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The Least Possible Value at Zero of Some Nonnegative Cosine Polynomials and Equivalent Dual Problems

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Dedicated to Professor Jean-Pièrre Kahane on the occasion of his sixtieth birthday.

ABSTRACT. *The present work is inspired by Edmund Landau's famous book, "Handbuch der Lehre von der Verteilung der Primzahlen", where he posed two extremal questions on cosine polynomials and deduced various estimates on the distribution of primes using known estimates of the extremal quantities. Although since then better theoretical results are available for the error term of the prime number formula, Landau's method is still the best in finding explicit bounds. In particular, Rosser and Schonfeld used the method in their work "Approximate formulas for some functions of prime numbers".*

In the present work we introduce general coefficient conditions, which enables us to handle the cases of fixed degree $n \in \mathbb{N}$ and that of free degree a unified way. Our goal is to describe the basic function $\alpha(a)$ that plays a crucial role in both extremal problems of Landau. We prove that $\alpha(a)$ is identical to another extremal quantity, that can be considered the dual of our basic function. While the approximation of this dual quantity was in the heart of the method of van der Waerden, our duality theorem reveals why his estimate was so sharp.

As we shall present in a forthcoming paper, the new insight into the structure of the problem also helps to improve upon the best known upper and lower estimates of French, Steckin and van der Waerden. The proof of the duality theorem uses functional analysis comparable to linear programming.

RESUMÉ. *Ce travail est inspiré par l'oeuvre célèbre d'Edmund Landau, dont le titre est "Handbuch der Lehre von der Verteilung der Primzahlen", et dans laquelle Landau a posé deux questions extrémales, concernantes des polynomes cosinus, dont il a déduit de différentes estimations pour la distribution des nombres premiers en utilisant quelques résultats sur des problèmes extrémales. On a obtenu de meilleurs résultats pour le terme d'erreur depuis-là, tout de même la méthode de Landau est la meilleur pour trouver des limites explicites. Rosser et Schonfeld ont utilisé cette méthode dans leur oeuvre "Approximate formulas for some functions of prime numbers".*

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Dans ce travail on introduit des conditions de coefficient générales, qui rendent possible la discussion uniforme des cas différants oú les degrés sont libres ou fixés $n \in \mathbb{N}$. On prend pour but de décrire la fonction $\alpha(a)$ qui joue un rôle important dans tous les deux problèmes extrémales de Landau. On démontre que $\alpha(a)$ est identique à une autre quantité extrémale, qui peut être considérée comme le duel de notre fonction de base. Comme l'essence du méthode de van der Waerden a été l'approximation de cette quantité duelle, le présent théoreme de réciprocité explique, pourquoi l'estimation de van der Waerden est si exacte.

Comme on va démontrer dans un article suivant, l'inspection de la structure du problème nous aide à améliorer les meilleures estimations connues; celles de French, Steckin et van der Waerden. La démonstration du théorème du réciprocité utilise l'analyse fonctionelle, avec un argument similaire à la programmation linéaire.

1. Introduction

1.1.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle group and $\mathbf{C}(\mathbb{T})$ and $\mathbf{BM}(\mathbb{T})$ be the sets of continuous functions and Borel measures on $\mathbb{T}$. Let $\mathcal{T}_n$ be the set of trigonometric polynomials of degree not exceeding n and $\mathcal{T}$ be the set of all trigonometric polynomials. Let us consider for any two subsets $M, L \subset \mathbb{N}_2 := \{2, 3, \ldots\} = \mathbb{Z} \cap [2, \infty)$ the set

$$\mathcal{F}(a, M, L) := \Big\{ f \in \mathbf{C}(\mathbb{T}) : f \geq 0,\ f(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx, \\ a_k \leq 0\ (k \notin M), a_k \geq 0\ (k \notin L) \Big\} \tag{1.1}$$

of nonnegative, even continuous functions and the set

$$\mathcal{T}(a, M, L) := \{ f \in \mathcal{T} : f \in \mathcal{F}(a, M, L) \} = \mathcal{T} \cap \mathcal{F}(a, M, L) \tag{1.2}$$

of nonnegative cosine polynomials.

In this paper we investigate the functions

$$\alpha(a, M, L) := \inf\{ f(0) : f \in \mathcal{F}(a, M, L) \} \tag{1.3}$$

and their twin pairs

$$\underline{\alpha}(a, M, L) := \inf\{ f(0) : f \in \mathcal{T}(a, M, L) \}. \tag{1.4}$$

For any particular $a \in \mathbb{R}$ and $M, L \subset \mathbb{N}_2$ the determination of α or $\underline{\alpha}$ constitutes an extremal problem, and this extremal problem is usually nontrivial. On the other hand, both in analytic number theory and analysis itself, there are many problems related to extremal problems of this kind, and valuable information on the values of α or $\underline{\alpha}$ have important corollaries. Since we are not going to present these applications here, let us refer to the book of Landau [3, §§65 and 79] and also to the papers of Rosser–Schonfeld [8], Ruzsa [9], and Stečkin [11].

In the present work we give an analysis of the above general definition and prove that some seemingly quite different extremal quantities of $\mathbf{BM}(\mathbb{T})$ are in fact equal to α. This dual description of α will provide essential help and new results in the analysis of these important extremal quantities and of extremal problems introduced by Landau in his Handbook [3] that we are going to publish in the subsequent paper [7].

1.2.

Throughout the paper we use the symbol of integration without specifying the domain of the integration for integrals over $\mathbb{T}$, i.e.,

$$\int = \int_{\mathbb{T}}. \tag{1.5}$$

The scalar product of $f \in \mathbf{C}(\mathbb{T})$ and $\mu \in \mathbf{BM}(\mathbb{T})$ is

$$\langle f, d\mu \rangle = \frac{1}{2\pi} \int f \, d\mu, \tag{1.6}$$

and Fourier coefficients of μ (or f) are always

$$\begin{aligned} a_0(\mu) &= \langle 1, d\mu \rangle, \\ a_k(\mu) &= 2\langle \cos kx, d\mu(x) \rangle \qquad (k = 1, 2, \ldots), \\ b_k(\mu) &= 2\langle \sin kx, d\mu(x) \rangle \qquad (k = 1, 2, \ldots) \end{aligned} \tag{1.7}$$

