorem A (*Turán [8, p. 90]*). *If* $p \in \mathcal{P}_n^{(0)}(D)$, *where* D *is the unit disk*, *then we have*

$$\|p'\|_D \geqslant \frac{n}{2} \|p\|_D . \tag{2}$$

Theorem B (*Turán [8, p. 91]*). *If* $p \in \mathcal{P}_n^{(0)}(I)$, *where* $I := [-1, 1]$, *then we have*

$$\|p'\|_I \geqslant \frac{\sqrt{n}}{6} \|p\|_I . \tag{3}$$

Theorem A is best possible. Regarding Theorem B, Turán pointed out by example of $(1 - x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [4], and the value of the constant is computed for all fixed n precisely in [3].

The key to Theorem A was the following observation, which had already been present implicitly in [8,3] and was later formulated explicitly in [4, Proposition 2.1].

Lemma C (*Turán*). *Assume that* $z \in \partial K$ *and that there exists a disc* D_R *of radius* R *so that* $z \in \partial D_R$ *and* $K \subset D_R$. *Then for all* $p \in \mathcal{P}_n^{(0)}(K)$ *we have*

$$|p'(z)| \geqslant \frac{n}{2R} |p(z)| . \tag{4}$$

Drawing from the work of Turán, Erőd [3, p. 74] already addressed the question: "For what kind of domains does the method of Turán apply?" Clearly, by applies he meant that it provides cn order of oscillation for the derivative. In particular, he showed

Theorem D (*Erőd [3, p. 73]*). *Let* $0 < b < 1$ *and let* E_b *denote the ellipse domain with major axes* $[-1, 1]$ *and minor axes* $[-ib, ib]$. *Then for all* $p \in \mathcal{P}_n^{(0)}(E_b)$ *we have*

$$\|p'\| \geqslant \frac{b}{2} n \|p\| . \tag{5}$$

[1] Namely, to each point z of K there exists another $w \in K$ with $|w - z| \geqslant \operatorname{diam}(K)/2$, and thus application of Markov's inequality on the segment $[z, w] \subset K$ yields $|p'(z)| \leqslant (4/\operatorname{diam}(K))n^2 \|p\|_K$.

Moreover, he elaborated on the inverse Markov factors belonging to domains with some favorable geometric properties, such as having positive curvature exceeding a given fixed positive bound at all boundary points, or at all boundary points with the exception of a given (finite) set of vertices, etc.

A lower estimate of the inverse Markov factor for any convex set and of at least the same order as for the interval was obtained in full generality only in 2002, see [4, Theorem 3.2].

Theorem E (*Levenberg–Poletsky*). *If* $K \subset \mathbb{C}$ *is a compact, convex set,* $d := \operatorname{diam} K$ *and* $p \in \mathcal{P}_n^{(0)}(K)$, *then we have*

$$\|p'\| \geqslant \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\| \ . \tag{6}$$

Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order. Namely, for bounded convex domains K and for all $p \in \mathcal{P}_n^{(0)}(K)$ first we found that at least $M(p) \geqslant C_1(K)n^{2/3}$, see [7]. Recall that here the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ *having nonempty interior*. Clearly, assuming boundedness is natural, since all polynomials of positive degree have $\|p\|_K = \infty$ when the set K is unbounded. Also, all convex sets with nonempty interior are *fat*, meaning that the closure of K equals the closure of the interior of it. Hence taking the closure does not change the sup norm of polynomials under study: also, $\mathcal{P}_n^{(0)}(K)$ is only closed when K is closed. Observe that the only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his lower estimate is of the right order.

The case of the unit disk and the example of $p(z) = 1 + z^n$ shows that in general the order of the inverse Markov factor cannot be higher than n. On the other hand, some general classes of domains were found to have order n inverse Markov factors. However, for convex domains in general only the order $\sqrt{n}$ estimate of Levenberg and Poletsky were known, and there is a big gap between. Here we will fill in the gap, but for historical completeness let us list the domains which were already known to have order n inverse Markov factors.

(1) All convex domains with C^2-smooth boundary and curvature above a given fixed parameter $\kappa > 0$ (Erőd [3, p. 75]).
(2) Convex domains bounded by finitely many C^2-smooth Jordan arcs and a finite number of vertices, with the curvature of any relative interior points of the arcs bounded away from 0 (Erőd [3, p. 75]).
(3) Convex domains of smooth boundary and curvature bounded away from 0, with the exception of straight line segments on the boundary having length $< \operatorname{diam}(K)/4$, and even if $< \Delta(K)/4$, where $\Delta(K)$ denotes the transfinite diameter (i.e., capacity) of K (Erőd [3, p. 77]).
(4) A square [2] (Erdélyi, [2]).
(5) Convex domains of *fixed positive depth*: in particular, smooth convex domains and also convex domains with boundary consisting of finitely many vertices connected by finitely many smooth Jordan arcs, with all the vertices having only acute supplementary angles [7].