(or the similar integral with $f(x)\,dx$ in place of $d\mu(x)$). Since we normalize measures and functions usually by requiring $a_0(\mu) = 1$ or $a_0(f) = 1$, in our terminology the Dirac delta measure is the normalized measure with

$$\delta_z(H) = \begin{cases} 2\pi, & z \in H, \\ 0, & z \notin H, \end{cases} \qquad (H \subset \mathbb{T} \text{ measurable}). \tag{1.8}$$

We write

$$\delta = \delta_0 \tag{1.9}$$

and define the "even atomic measure at z" to be

$$\nu_z = \frac{1}{2}(\delta_z + \delta_{-z}). \tag{1.10}$$

These have the Fourier series expansions

$$d\delta_z(x) \sim 1 + 2\sum_{k=1}^{\infty}(\cos kz \cos kx + \sin kz \sin kx), \tag{1.11}$$

$$d\delta(x) \sim 1 + 2\sum_{k=1}^{\infty} \cos kx, \tag{1.12}$$

$$d\nu_z(x) \sim 1 + 2\sum_{k=1}^{\infty} \cos kz \cos kx. \tag{1.13}$$

For the convolution operator we use the same normalized integration, hence

$$(f * \mu)(x) = \frac{1}{2\pi}\int f(x-t)\, d\mu(t) = \big\langle f(x-t), d\mu(t)\big\rangle_t\,; \tag{1.14}$$

and for *even* measures

$$(f * \mu)(x) \sim 1 + \sum_{k=1}^{\infty} \frac{a_k(\mu)}{2}\big(a_k(f)\cos kx + b_k(f)\sin kx\big) \qquad \big(d\mu(x) = d\mu(-x)\big). \tag{1.15}$$

In particular, convolution by $\delta = \delta_0 = \nu_0$ is the identity operator. The Dirichlet, Fejér, and Poisson kernels are

$$D_N(x) = 1 + 2\sum_{n=1}^{N} \cos nx = \frac{\sin((N+1/2)\,x)}{\sin(x/2)}, \tag{1.16}$$

$$F_N(x) = 1 + 2\sum_{n=1}^{N}\left(1 - \frac{n}{N+1}\right)\cos nx = \frac{1}{N+1}\left(\frac{\sin((N+1)\,x/2)}{\sin\left(\frac{x}{2}\right)}\right)^2, \tag{1.17}$$

$$P_r(x) = 1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1-r^2}{1 - 2r\cos x + r^2}, \tag{1.18}$$

respectively. For any function φ from $\mathbb{R}$ to $\mathbb{R}$ the domain of definition and the range of the values are $\mathcal{D}(\varphi)$ and $\mathcal{R}(\varphi)$, respectively. We note that we define $\inf \emptyset = +\infty$; hence we attribute a meaning to (1.3) or (1.4) even if $\mathcal{F}(a, M, L) = \emptyset$ or $\mathcal{T}(a, M, L) = \emptyset$, but in these cases the values of $\alpha(a, M, L)$ (resp. $\underline{\alpha}(a, M, L)$) are not in $\mathbb{R}$ and $a \notin \mathcal{D}\big(\alpha(\ , M, L)\big)$ or $\mathcal{D}\big(\underline{\alpha}(\ , M, L)\big)$, respectively.

Note that the most important special cases are when $L = \emptyset$ and $M = \mathbb{N}_2$ or $M = [2, n]$. For easier use we write

$$\mathcal{F}(a) := \mathcal{F}(a, M, L), \qquad \alpha(a) := \alpha(a, M, L) \tag{1.19}$$

and similarly

$$\mathcal{T}(a) := T(a, M, L), \qquad \underline{\alpha}(a) = \underline{\alpha}(a, M, L) \tag{1.20}$$

whenever no confusion can arise as for the sets M and L. We use T_π for the translation operator by π as

$$\begin{aligned} (T_\pi f)(x) &= f(x+\pi) \qquad \big(f \in \mathbf{C}(\mathbb{T})\big), \\ d(T_\pi \mu)(x) &= d\mu(x+\pi) \qquad \big(\mu \in \mathbf{BM}(\mathbb{T})\big). \end{aligned} \tag{1.21}$$

Naturally we can apply T_π to subsets of $\mathbf{C}(\mathbb{T})$ or $\mathbf{BM}(\mathbb{T})$ as well. Now if a measure, a function, or a set of those satisfy some conditions of the coefficients, then the translated objects satisfy some coefficient conditions of a similar type. More precisely, if the coefficient conditions of the original object can be described by $M, L \subset \mathbb{N}_2$ as in (1.1), then the translated object satisfies coefficient conditions with a new pair of index sets $T(M, L) = (M', L')$, where

$$M' = (M \cap 2\mathbb{N}) \cup \big(L \cap (2\mathbb{N}+1)\big), \qquad L' = \big(M \cap (2\mathbb{N}+1)\big) \cup (L \cap 2\mathbb{N}). \tag{1.22}$$

In particular, one can check easily that

$$T_\pi \mathcal{F}(a, M, L) = \mathcal{F}(-a, M', L'). \tag{1.23}$$

For later reference we introduce the relative complementer set

$$\overline{K} = \mathbb{N}_2 \setminus K \qquad (K \subseteq \mathbb{N}_2). \tag{1.24}$$

2. Properties of $\alpha\,(a, M, L)$ and $\underline{\alpha}\,(a, M, L)$

2.1. Proposition

i. *We always have* $[-1, 1] \subseteq \mathcal{D}\big(\underline{\alpha}(\ , M, L)\big) \subseteq \mathcal{D}\big(\alpha(\ , M, L)\big) \subsetneq (-2, 2)$.
ii. $\mathcal{D}\big(\alpha(\ , M, L)\big)$ *is a finite interval.*
iii. $D\big(\underline{\alpha}(\ , M, L)\big)$ *is an interval with the same endpoints as* $D\big(\alpha(\ , M, L)\big)$.

Proof. (i) For $-1 \le a \le 1$ the function $1 + a\cos x$ always belongs to the set (1.2), hence $a \in \mathcal{D}(\underline{\alpha}(\ , M, L))$. The second inclusion is obvious from the definitions (1.1)–(1.4). The last assertion is also well known, but let us prove it for completeness. Let $a \in \mathbb{R}$, $f \in \mathcal{F}(a, M, L)$. Then $f \ge 0$ and $f \in \mathbf{C}(\mathbb{T})$ is not identically 0. From these properties we deduce for arbitrary $n \in \mathbb{N}$ the relation

$$|a_n(f)| < 2a_0(f) \qquad (f \in \mathbf{C}(\mathbb{T}),\ f \ge 0,\ f \not\equiv 0,\ n \in \mathbb{N}). \tag{2.1}$$

Considering the scalar product of the nonnegative functions

$$0 \le \langle 1 \pm \cos nx, f\rangle = a_0(f) \pm \frac{1}{2}a_n(f), \tag{2.2}$$

we immediately obtain (2.1) with $\le$ in place of $<$. Now let us observe that in case of equality the corresponding scalar product must vanish, hence supp f is contained in the set of zeros of $1+\cos nx$ or $1 - \cos nx$. Since $f \in \mathbf{C}(\mathbb{T})$ vanishes outside of a discrete set, f is identically zero. Hence for $f \not\equiv 0$ (2.1) holds with strict inequality. Plainly for $f \in \mathcal{F}(a, M, L)$, $a_0(f) = 1$, $a_1(f) = a$, and (i) follows.

(ii) and (iii) Let $0 < a' < a$. We prove the following two assertions.

If $a \in \mathcal{D}(\alpha(\ , M, L))$, then $a' \in \mathcal{D}(\underline{\alpha}(\ , M, L))$ ("positive part"). Similarly, if $-a \in \mathcal{D}(\alpha(\ , M, L))$, then $-a' \in \mathcal{D}(\underline{\alpha}(\ , M, L))$ ("negative part").

First let $a \in \mathcal{D}(\alpha(\ , M, L))$ and choose N to satisfy

$$N > \frac{a}{a - a'}. \tag{2.3}$$

We also put

$$r = \frac{(N+1)a'}{Na}. \tag{2.4}$$

Observe that

$$r < \left(1 + \frac{a - a'}{a}\right)\frac{a'}{a} = \left(2 - \frac{a'}{a}\right)\frac{a'}{a} < 1, \tag{2.5}$$

hence with the values (2.3), (2.4) the positive definite kernels of Fejér (1.17) and Poisson (1.18) can be used. By condition, $\mathcal{F}(a, M, L)$ is not empty, hence we can choose a function f belonging to it. Now consider the convolution

$$g(x) = (f * F_N * P_r)(x). \tag{2.6}$$

Plainly $g \in \mathcal{T}_N$ because of F_N, and $g \geq 0$ since $f \geq 0$, $F_N \geq 0$, and $P_r \geq 0$. For the Fourier series of g we have

$$g(x) = 1 + \sum_{k=1}^{N} a_k(f) \cdot \left(1 - \frac{k}{N+1}\right) r^k \cos kx, \tag{2.7}$$

hence using $a_1(f) = a$ and (2.4) we have $a_1(g) = a'$. Obviously $a_k(g)$ is of the same sign as $a_k(f)$ $(k \in \mathbb{N}_2)$; that is, we are led to $g \in \mathcal{T}(a', M, L)$ and $a' \in \mathcal{D}\big(\underline{\alpha}(\ , M, L)\big)$, as stated.

The "negative part" can be proved similarly or deduced from the "positive part" proven above. Indeed, $-a \in \mathcal{D}\big(\alpha(\ , M, L)\big)$ means $\mathcal{F}(-a, M, L) \neq \emptyset$, hence from (1.23) we conclude that $\mathcal{F}(a, M', L') \neq \emptyset$ and $a \in \mathcal{D}\big(\alpha(\ , M', L')\big)$. Applying the "positive part" for M' and L', we conclude that $\mathcal{T}(a', M', L') \neq \emptyset$. Note that similarly to (1.23) we also have

$$T_\pi \mathcal{T}(-a', M, L) = \mathcal{T}(a', M', L'), \tag{2.8}$$

hence $\mathcal{T}(-a', M, L) \neq \emptyset$ and the "negative part" follows.

Now the above two assertions really imply (ii) and (iii) since if

$$\begin{aligned} A &:= A(M, L) := \inf \mathcal{D}\big(\alpha(\ , M, L)\big), \\ B &:= B(M, L) := \sup \mathcal{D}\big(\alpha(\ , M, L)\big), \end{aligned} \tag{2.9}$$

then $-2 \leq A$ (≤ -1) and $(1 \leq)$ $B \leq 2$ according to (i) and all numbers $a' \in (A, B)$ belong to $D\big(\underline{\alpha}(\ , M, L)\big) \subseteq \mathcal{D}\big(\alpha(\ , M, L)\big)$. □

2.2.

To determine the quantities (2.9) provides a nontrivial extremal problem itself. The translation operator T_π relates these type of quantities since it follows from (1.23) that

$$A(M, L) = -B(M', L'). \tag{2.10}$$

Another useful relation is the duality-type formula

$$B(M, L) = \frac{2}{B(M^*, L^*)}, \tag{2.11}$$

where the "conjugate pair of sets to (M, L)" are defined as

$$M^* = \overline{M'}, \qquad L^* = \overline{L'} \tag{2.12}$$

with the notation of (1.22) and (1.24). The result (2.11) can be found in [4].[1]

Trivial examples of $M = L = \mathbb{N}_2$ and $M = L = \emptyset$ show that $\mathcal{D}\big(\underline{\alpha}(\ ,M,L)\big) = (-2,2)$ and $\mathcal{D}\big(\alpha(\ ,M,L)\big) = [-1,1]$ can occur. Still, the exact values of A and B are known only for some particular cases. For example, for the most important special case $M = \mathbb{N}_2$, $L = \emptyset$ we know

$$B(\mathbb{N}_2, \emptyset) = 2, \tag{2.13}$$

but we do not know the exact value of

$$A(\mathbb{N}_2, \emptyset) = -B(2\mathbb{N}, 2\mathbb{N}+1) = \frac{-2}{B(\emptyset, \mathbb{N}_2)} \tag{2.14}$$

(cf. [6]).

Finally let us see some examples. First we see that an endpoint of the domain, say $B(M,L)$, may lie out of the domain $\mathcal{D}\big(\alpha(\ ,M,L)\big)$ even for $B(M,L) < 2$. Let $M = L = 2\mathbb{N}+1$; then

$$\begin{aligned} &B(2\mathbb{N}+1, 2\mathbb{N}+1) \\ &\quad = \sup\left\{a : \exists\, f \geq 0,\ f(x) \sim 1 + a\cos x + \sum_{k=0}^{\infty} a_k \cos((2k+1)x)\right\}, \end{aligned} \tag{2.15}$$

and it is well known (cf. the end of [10]) that $B = \frac{4}{\pi}$ and the unique extremal function is

$$F(x) = 1 + \sum_{k=0}^{\infty} \frac{4(-1)^k}{\pi(2k+1)} \cos\big((2k+1)x\big) = \begin{cases} 2, & |x| < \frac{\pi}{2}, \\ 1, & |x| = \frac{\pi}{2}, \\ 0, & |x| > \frac{\pi}{2}, \end{cases} \tag{2.16}$$

which is almost in $\mathcal{F}\big(\frac{4}{\pi}, 2\mathbb{N}+1, 2\mathbb{N}+1\big)$, except that it is not continuous. Hence here $B(2\mathbb{N}+1, 2\mathbb{N}+1) \notin \mathcal{D}\big(\alpha(\ ,2\mathbb{N}+1, 2\mathbb{N}+1)\big)$.