On the other hand, it was not known whether the inverse Markov factor can be $o(n)$ or not.

[2] Erdélyi also proves similar results on rhombuses, under the further condition of some symmetry of the polynomials in consideration — e.g. if the polynomials are real, or odd. Note also that his work on the topic preceded ours and apparently was accomplished without being aware of details of [3].

To study (1) some geometric parameters of the convex domain K are involved naturally. We write $d := d(K) := \operatorname{diam}(K)$ for the *diameter* of K, and $w := w(K) := \operatorname{width}(K)$ for the *minimal width* of K. That is,

$$w(K) := \min_{\gamma\in[-\pi,\pi]} \left(\max_{z\in K} \Re(ze^{-i\gamma}) - \min_{z\in K} \Re(ze^{-i\gamma}) \right). \tag{7}$$

Note that a (closed) convex domain is a (closed), bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leqslant d(K) < \infty$. Our main result is the following.

Theorem 1. *Let $K \subset \mathbb{C}$ be any bounded convex domain. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$\frac{\|p'\|}{\|p\|} \geqslant C(K)n \quad \text{with} \quad C(K) = 0.0003\,\frac{w(K)}{d^2(K)}\,. \tag{8}$$

Clearly this result contains all the above results apart from the precise value of the absolute constant factor. Moreover, the result is essentially sharp for all convex domains K: see §2 below.

1. Proof of theorem 1

Idea of proof. Throughout we will assume, as we may, that K is also closed, hence a compact convex set with nonempty interior. Our proof will follow the argument of [7], with one key alteration, suggested to us by Gábor Halász. Let us first describe the original idea and then the additional suggestion of Halász, even if the reader may understand the proof below without these notes as well.

We start with picking up a boundary point $\zeta \in \partial K$ of maximality of $|p|$, and consider a supporting line at ζ to K. Our original argument of [7] then used a normal direction and compared values of p at ζ and on the intersection of K and this normal line. Essential use were made of the fact that in case the length h of this intersection is small (relative to w), then, due to convexity, the normal line cuts K into half unevenly: one part has to be small (of the order of h). That was explicitly formulated in [7], and is used implicitly even here through various calculations with the angles.

However, here we compare the values of p at ζ and *on a line slightly slanted off from the normal*. Comparing the calculations here and in [7] one can observe how this change led to a further, essential improvement of the result through improving the contribution of the factors belonging to zeroes close to the supporting line. In [7] we could get a square term (in h there) only, due to orthogonality and the consequent use of the Pythagorean Theorem in calculating the distances. However, here we obtain *linear dependence* in δ via the general cosine theorem for the slanted segment J. (That insightful observation was provided by G. Halász.)

One of the major geometric features still at our help is the fact, that when h is small, then one portion of K, cut into half by our slightly tilted line, is also small. This is the key feature which allows us to bend the direction of the normal a bit *towards the smaller portion* of K[3].

As a result of the improved estimates squeezed out this way, we do not need to employ the second usual technique, also going back to Turán, i.e. integration of $(p'/p)'$ over a suitably chosen

[3] If we try tilting the other way we would fail badly, even if the reader may find it difficult to distill from the proof where, and how. But if there were zeroes close to (or on) the supporting line and far from ζ *in the direction of the tilting*, then these zeroes were farther off from ζ, than from the other end of the intersecting segment. That would spoil the whole argument. However, since K is small in one direction of the supporting line, tilting towards this smaller portion does work.

interval. As pointed out already in [7], this part of the proof yields weaker estimates than cn, so avoiding it is not only a matter of convenience, but is an essential necessity.

Proof. We list the zeroes of a polynomial $p \in \mathcal{P}_n^{(0)}(K)$ according to multiplicities as $z_1, \dots, z_n$, and the set of these zero points is denoted as $\mathcal{Z} := \mathcal{Z}(p) := \{z_j \ : \ j = 1, \dots, n\} \subset K$. (It suffices to assume that all z_j are distinct, so we do not bother with repeatedly explaining multiplicities, etc.) Assume, as we may, $p(z) = \prod_{j=1}^n (z - z_j)$.

We start with picking up a point ζ of K, where p attains its norm. By the maximum principle, $\zeta \in \partial K$, and by convexity there exists a supporting line to K at ζ with inward normal vector $\boldsymbol{v}$, say. Without loss of generality we can take $\zeta = 0$ and $\boldsymbol{v} = i$. Now by definition of the minimal width $w = w(K)$, there exists a point $A \in K$ with $\Im A \geqslant w$; by symmetry, we may assume $\Re A \leqslant 0$, say.