Next we present an example with $\mathcal{D}\big(\underline{\alpha}(\ ,M,L)\big) \neq \mathcal{D}\big(\alpha(\ ,M,L)\big)$, to which the only possibility is that an endpoint of the domain, say now $A(M,L)$, belongs to $\mathcal{D}\big(\alpha(\ ,M,L)\big)$ but not to $\mathcal{D}\big(\underline{\alpha}(\ ,M,L)\big)$. Now let $M = L = 2\mathbb{N}$, and consider

$$f(x) = \frac{\pi}{2}\big(|\cos x| - \cos x\big) = 1 - \frac{\pi}{2}\cos x + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1}\cos(2kx). \tag{2.17}$$

[1] Let us take the opportunity to correct a mistake in formula (6) of [4], where in the definition of the conjugate sets the role of the even and odd integers (or, equivalently, the two sets) are changed. The correct formulation is given in (1.22) and (2.12) of the present work.

Plainly $f \in \mathcal{F}(-\frac{\pi}{2}, 2\mathbb{N}, 2\mathbb{N})$ and hence $A(2\mathbb{N}, 2\mathbb{N}) \leq -\pi/2$. If g belongs to $\mathcal{F}(a, 2\mathbb{N}, 2\mathbb{N})$ with $a \in \mathcal{D}(\alpha(\ , 2\mathbb{N}, 2\mathbb{N}))$, then scalar multiplication of g and F of (2.16) gives

$$0 \leq \langle g, F \rangle = 1 + \frac{2}{\pi} a; \tag{2.18}$$

hence $a \geq -\pi/2$ and equality can occur only if $\langle g, F \rangle = 0$ and g vanishes on $[-\pi/2, \pi/2]$. That is, $A(2\mathbb{N}, 2\mathbb{N}) = -\pi/2$, and $\mathcal{F}(-\pi/2, 2\mathbb{N}, 2\mathbb{N})$ contains no polynomials; hence $-\pi/2 \in \mathcal{D}(\alpha(\ , 2\mathbb{N}, 2\mathbb{N}))$, but not in $\mathcal{D}(\underline{\alpha}(\ , 2\mathbb{N}, 2\mathbb{N}))$.

2.3. Proposition

Both $\alpha(\ , M, L)$ and $\underline{\alpha}(\ , M, L)$ are convex functions on $D(\alpha(\ , M, L))$ and $\mathcal{D}(\underline{\alpha}(\ , M, L))$, respectively.

Proof. We prove the proposition only for $\alpha = \alpha(\ , M, L)$, the other case being similar. If $f \in \mathcal{F}(a)$ and $g \in \mathcal{F}(a')$, then for any $0 \leq \lambda \leq 1$ we have

$$h = \lambda f + (1 - \lambda) g \in \mathcal{F}(\lambda a + (1 - \lambda) a'). \tag{2.19}$$

Substituting 0 and taking the infinum over $\mathcal{F}(\lambda a + (1 - \lambda) a')$, definition (1.3) gives

$$\alpha(\lambda a + (1 - \lambda) a') \leq \lambda f(0) + (1 - \lambda) g(0); \tag{2.20}$$

and as (2.20) holds for any $f \in \mathcal{F}(a)$ and $g \in \mathcal{F}(a')$, taking the infinum over those sets we obtain the proposition. □

Note that convexity entails that these functions are continuous and have one-sided derivatives everywhere in the interior of their domain. To extend the result to the endpoints (2.9) requires some more consideration and will be analyzed only later (cf. Proposition 2.6).

2.4. Proposition

$\underline{\alpha}(a, M, L) = \alpha(a, M, L)$ for all $a \in \mathcal{D}(\underline{\alpha}(\ , M, L)))$.

Proof. Suppose $a = 0$ first. If $L = \emptyset$, then both functions obviously take the value 1; and if $L \neq \emptyset$, then for any $n \in L$, $1 - \cos nx \in \mathcal{T}(0, M, L)$ and hence both functions α and $\underline{\alpha}$ take the value zero. We can now suppose that $a > 0$, since the case $a < 0$ can be reduced to this case using the translation operator T_π similarly to the proof of Proposition 2.1. Because of $\mathcal{T}(a, M, L) \subset \mathcal{F}(a, M, L)$, we trivially have $\alpha(a) \leq \underline{\alpha}(a)$. We will prove that

$$\lim_{a' \to a-} \underline{\alpha}(a') \leq \alpha(a). \tag{2.21}$$

As $\underline{\alpha}$ is continuous, (2.21) entails $\underline{\alpha}(a) \le \alpha(a)$ and the proposition follows. Now let $f \in \mathcal{F}(a)$ be arbitrary. Since $f \in \mathbf{C}(\mathbb{T})$, we know that f is Cesaro–Fejér summable and so

$$(F_N * f)(0) \to f(0) \qquad (N \to \infty). \tag{2.22}$$

However, $F_N * f \in \mathcal{T}\left(\frac{Na}{N+1}\right)$ and

$$\underline{\alpha}\left(\frac{Na}{N+1}\right) \le (F_N * f)(0). \tag{2.23}$$

Since $\underline{\alpha}$ is continuous, (2.22) and (2.23) yield

$$\lim_{a' \to a-} \underline{\alpha}(a') = \lim_{N\to\infty} \underline{\alpha}\left(\frac{Na}{N+1}\right) \le \lim_{N\to\infty} (F_N * f)(0) = f(0). \tag{2.24}$$

Since (2.24) is valid for all $f \in \mathcal{F}(a)$, (1.3) entails (2.21), proving the proposition. □