Sometimes we write the zeroes in their polar form

$$z_j = r_j e^{i\varphi_j} \quad \left(r_j := |z_j|, \quad \varphi_j := \arg z_j \quad (j = 1, \dots, n)\right) . \tag{9}$$

Throughout the proofs with $[(\varphi, \psi)]$ being any open, closed, halfopen-halfclosed or halfclosed-halfopen interval we use the notations

$$S[(\varphi, \psi)] := \{z \in \mathbb{C} \ : \ \arg(z) \in [(\varphi, \psi)]\} \tag{10}$$

and

$$\mathcal{Z}[(\varphi, \psi)] := \mathcal{Z} \cap S[(\varphi, \psi)], \quad n[(\varphi, \psi)] := \#\mathcal{Z}[(\varphi, \psi)], \tag{11}$$

for the sectors, the zeroes in the sectors, and the number of zeroes in the sectors determined by the angles φ and ψ.

Let us formulate a well-known but useful fact in advance. □

Lemma 1 (*Chebyshev*). *Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$ and let $J \subset R \subset \mathbb{C}$ be any set containing J. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in R} \ \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geqslant 2 \left(\frac{|J|}{4} \right)^k . \tag{12}$$

Proof. This is essentially the classical result of Chebyshev for a real interval, cf. [1,5], and it holds for much more general situations (perhaps with the loss of the factor 2) from the notion of Chebyshev constants and capacity, cf. Theorem 5.5.4. (a) in [6]. Note that already Erőd brought into the subject the use of this lemma, cf. [3, p. 76]. □

In all our proof we fix the angles

$$\psi := \arctan\left(\frac{w}{d}\right) \in (0, \pi/4] \quad \text{and} \quad \theta := \psi/20 \in (0, \pi/80]. \tag{13}$$

Since $|p(0)| = \|p\|$, $M \geqslant |p'(0)/p(0)|$. Observe that for any subset $\mathcal{W} \subset \mathcal{Z}$ we then have

$$M \geqslant \left| \frac{p'}{p}(0) \right| \geqslant \Im \frac{p'}{p}(0) = \sum_{j=1}^{n} \Im \frac{-1}{z_j} \geqslant \sum_{z_j \in \mathcal{W}} \Im \frac{-1}{z_j} = \sum_{z_j \in \mathcal{W}} \frac{\sin \varphi_j}{r_j}, \tag{14}$$

since all terms in the full sum are nonnegative.

Let us consider now the ray (straight half-line) emanating from $\zeta = 0$ in the direction of $e^{i(\pi/2-2\theta)}$. This ray intersects K in a line segment $[0, D]$, and if $D = 0$, then $K \subset S[\pi/2-2\theta, \pi]$ and a standard argument using e.g. Turán's Lemma C yields $M \geqslant n/(2d)$. Hence we may assume $D \neq 0$.

Consider now any point $B \in K$ with maximal real part, and take $B' := \Re B = \max\{\Re z : z \in K\}$. Since $D \neq 0$, $B' > 0$, and as $\Re A \leqslant 0$ and $\Re B$ is maximal, $[A, B']$ intersects $[0, D]$ in a point $D' \in [0, D]$, i.e. $[0, D'] \subset [0, D] \subset K$. Moreover, the angle at B' between the real line and AB' is $-\arg(B'-A) = -\arg(B'-D') \in [\psi, \pi/2)$. Indeed, $\Im(A-B') \geqslant w$ and $\Re(B'-A) = \Re(B-A) \leqslant d$ (resulting from $A, B \in K$) imply $-\arg(B'-A) \geqslant \arctan(w/d) = \psi$.

In the following let us write $\delta := |D'| > 0$; it cannot vanish, as $B' \neq 0$ and the line segment $[B', A]$ intersects the real line only in B'. Consider the point $B'' \in \mathbb{R}$ with $B'' \geqslant B' > 0$ and $-\arg(B''-D') = \psi$. We can say now that K lies both in the upper half of the disk with radius d around 0 (which we denote by U), and the halfplane $\Re z \leqslant B''$ (which we denote by H); moreover, $[0, D'] \subset K \subset (U \cap H)$.

Now we put $D'' := 3D'/4$ and take

$$J := \left[D'', D'\right] \subset K, \quad \text{i.e. } J := \{\tau := te^{i(\pi/2-2\theta)}\delta \ : \ 3/4 \leqslant t \leqslant 1\}. \tag{15}$$

Denoting $D_r(0) := \{z : |z| \leqslant r\}$ we split the set $\mathcal{Z}$ into the following parts:

$$\begin{aligned}
\mathcal{Z}_1 &:= \mathcal{Z}[0,\theta], \quad \mu := \#\mathcal{Z}_1 = n[0,\theta],\\
\mathcal{Z}_2 &:= \mathcal{Z}(\theta, \pi-\theta) \cap \left\{\Im(e^{i2\theta}z) < \frac{3}{8}\delta\right\}, \quad \nu := \#\mathcal{Z}_2,\\
\mathcal{Z}_3 &:= \mathcal{Z}(\theta, \pi-\theta) \cap \left\{\Im(e^{i2\theta}z) \geqslant \frac{3}{8}\delta\right\} \cap D_{2\delta}(0), \quad \kappa := \#\mathcal{Z}_3,\\
\mathcal{Z}_4 &:= \mathcal{Z}(\theta, \pi-\theta) \cap \left\{\Im(e^{i2\theta}z) \geqslant \frac{3}{8}\delta\right\} \Big\backslash D_{2\delta}(0)\\
&= \mathcal{Z}(\theta, \pi-\theta) \setminus (\mathcal{Z}_2 \cup \mathcal{Z}_3), \quad k := \#\mathcal{Z}_4,\\
\mathcal{Z}_5 &:= \mathcal{Z}[\pi-\theta, \pi], \quad m := \#\mathcal{Z}_5 = n[\pi-\theta, \pi].
\end{aligned} \tag{16}$$

In the following we establish an inequality from condition of maximality of $|p(0)|$. First we estimate the distance of any $z_j \in \mathcal{Z}_1$ from J. In fact, taking any point $z = re^{i\varphi} \in H \cap S[0,\theta]$ the sine theorem yields $r\cos\varphi = \Re z \leqslant |B''| = \delta\sin(\pi/2+2\theta-\psi)/\sin\psi = \delta\cos(\psi-2\theta)/\sin\psi < \delta\cot(18\theta)$, and so

$$r\sin\theta < \frac{\sin\theta}{\cos\varphi}\frac{\delta}{\tan(18\theta)} \leqslant \delta\frac{\tan\theta}{\tan(18\theta)} < \frac{\delta}{18}. \tag{17}$$

Now $\operatorname{dist}(z, J) = \min_{3/4\leqslant t\leqslant 1} |z-\tau|$, (where $\tau := te^{i(\pi/2-2\theta)}\delta$) and by the cosine theorem $|z-\tau|^2 = t^2\delta^2 + r^2 - 2\cos(\pi/2-\varphi-2\theta)\, rt\delta$. Because of $\cos(\pi/2-\varphi-2\theta) = \sin(\varphi+2\theta) \leqslant \sin(3\theta) \leqslant 3\sin\theta$, (17) implies $|z-\tau|^2 \geqslant t^2\delta^2 + r^2 - 6t\delta\sin\theta\, r \geqslant t^2\delta^2 + r^2 - \left(\frac{1}{3}\right)t\delta^2$, and thus $\min_{3/4\leqslant t\leqslant 1} |z-\tau|^2 \geqslant \min_{3/4\leqslant t\leqslant 1} t^2\delta^2 + r^2 - \left(\frac{1}{3}\right)t\delta^2 = r^2 + \left(\frac{5}{16}\right)\delta^2$. It follows that we have

$$\frac{|z-\tau|^2}{|z|^2} \geqslant \frac{r^2 + (5/16)\delta^2}{r^2} > 1 + \frac{(90/16)\sin\theta\ \delta}{r} > 1 + \frac{5\sin\theta\ \delta}{d} \quad (\tau \in J),$$

applying also (17) to estimate δ/r in the last but one step. Now $\delta/d \leqslant 1$ and $5\sin\theta < 0.2$, hence we can apply $\log(1+x) \geqslant x - x^2/2 \geqslant 0.9x$ for $0 < x < 0.2$ to get

$$\frac{|z-\tau|^2}{|z|^2} \geqslant \exp\left(0.9\,\frac{5\sin\theta\,\delta}{d}\right) > \exp\left(\frac{4\sin\theta\,\delta}{d}\right) \quad (\tau \in J)\,.$$

Applying this estimate for all the μ zeroes $z_j \in \mathcal{Z}_1$ we finally find

$$\prod_{z_j\in\mathcal{Z}_1}\left|\frac{z_j-\tau}{z_j}\right| \geqslant \exp\left(\frac{2\sin\theta\,\delta\mu}{d}\right) \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right). \tag{18}$$

The estimate of the contribution of zeroes from $\mathcal{Z}_5$ is somewhat easier, as now the angle between z_j and τ exceeds $\pi/2$. By the cosine theorem again, we obtain for any $z = re^{i\varphi} \in S[\pi-\theta,\pi] \cap U$ the estimate

$$\begin{aligned}|z-\tau|^2 &= r^2 + t^2\delta^2 - 2\cos(\varphi - (\pi/2-2\theta))\,rt\delta \\ &\geqslant r^2 + t^2\delta^2 + 2\sin\theta\,rt\delta > r^2\left(1 + \frac{3\sin\theta\,\delta}{2d}\right) \quad (\tau \in J)\,,\end{aligned} \tag{19}$$

as $t \geqslant \frac{3}{4}$ and $r \leqslant d$. Hence using again $\delta/d \leqslant 1$ and $1.5\sin\theta < 0.06$ we can apply $\log(1+x) \geqslant x - x^2/2 \geqslant 0.97x$ for $0 < x < 0.06$ to get

$$\frac{|z-\tau|}{|z|} \geqslant \exp\left(\frac{0.97}{2}\,\frac{3\sin\theta\,\delta}{2d}\right) \geqslant \exp\left(\frac{18\sin\theta\,\delta}{25d}\right) \quad (\tau \in J)\,,$$

whence

$$\prod_{z_j\in\mathcal{Z}_5}\left|\frac{z_j-\tau}{z_j}\right| \geqslant \exp\left(\frac{18\sin\theta\,\delta m}{25d}\right) \quad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right). \tag{20}$$