2.5. Proposition

Let us suppose that $L \subset \mathbb{N}_2$ is a finite index set, while $M \subset \mathbb{N}_2$ is arbitrary. If $\mathcal{F}(a, M, L) \neq \emptyset$, i.e. $a \in \mathcal{D}(\alpha(\ , M, L))$, then

$$\alpha(a, M, L) = \min\{f(0) \colon f \in \mathcal{F}(a, M, L)\}. \tag{2.25}$$

Proof. We define

$$d\mu_n(x) = f_n(x)\, dx \tag{2.26}$$

where $f_n \in \mathcal{F}(a)$ satisfy $f_n(0) \to \alpha(a)$. Clearly for the total variation norm of $\mathbf{BM}(\mathbb{T})$ we have

$$\|\mu_n\|_{\mathbf{BM}(\mathbb{T})} = \frac{1}{2\pi}\int |f_n| = \frac{1}{2\pi}\int f_n = 1, \tag{2.27}$$

hence (μ_n) is a bounded sequence in $\mathbf{BM}(\mathbb{T})$. Recall that every norm-bounded subset of $\mathbf{BM}(\mathbb{T})$ is relatively compact in the weak [1], or, as other authors term it, the weak $*$ [2] topology of $\mathbf{BM}(\mathbb{T})$ defined to be the weakest topology with the property that all functionals $\langle f, \cdot\rangle \colon \mathbf{BM}(\mathbb{T}) \to \mathbb{R}$ with $f \in \mathbf{C}(\mathbb{T})$ are continuous functionals. (See, e.g., [1, 4.10.3. Theorem and Remarks]. Consequently, we can select a subsequence of (μ_n) that converges weakly $*$ to some measure $\mu \in \mathbf{BM}(\mathbb{T})$.

For convenience, let us assume that (μ_n) is already convergent. Since all μ_n are positive and even, so is μ; and in view of that $\langle 1, \mu_n\rangle = 1$ and $\langle \cos x, \mu_n\rangle = a/2$ identically, we get for the Fourier expansion of μ the form

$$d\mu(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx. \tag{2.28}$$

Moreover, since the functionals $\langle \cos kx, \cdot \rangle$ are continuous on $\mathbf{BM}(\mathbb{T})$ (in the weak $*$ topology), all a_k are the limits of $a_k(\mu_n)$ $(n \to \infty)$ and thus a_k are subject to the same coefficient conditions as all the $a_k(\mu_n)$.

Next we make use of the finiteness of L. If l is the maximal element of $L \cup \{1\}$, then for $k > l$, $k \neq 1$ and $k \notin L$; hence $a_k(\mu_n) \geq 0$ and $a_k \geq 0$ by definition of $\mathcal{F}(a, M, L)$. Now define

$$P_n(x) = a\cos x + \sum_{k=2}^{l} a_k(\mu_n)\cos kx, \qquad P(x) = a\cos x + \sum_{k=2}^{l} a_k \cos kx \tag{2.29}$$

and

$$d\nu_n(x) = d\mu_n(x) - P_n(x)\,dx, \qquad d\nu(x) = d\mu(x) - P(x)\,dx. \tag{2.30}$$

Clearly $\nu_n \in \mathbf{BM}(\mathbb{T})$ and $\nu_n \to \nu$ in the weak $*$ topology, and $P_n \to P$ is valid even in the supremum norm as l is fixed and $a_k(\mu_n) \to a_k$ $(n \to \infty)$. Similarly to (2.2) we also have

$$\left|a_k(\mu_n)\right| \leq 2, \qquad |a_k| \leq 2 \qquad (k \in \mathbb{N}), \tag{2.31}$$

and this entails that

$$\|P_n\|_\infty \leq 2l, \qquad \|P\|_\infty \leq 2l \tag{2.32}$$

while the Fourier expansions

$$d\nu_n(x) \sim 1 + \sum_{k=l+1}^{\infty} a_k(\mu_n)\cos kx, \qquad d\nu(x) \sim 1 + \sum_{k=l+1}^{\infty} a_k \cos kx \tag{2.33}$$

have only nonnegative coefficients. Now for any $N \in \mathbb{N}$

$$\begin{aligned} 1 + \sum_{k=l+1}^{N}\left(1 - \frac{k}{N+1}\right)a_k &= \langle F_N, \nu\rangle = \lim_{n\to\infty}\langle F_N, \nu_n\rangle = \lim_{n\to\infty}\langle F_N, f_n - P_n\rangle \\ &\leq \lim_{n\to\infty}\|f_n - P_n\|_\infty = \lim_{n\to\infty}\big(f_n(0) - P_n(0)\big) \\ &= \alpha(a) - P(0) \leq \alpha(a) + 2l, \end{aligned} \tag{2.34}$$

since $f_n - P_n$ is positive definite and its norm is attained at 0. From (2.34) and $a_k \geq 0$ $(k > l)$, letting $N \to \infty$ we conclude

$$1+\sum_{k=l+1}^{\infty} a_k \le \alpha(a)+2l; \tag{2.35}$$

hence the series expansion of $d\nu$ in (2.33) is absolutely convergent and $d\nu(x) = g(x)\,dx$, where $g \in \mathbf{C}(\mathbb{T})$ with the same absolutely convergent series expansion. Thus $d\mu(x) = f(x)\,dx$ with $f = g + P$. We trivially have $f \in \mathbf{C}(\mathbb{T})$ and $f \in \mathcal{F}(a, M, L)$. Finally, if $N > l$, $a_k(\mu_n) \ge 0$ for $k > N$ and all $n \in \mathbb{N}$; hence $\langle D_N, f_n\rangle \le f_n(0)$ and $n \to \infty$ yields $\langle D_N, f\rangle \le \lim_{n\to\infty} f_n(0) = \alpha(a)$. The absolute convergence of the Fourier series of f implies after $N \to \infty$ that $f(0) \le \alpha(a)$, and in view of definition (1.3) and $f \in \mathcal{F}(a, M, L)$, $f(0) = \alpha(a)$. In other words, $\alpha(a)$ is attained for f and the infinum is actually a minimum.

2.6. Proposition

Let us suppose that $L \subset \mathbb{N}_2$ is a finite index set while $M \subset \mathbb{N}_2$ is arbitrary. With the notation (2.9) *we have that*

i. *either* $\lim_{A+} \alpha(a) = +\infty$ *or* $A \in \mathcal{D}\big(\alpha(\ \,, M, L)\big)$ *and* $\lim_{A+} \alpha(a) = \alpha(A)$*;*

ii. *either* $\lim_{B-} \alpha(a) = +\infty$ *or* $B \in \mathcal{D}\big(\alpha(\ \,, M, L)\big)$ *and* $\lim_{B-} \alpha(a) = \alpha(B)$.