Observe that zeroes belonging to $\mathcal{Z}_2$ have the property that they fall to the opposite side of the line $\Im(e^{i2\theta}z) = 3\delta/8$ than *J*, hence they are closer to 0 than to any point of *J*. It follows that

$$\prod_{z_j\in\mathcal{Z}_2}\left|\frac{z_j-\tau}{z_j}\right| \geqslant 1 \quad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right). \tag{21}$$

Next we use Lemma 1 to estimate the contribution of zero factors belonging to $\mathcal{Z}_3$. We find

$$\max_{\tau\in J}\prod_{z_j\in\mathcal{Z}_3}\left|\frac{z_j-\tau}{z_j}\right| \geqslant 2\left(\frac{|J|}{4}\right)^{\kappa}\prod_{z_j\in\mathcal{Z}_3}\frac{1}{r_j} > \left(\frac{1}{32}\right)^{\kappa} > \exp(-3.5\kappa), \tag{22}$$

in view of $|J| = \delta/4$ and $r_j \leqslant 2\delta$.

Note that for any point $z = re^{i\varphi} \in D_{2\delta}(0) \cap \{\Im(e^{i2\theta}z) \geqslant 3\delta/8\}$ we must have

$$\frac{3\delta}{8} \leqslant \Im(e^{i2\theta}re^{i\varphi}) = r\sin(\varphi+2\theta),$$

hence by $r \leqslant 2\delta$ also

$$\sin(\varphi+2\theta) \geqslant \frac{3\delta}{8r} \geqslant \frac{3}{16}$$

and $\sin\varphi \geqslant \sin(\varphi+2\theta)-2\theta \geqslant \frac{3}{16}-\pi/40 > \frac{1}{10}$. Applying this for all the zeroes $z_j \in \mathcal{Z}_3$ we are led to

$$1 \leqslant \frac{2\delta}{r_j} \leqslant 20\delta \frac{\sin\varphi_j}{r_j} \qquad \left(z_j \in \mathcal{Z}_3\right). \tag{23}$$

On combining (22) with (23) we are led to

$$\max_{\tau\in J} \prod_{z_j\in\mathcal{Z}_3} \left|\frac{z_j-\tau}{z_j}\right| \geqslant \exp\left(-70\delta \sum_{z_j\in\mathcal{Z}_3} \frac{\sin\varphi_j}{r_j}\right). \tag{24}$$

Finally we consider the contribution of the zeroes from $\mathcal{Z}_4$, i.e. the "far" zeroes for which we have $\Im(z_j e^{2i\theta}) \geqslant 3\delta/8$, $\varphi_j \in (\theta, \pi-\theta)$ and $|r_j| \geqslant 2\delta$. Put now $Z := z_j e^{2i\theta} = u+iv = re^{i\alpha}$, and $s := |\tau| = t\delta$, say. We then have

$$\begin{aligned} \left|\frac{z_j-\tau}{z_j}\right|^2 &= \frac{|Z-t\delta i|^2}{r^2} = \frac{u^2+(v-s)^2}{r^2} = 1-\frac{2vs}{r^2}+\frac{s^2}{r^2} \\ &> 1-\frac{2vs}{r^2}+\frac{s^2}{r^2}\frac{v^2}{r^2} = \left(1-\frac{vs}{r^2}\right)^2 \geqslant \left(1-\frac{|v|\delta}{r^2}\right)^2 = \left(1-\frac{\delta|\sin\alpha|}{r}\right)^2. \end{aligned} \tag{25}$$

Recall that $\log(1-x) > -x-\frac{x^2}{2}\frac{1}{1-x} \geqslant -x(1+\frac{1}{2})$ whenever $0 \leqslant x \leqslant \frac{1}{2}$. We can apply this for $x := \delta|\sin\alpha|/r_j \leqslant \delta/r_j \leqslant \frac{1}{2}$ using $r = r_j = |z_j| \geqslant 2\delta$. As a result, (25) leads to

$$\left|\frac{z_j-\tau}{z_j}\right| \geqslant \exp\left(-\frac{3}{2}\,\delta\,\frac{|\sin(\varphi_j+2\theta)|}{r_j}\right), \tag{26}$$

and using $|\sin(\varphi_j+2\theta)| \leqslant \sin(\varphi_j)+\sin(2\theta) \leqslant 3\sin\varphi_j$ (in view of $\varphi_j \in (\theta, \pi-\theta)$), finally we get

$$\prod_{z_j\in\mathcal{Z}_4} \left|\frac{z_j-\tau}{z_j}\right| \geqslant \exp\left(-\frac{9\delta}{2} \sum_{z_j\in\mathcal{Z}_4} \frac{\sin\varphi_j}{r_j}\right) \qquad \left(\tau = t\delta e^{i(\pi/2-2\theta)} \in J\right). \tag{27}$$