Proof. We prove only (i) since (ii) is quite similar. Observe that the convex function α always has a finite or infinite limit at A from the right, and suppose that $\lim_{A+} \alpha(a) = C < \infty$. We argue similarly to the proof of Proposition 2.5. Namely, we can select a sequence $d\mu_n(x) = f_n(x)\,dx$, weakly $*$ convergent in $\mathbf{BM}(\mathbb{T})$ to some measure μ, so that $f_n \in \mathcal{F}(A_n)$ with $A_n \to A$ and $f_n(0) \to C$. (According to Proposition 2.5, we even can take $f_n(0) = \alpha(A_n)$, but that is not necessary.) Then the same proof works with the only changes that here $\langle \cos x, \mu_n\rangle = A_n/2$ is not identically $A/2$, but only converges to it, and in (2.29) in the first place A_n is to be put in place of a while in the second expression A is to be put in place of a. Then the same argument gives $d\mu(x) = f(x)\,dx$ with $f \in \mathcal{F}(A)$ and $f(0) = C$. Thus we obtain that in the case of a finite limit $A \in \mathcal{D}(\alpha)$ and $\alpha(A) \le f(0) = C$, while $\lim_{A+} \alpha(a) \le \alpha(A)$ is obvious (cf. Proposition 2.4 and (2.21)).

2.7.

If $|L| = \infty$, Proposition 2.6 is no longer valid, as the case $M = L = 2\mathbb{N} + 1$, $B = 4/\pi$ and (2.16) shows. Indeed, here the limit of $\alpha(a)$ at $B-$ is 2, while $F \notin \mathbf{C}(\mathbb{T})$ and $B \notin \mathcal{D}(\alpha)$.

2.8. Lemma

Suppose that $F\colon \mathbb{T} \to \mathbb{R}$ is a measurable, nonnegative, bounded, even function having Fourier series expansion

$$F(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx \tag{2.36}$$

where $a_k \leq 0$ for $k \notin M$ and $a_k \geq 0$ for $k \notin L$. Denote

$$\overline{F}(0) = \limsup_{0} F. \tag{2.37}$$

Then we have

$$\lim_{a' \to a-} \alpha(a', M, L) \leq \overline{F}(0) \tag{2.38}$$

in case $a \geq 0$ and

$$\lim_{a' \to a+} \alpha(a', M, L) \leq \overline{F}(0) \tag{2.39}$$

in case $a \leq 0$.

Proof. Let $r < 1$ be arbitrary, and consider (cf. (1.14)–(1.18))

$$f_r := F * P_r. \tag{2.40}$$

We have $f_r \geq 0$, $f_r \in \mathbf{C}(\mathbb{T})$ and from (2.36) and (1.15)–(1.18)

$$F_r(x) \sim 1 + ar \cos x + \sum_{k=2}^{\infty} a_k r^k \cos kx; \tag{2.41}$$

hence $f_r \in \mathcal{F}(ar, M, L)$. As is well known,

$$\limsup_{r \to 1-0} f_r(0) \leq \overline{F}(0). \tag{2.42}$$

If $a = 0$, $f_r \in \mathcal{F}(0, M, L)$ and $\alpha(0) \leq f_r(0)$; hence from (2.42) we gain

$$\alpha(0) \leq \overline{F}(0) \qquad (a = 0). \tag{2.43}$$

Since $a = 0$ is in the interior of $\mathcal{D}(\alpha)$ (Proposition 2.1) and α is convex, hence continuous according to Proposition 2.3, (2.43) clearly entails both (2.38) and (2.39) for $a = 0$. If $a \neq 0$, $f_r \in \mathcal{F}(ar, M, L)$ entails $\alpha(ra) \leq f_r(0)$ and so (2.42) implies (2.38) in case $a > 0$ and (2.39) in case $a < 0$. □

2.9.

We do not know if Proposition 2.5 remains valid for $|L| = \infty$ or not.

The above analysis gave that α is convex and, thus, is continuous in the interior of the domain. For $|L| < \infty$ we essentially proved continuity at the endpoints, too in 2.6. However, the above considerations do not answer the problem in general. From convexity, or from 2.8 we have

$$\lim_{A+} \alpha \leq \alpha(A) \text{ if } A \in \mathcal{D}(\alpha) \quad \text{and} \quad \lim_{B-} \alpha \leq \alpha(B) \text{ if } B \in \mathcal{D}(\alpha),$$

but equality does not follow from these arguments.

Another natural question is if $A \notin \mathcal{D}(\alpha)$ but $\lim_{A+} \alpha = C < \infty$, then there exist functions F like (2.16) and in 2.8 or not. In particular, if they exist, do we always have $C = \inf \overline{F}(0)$ or even $C = \min \overline{F}(0)$?

Let us point out one more useful side result of the argument in 2.8. With the construction (2.40), (2.41) we have found a strictly positive function f_r, since we have $f_r(x) = \langle F(t), P_r(x-t)\rangle_t \geq \langle F(t), \frac{1-r}{1+r}\rangle = \frac{1-r}{1+r} > 0$. As a consequence, we see that for $A(M, L) < a < B(M, L)$ the set $\mathcal{F}(a, M, L)$ contains strictly positive functions, too.

2.10. Lemma

Suppose that $f \in \mathcal{F}(a, M, L)$ is strictly positive. Then either $|a| < 1$ and $L = \emptyset$ or $f(0) > \alpha(a, M, L)$ and $A(M, L) < a < B(M, L)$.

Proof. If $f(x) = 1 + a\cos x$, then positivity of f implies $|a| < 1$. Moreover, if for this particular f $\alpha = f(0) = 1 + a$, then $L = \emptyset$ follows from the minimality of α. Otherwise the Fourier expansion of f, as given in (1.1), has other nonzero terms and thus $a_k = a_k(f) \neq 0$ for some $k \in \mathbb{N}_2$. Let $m := \min f$. By assumption, $0 < m \leq 1$. Also $|a_k(f)| \leq 2$ (cf. (2.1)). Hence $g(x) := f(x) - \frac{m}{2}|a_k| \cos kx \geq f(x) - m \geq 0$, and Fourier coefficients of g are also of the type (1.1); hence $g \in \mathcal{F}(a, M, L)$. Now we get $f(0) > g(0) \geq \alpha(a, M, L)$ in view of the definitions of g and (1.3).

Finally, it is clear that $0 \leq f(x) \pm m\cos x \in \mathcal{F}(a \pm m, M, L)$; hence a cannot be an endpoint of $\mathcal{D}(\alpha(\ , M, L))$, and the last assertion of the lemma follows. □

3. The Extremal Quantity $\omega(a, M, L)$

3.1.