If we collect the estimates (18) (20) (21) (24) and (27), we find for a certain point of maxima $\tau_0 \in J$ in (24) the inequality

$$\begin{aligned} 1 &\geqslant \frac{|p(\tau_0)|}{|p(0)|} = \prod_{z_j\in\mathcal{Z}} \left|\frac{z_j-\tau_0}{z_j}\right| \\ &> \exp\left\{\frac{18}{25}\sin\theta\,\delta\,\frac{\mu+m}{d} - 70\delta \sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4} \frac{\sin\varphi_j}{r_j}\right\}, \end{aligned} \tag{28}$$

or, after taking logarithms and cancelling by $18\delta/25$

$$\sin\theta\,\frac{\mu+m}{d} < \frac{875}{9} \sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4} \frac{\sin\varphi_j}{r_j}. \tag{29}$$

Observe that for the zeroes in $\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4$ we have $\sin\varphi_j > \sin\theta$, whence also

$$(\nu+\kappa+k)\,\frac{\sin\theta}{d} \leqslant \sum_{z_j\in\mathcal{Z}_2\cup\mathcal{Z}_3\cup\mathcal{Z}_4} \frac{\sin\varphi_j}{r_j}. \tag{30}$$

Adding (29) and (30) and taking into account $\#\mathcal{Z} = \sum_{j=1}^{5} \#Z_j$, we obtain

$$\sin\theta \frac{n}{d} = \sin\theta \frac{\mu + m + v + \kappa + k}{d} < \frac{884}{9} \sum_{z_j \in \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4} \frac{\sin\varphi_j}{r_j} . \tag{31}$$

Making use of (14) with the choice of $\mathcal{W} := \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$ we arrive at

$$\sin\theta \frac{n}{d} < \frac{884}{9} M ,$$

that is,

$$M > \frac{9 \sin\theta}{884 d} n . \tag{32}$$

It remains to recall (13) and to estimate

$$\sin\theta = \sin\left(\frac{\arctan(w/d)}{20}\right) .$$

As $\theta \in (0, \pi/80]$, $\sin\theta > \theta(1 - \theta^2/6) \geqslant \theta(1 - \pi/240) > 0.98\theta$ and as $0 < w/d \leqslant 1$, $\arctan(w/d) \geqslant (w/d)(\pi/4)$, whence

$$\sin\theta \geqslant 0.98 \frac{\arctan(w/d)}{20} \geqslant \frac{0.98\pi}{80} \frac{w}{d} .$$

If we substitute this last estimate into (32) we get

$$M > \frac{9}{884} \cdot \frac{0.98\pi}{80} \cdot \frac{w}{d^2} \cdot n > 0.0003 \frac{w}{d^2} n,$$

concluding the proof. □

2. On sharpness of the main result

Theorem 2. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$ there exists a polynomial $p \in \mathcal{P}_n^{(0)}(K)$ of degree exactly n satisfying*

$$\|p'\| \leqslant C'(K)\, n\, \|p\| \quad \text{with } C'(K) := 600 \frac{w(K)}{d^2(K)}. \tag{33}$$

Remark 1. Note that here we do not assume that K be convex, but only that it is a connected, closed (compact) subset of $\mathbb{C}$. (Clearly the condition of boundedness is not restrictive, $\|p\|$ being infinite otherwise.)

Proof. Take $a, b \in K$ with $|a - b| = d$ and $m \in \mathbb{N}$ with $m > m_0$ to be determined later. Consider the polynomials $q(z) := (z - a)(z - b)$, $p(z) = (z - a)^m (z - b)^m = q^m(z)$ and $P(z) = (z - a)^m (z - b)^{m+1} = (z - b) q^m(z)$. Clearly, $p, P \in \mathcal{P}_n^{(0)}(K)$ with $n = \deg p = 2m$ and $n = \deg P = 2m + 1$, respectively. We claim that for appropriate choice of m_0 these polynomials satisfy inequality (33) for all $n > 2m_0$.