Let us introduce for $L, M \subset \mathbb{N}_2$ the sets

$$\mathcal{M}(0, M, L) := \Big\{ \tau \in \mathbf{BM}(\mathbb{T})\colon d\tau(x) \sim \sum_{k=1}^{\infty} t_k \cos kx,$$

$$t_1 \in \mathbb{R},\ t_k \le 0\,(k \notin M),\ t_k \ge 0\,(k \notin L) \Big\},$$

$$\mathcal{M}(a, M, L) := \Big\{ \tau \in \mathbf{BM}(\mathbb{T})\colon d\tau(x) \sim b\left(1 - \frac{2}{a}\cos x\right) + \sum_{k=2}^{\infty} t_k \cos kx, \tag{3.1}$$

$$b \in \mathbb{R},\ t_k \le 0\,(k \notin M),\ t_k \ge 0\,(k \notin L) \Big\}$$

$$= \Big\{ \tau \in \mathbf{BM}(\mathbb{T})\colon\ \tau = -\frac{a}{2} t_1(\tau_0)\lambda + \tau_0,\ \ \tau_0 \in \mathcal{M}(0, M, L) \Big\},$$

the second alternative being defined for all $a \in \mathbb{R}$ and $a \ne 0$ with using $d\lambda(x) = dx$ for the Lebesgue measure. Recalling (1.9) we define

$$\omega(a, M, L) := \sup\{t \in \mathbb{R}\colon \exists\, \tau \in \mathcal{M}(a, M, L) \text{ with } \tau + \delta \ge t \cdot \lambda\}. \tag{3.2}$$

Note that the sets (3.1) are never empty, since the zero measure $\mathbf{0}$ is an element of $\mathcal{M}(0, M, L)$ and thus (with $b = 0$) belongs to all $\mathcal{M}(a, M, L)$. As a consequence, $\omega(a, M, L) \ge 0$ $(a \in \mathbb{R})$, and we can restrict the supremum in (3.2) for measures satisfying $\tau + \delta \ge \mathbf{0}$, i.e. $\tau \ge -\delta$. Any such measure generates the set

$$\{t \ge 0\colon \delta + \tau \ge t \cdot \lambda\} \tag{3.3}$$

and plainly this set contains zero, hence it is nonvoid. Also it is bounded and closed. Thus it has a maximum

$$t(\tau) := \max\{t\ (\ge 0)\colon \delta + \tau \ge t \cdot \lambda\}, \tag{3.4}$$

and we can write in place of (3.2) the expression

$$\omega(a, M, L) := \sup\{t(\tau)\colon \tau \in \mathcal{M}(a, M, L),\ \tau \ge -\delta\}. \tag{3.5}$$

Note that (3.2) or (3.5) is defined for all a, but $\mathcal{D}\big(\omega(\ \ , M, L)\big) = \{a \in \mathbb{R}\colon\ \omega(a, M, L) < +\infty\}$.

3.2.

Now let us take a function F satisfying the conditions of Lemma 2.8. (In particular, all functions of the class $\mathcal{F}(a, M, L)$ can be taken.) Using also the notation (1.24), consider also a measure $\tau \in \mathcal{M}(a, \overline{M}, \overline{L})$ satisfying $\tau \geq -\delta$. Note that

$$\omega(a, \overline{M}, \overline{L}) = \sup\{t(\tau) \colon \tau \in \mathcal{M}(a, \overline{M}, \overline{L}), \tau \geq -\delta\} \tag{3.6}$$

according to (3.5). Observe that if $a \neq 0$ and $b \in \mathbb{R}$ is arbitrary, we have

$$\left\langle F, b\left(1 - \frac{2}{a}\cos x\right)\right\rangle = 0;$$

therefore for arbitrary $a \in \mathbb{R}$,

$$\langle F, \tau \rangle = 0 + \frac{1}{2}\sum_{k=2}^{\infty} a_k t_k \leq 0, \tag{3.7}$$

as can easily be seen from the coefficient conditions. Now using (3.7) and the nonnegativity of F we obtain

$$\langle F, \delta \rangle \geq \langle F, \delta + \tau \rangle \geq \langle F, t(\tau) \cdot \lambda \rangle = t(\tau), \tag{3.8}$$

and (3.6) immediately gives

$$F(0) = \langle F, \delta \rangle \geq \omega(a, \overline{M}, \overline{L}). \tag{3.9}$$

Since (3.9) is valid for all $F \in \mathcal{F}(a, M, L)$, (1.3) also gives

$$\alpha(a, M, L) \geq \omega(a, \overline{M}, \overline{L}). \tag{3.10}$$

Note that (3.10) immediately gives $\mathcal{D}(\omega(\ , \overline{M}, \overline{L})) \supseteq \mathcal{D}(\alpha(\ , M, L))$, and example (2.16) shows that inequality of these sets is possible.

3.3. Lemma

Suppose that there exists a nonnegative normalized measure σ in $\mathcal{M}(a, \overline{M}, \overline{L})$, that is,

$$0 \leq d\sigma(x) \sim 1 - \frac{2}{a}\cos x + \sum_{k=2}^{\infty} s_k \cos kx, \qquad s_k \leq 0 \ (k \notin \overline{M}), \quad s_k \geq 0 \ (k \notin \overline{L}). \tag{3.11}$$

i. *If* $a > 0$, *then for all* $a' > a$ *we have* $\omega(a', \overline{M}, \overline{L}) = +\infty$.

ii. *If* $a < 0$, *then for all* $a' < a$ *we have* $\omega(a', \overline{M}, \overline{L}) = +\infty$.

Proof. First let us note that $a = 0$ cannot occur since $\mathcal{M}(0, \overline{M}, \overline{L})$ contains only measures with constant term 0. Now put $r := a/a'$, and with this $r \in (0, 1)$ consider the measure $\rho = \sigma * P_r$ with the Poisson kernel (1.18). Clearly

$$0 \le d\rho(x) \sim 1 - \frac{2}{a'} \cos x + \sum_{k=2}^{\infty} r^k s_k \cos kx \in \mathcal{M}(a', \overline{M}, \overline{L}), \tag{3.12}$$

and as $P_r(x) \ge \frac{1-r}{1+r}$ we get the estimate

$$\rho = \sigma * P_r \ge \frac{1-r}{1+r}\lambda. \tag{3.13}$$

For any given $K > 0$ consider the measure

$$\tau := b \cdot \rho \quad \text{with} \quad b := \frac{1+r}{1-r}K. \tag{3.14}$$

Plainly $\tau \in \mathcal{M}(a', \overline{M}, \overline{L})$ and $\tau + \delta \ge \tau \ge K\lambda$; hence $t(\tau) \ge K$, and letting $K \to +\infty$ we obtain the lemma. □

3.4. Lemma

The following assertions are equivalent.

i. $a \notin (A, B)$ *(i.e., either* $a \le A(M, L)$ *or* $a \ge B(M, L)$*).*

ii. *There exists a nonnegative, normalized measure* (3.11) *in* $\mathcal{M}(a, \overline{M}, \overline{L})$.