Without loss of generality we may assume $a = -1$, $b = 1$ and thus $d = 2$, as substitution by the linear function $\Phi(z) := \frac{2}{b-a}z - \frac{a+b}{b-a}$ shows. Indeed, if we prove the assertion for $\widetilde{K} := \Phi(K)$ and for $\widetilde{p}(z) = (z+1)^m(z-1)^m$, $\widetilde{P}(z) = (z+1)^m(z-1)^{m+1}$ defined on $\widetilde{K}$, we also obtain estimates for $p = \widetilde{p} \circ \Phi$ and $P = \widetilde{P} \circ \Phi$ on K. The homothetic factor of the inverse substitution Φ^{-1} is $\Lambda := \left|\frac{b-a}{2}\right| = d(K)/2$, and width changes according to $w(\widetilde{K}) = 2w(K)/d(K)$. Note also that under the linear substitution Φ the norms are unchanged but for the derivatives $\|p'\| = \Lambda^{-1}\|\widetilde{p}'\|$ and $\|P'\| = \Lambda^{-1}\|\widetilde{P}'\|$. So now we restrict to $a = -1$, $b = 1$, $d = 2$ and $q(z) := z^2 - 1$ etc.

First we make a few general observations. One obvious fact is that the imaginary axes separates $a = -1$ and $b = 1$, and as K is connected, it also contains some point $c = it$ of K. Therefore, $\|q\| \geqslant |q(c)| = 1 + t^2 \geqslant 1$. Also, it is clear that $q'(z) = 2z = (z-1) + (z+1)$: thus, by definition of the diameter

$$\|q'\| \leqslant \| |z-1| + |z+1| \| \leqslant 4. \tag{34}$$

Let us put $w^+ := \sup_{z\in K} \Im z$ and $w^- := -\inf_{z\in K} \Im z$. We can estimate $w' := \max(w^+, w^-)$ from above by a constant times w. That is, we claim that for any point $\omega = \alpha + i\beta \in K$ we necessarily have $|\beta| \leqslant \sqrt{2}w$ and so the domain K lies in the rectangle $R := \operatorname{con}\{-1 - i\sqrt{2}w, 1 - i\sqrt{2}w, 1 + i\sqrt{2}w, -1 + i\sqrt{2}w\}$.

To see this first note that $\beta \leqslant \sqrt{3}$, since $d(K) = 2$ by assumption. Recalling (7), take $e^{i\gamma}$ be the direction of the minimal width of K: by symmetry, we may take $0 \leqslant \gamma < \pi$. Then there is a strip of width w and direction $ie^{i\gamma}$ containing K, hence also the segments $[-1, 1]$ and $[\alpha, \alpha + i\beta]$. It follows that $2|\cos\gamma| \leqslant w$ and $\beta \sin\gamma \leqslant w$. The second inequality immediately leads to $\beta \leqslant \sqrt{2}w$ if $\gamma \in [\pi/4, 3\pi/4]$. So let now $\gamma \in [0, \pi/4) \cup [3\pi/4, \pi)$, i.e. $|\cos\gamma| \geqslant 1/\sqrt{2}$. Applying also $\beta \leqslant \sqrt{3}$ now we deduce $\beta \leqslant \sqrt{3} \leqslant \sqrt{3/2}\, 2|\cos\gamma| \leqslant \sqrt{3/2}w$, whence the asserted $w^{\pm} \leqslant \sqrt{2}w$ is proved.

Consider now the norms of the derivatives. As for p, we have $p' = mq'q^{m-1}$, hence

$$\|p'\| \leqslant m\|q'\|\|q\|^{m-1} \leqslant m4\frac{\|p\|}{\|q\|} \leqslant 4m\|p\|. \tag{35}$$

Concerning P we can write using also (35) above

$$\|P'\| \leqslant \|p\| + \|p'\|\|z-1\| \leqslant \|p\| + 2\|p'\| \leqslant (8m+1)\|p\|. \tag{36}$$

Consider any point $z \in K$ where $\|q\|$, and thus also $\|p\|$ is attained. We clearly have $\|P\| \geqslant |P(z)| = |z-1|\|p\|$. But here $|z-1| \geqslant \frac{2}{5}$: for in case $|z-1| \leqslant \frac{2}{5}$ we also have $|z+1| \leqslant \frac{12}{5}$ and thus $|q(z)| \leqslant \frac{24}{25} < \|q\|$, as $\|q\| \geqslant 1$ was shown above. We conclude $\|P\| \geqslant (\frac{2}{5})\|p\|$ and (36) leads to

$$\|P'\| \leqslant \frac{5(8m+1)}{2}\|P\| < 10n\|P\| \quad (n := 2m+1 = \deg P). \tag{37}$$

Now consider first the case $w \geqslant \frac{2}{25}$. Using $(25w/2) \geqslant 1$ we obtain both for p and for P the estimate

$$M(p), M(P) \leqslant 10n \leqslant 125wn \quad (n := \deg p \text{ or} \deg P,\ \text{ respectively}). \tag{38}$$

Note that here we have these estimates for any $n \in \mathbb{N}$, without bounds on n.

Let now $w < \frac{2}{25}$. For the central part $Q := \{\alpha + i\beta \in R\ :\ |\alpha| \leqslant 10w\}$ of R we have

$$\|q'\|_{K\cap Q} = \|2z\|_{K\cap Q} \leqslant 2\sqrt{(10w)^2 + (\sqrt{2}w)^2} \leqslant 2\sqrt{102w^2} < 21w, \tag{39}$$

while for the remaining part (34) remains valid as above.

Next we estimate q in $K \setminus Q$. It is easy to see that here $\|q\|_{K\setminus Q} \leqslant \|q\|_{R\setminus Q} = \left|q\left(10w + i\sqrt{2}w\right)\right|$, hence using also $w \leqslant \frac{2}{25}$ we are led to

$$\begin{aligned}\|q\|^2_{K\setminus Q} &\leqslant \left[(1+10w)^2 + (\sqrt{2}w)^2\right]\left[(1-10w)^2 + (\sqrt{2}w)^2\right]\\ &= 1 - 196w^2 + 10404w^4 \leqslant 1 - 196w^2 + 10000\left(\frac{2}{25}\right)^2 w^2 + 404w^4\\ &= 1 - 132w^2 + 404w^4 \leqslant 1 - 128w^2 + 4096w^4 = \left[1-(8w)^2\right]^2. \end{aligned} \tag{40}$$

Now for $z \in K \cap Q$ we have in view of (39) and $\|q\|_K \geqslant 1$

$$|p'(z)| = m \cdot |q'(z)| \cdot |q^{m-1}(z)| \leqslant m21w\|q\|^m = \frac{21}{2}wn\|p\|, \tag{41}$$

and for $z \in K \setminus Q$ using $\|p\|_K = \|q\|^m_K$, (34) and (40) we get

$$|p'(z)| \leqslant m \cdot 4 \cdot \|q\|^{m-1}_{K\setminus Q} \leqslant 4m\|p\|\left[1-(8w)^2\right]^{m-1}. \tag{42}$$

In view of $w < \frac{2}{25}$, a standard calculation shows that

$$\left[1-(8w)^2\right]^{m-1} \leqslant \frac{25}{2}w \quad \text{if } m \geqslant m_0 := \left(\frac{1}{8w}\right)^2 \log\left(\frac{1}{8w}\right). \tag{43}$$

Indeed, as $\log(1-x) < -x$ for all $0 < x < 1$, using $w < \frac{2}{25}$ we find

$$(m-1)\log\left[1-(8w)^2\right] < -(m-1)\,(8w)^2 < -m\,(8w)^2 + 0.41,$$

which entails for $m \geqslant m_0$ that

$$\left[1-(8w)^2\right]^{m-1} < e^{-m_0(8w)^2+0.41} = e^{-\log\left(\frac{1}{8w}\right)+0.41} < \frac{25w}{2}.$$

It follows from (42) and (43) that

$$\|p'\|_{K\setminus Q} \leqslant 25wn\|p\|. \tag{44}$$

Collecting (41) and (44) we get also in this case of $w < \frac{2}{25}$ the estimate

$$\|p'\| \leqslant 25wn\|p\| \quad (n = 2m = \deg p,\ m \geqslant m_0). \tag{45}$$

It remains to consider the odd degree case of $n = 2m+1$, i.e. P. Now write

$$|P'(z)| \leqslant |p(z)| + |p'(z)| \cdot |z-1| \leqslant \|p\| + 2\|p'\| \leqslant (1 + 100wm)\|p\| \quad (m \geqslant m_0), \tag{46}$$

in view of (45). As shown above, we have $\|P\| \geqslant \|p\|/(\frac{2}{5})$, while $m \geqslant m_0$ entails $1 \leqslant m/m_0 < m(8w)(\frac{16}{25})(1/\log(\frac{25}{16})) < 12mw$, hence (46) yields

$$\|P'\| \leqslant 112mw\|p\| \leqslant 280mw\|P\|.$$

Since now $n = 2m+1 > 2m$, we finally find

$$\|P'\| < 140wn\|P\| \quad (n = 2m+1 = \deg P,\ m > m_0). \tag{47}$$

Collecting (38), (45) and (47), in view of $\max(125, 25, 140) < 150$ we always get

$$M(p), M(P) < 150wn \quad (n := \deg p \text{ or} \deg P, \text{ respectively}). \tag{48}$$

As remarked at the outset, for the general case the homothetic substitution Φ can be considered. That yields $< 600w/d^2$ on the right hand side of (48). □

3. Comments

In the case of the unit interval also Turán type L^p estimates were studied, see [9] and the references therein. It would be interesting to consider the analogous question for convex domains on the plane.

Because [7] will not be published in a journal, a full, self-contained proof was presented here. At the same time, this was meant to provide also a clear explanation and documentation of the origin and development of the various ideas that have led to the result.

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