Proof. (ii) implies (i) in view of (3.10) and Lemma 3.3, as for $A(M, L) < a' < B(M, L)$, $\omega(a', \overline{M}, \overline{L}) \le \alpha(a', M, L) < +\infty$. Now we prove the converse. For the proof we fix any $a \ge B(M, L)$ and construct a σ satisfying (3.11). (As the case $a \le A(M, L)$ is similar, or can be transformed to a positive case using T_π, we omit the proof for this negative case.) Note that in view of the construction in Lemma 3.3, it suffices to present our construction for $a = B$; as for $a' > a$, ρ will be a measure of the required form once σ has been constructed. However, we do not use this remark and take $a \ge B(M, L)$ arbitrary. We consider the Banach space $\mathbf{C}(\mathbb{T})$, equipped with the usual infinity norm, and take the two sets

$$\mathcal{P} := \{f \in \mathbf{C}(\mathbb{T}) : f > 0\} \tag{3.15}$$

and

$$\mathcal{G} := \left\{ g \in \mathbf{C}(\mathbb{T}) \colon g(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx, \right.$$
$$\left. a_k \le 0\,(k \notin M),\, a_k \ge 0\,(k \notin L) \right\}. \tag{3.16}$$

$\mathcal{P}$ is an open, nonempty convex cone of the Banach space $\mathbf{C}(\mathbb{T})$, while $\mathcal{G}$ is also convex and is closed in $\mathbf{C}(\mathbb{T})$. Note that $1 + a\cos x \in \mathcal{G}$, hence $\mathcal{G} \neq \emptyset$. Next we observe that $\mathcal{P}$ and $\mathcal{G}$ are disjoint sets. Indeed, $\mathcal{P} \cap \mathcal{G} = \mathcal{P} \cap \mathcal{F}(a, M, L) = \emptyset$ in view of $a \ge B(M, L)$, Proposition 2.1, and Lemma 2.10.

Now we can apply the separation theorem of convex sets (cf. e.g., [1, Corollary 2.2.2]) to obtain a nontrivial linear functional I and a constant $w \in \mathbb{R}$ so that

$$I\mathcal{G} \le w \le I\mathcal{P}. \tag{3.17}$$

Since $\mathcal{P}$ is a convex cone, so is $I\mathcal{P} \subset \mathbb{R}$ and, as $I\mathcal{P}$ is bounded from below, we have $I\mathcal{P} \subseteq [0, \infty)$. As I is nontrivial and $\mathbf{C}(\mathbb{T}) = \mathcal{P} - \mathcal{P}$, $I\mathcal{P} \neq \{0\}$.

Hence we can select some $p \in \mathcal{P}$ with $Ip > 0$. Now $0 < p \le K := \max p$, the function $q := 2K - p$ is also strictly positive, and so $Iq \ge 0$. Using linearity we get $0 < Ip \le Ip + Iq = 2KI\mathbf{1}$, hence $I\mathbf{1} \neq 0$ and we can normalize by supposing $I\mathbf{1} = 1$. Similarly, for any $f \in \mathcal{P}$, $f \ge m := \min f > 0$ entails $If \ge m > 0$; whence $I\mathcal{P} = (0, \infty)$, and w can be taken 0 in (3.17). Finally substituting I by the even functional $\frac{1}{2}\big(I(f(x)) + I(f(-x))\big)$ if necessary, we can suppose that I is even in the sense that $\big(If(x)\big) = I\big(f(-x)\big)$ for all $f \in \mathbf{C}(\mathbb{T})$.

Now we can apply the Riesz representation theorem (cf., e.g., [1, Theorem 4.10.1]) to obtain a measure $\mu \in \mathbf{BM}(\mathbb{T})$ satisfying

$$I = \langle\ , d\mu\rangle, \qquad I\mathcal{G} \le 0 < I\mathcal{P}, \qquad d\mu(x) \sim 1 + \sum_{k=1}^{\infty} b_k \cos kx. \tag{3.18}$$

Note that the Fourier series expansion of μ is a pure cosine series with constant term 1 since I is even and normalized. Also μ is even, normalized, and positive, and similarly to (2.2) or (2.31) we immediately have $|b_k| \le 2$ $(k \in \mathbb{N})$. Next we deduce more precise bounds for these coefficients using $I\mathcal{G} \le 0$.

First, take $1 + a\cos x \in \mathcal{G}$ to deduce

$$0 \ge I(1 + a\cos x) = 1 + \frac{ab_1}{2}. \tag{3.19}$$

Second, for any $m \in M$ and $K > 0$ consider $1 + a\cos x + K\cos mx \in \mathcal{G}$ and obtain

$$0 \ge I(1 + a\cos x + K\cos mx) = 1 + \frac{ab_1}{2} + \frac{K}{2}b_m \qquad (m \in M, K > 0). \tag{3.20}$$

Third, for any $l \in L$ and $K > 0$ consider $1 + a\cos x - K\cos lx \in \mathcal{G}$ to get

$$0 \geq I(1 + a\cos x - K\cos lx) = 1 + \frac{ab_1}{2} - Kb_\ell \qquad (l \in L,\ K > 0). \tag{3.21}$$

Dividing by K and letting $K \to +\infty$ yield from (3.20) and (3.21) the bounds

$$b_m \leq 0 \leq b_\ell \qquad (\forall m \in M,\ \forall l \in L), \tag{3.22}$$

while (3.19) gives

$$b_1 \leq \frac{-2}{a} (< 0). \tag{3.23}$$

In all, the measure μ is normalized and nonnegative, and from (3.18) and (3.22) it is clear that $\mu \in \mathcal{M}\left(\frac{-2}{b_1}, \overline{M}, \overline{L}\right)$. Hence if (3.23) holds with equality, that is, $a = -2/b_1$, then the measure μ itself is of the type (3.11) and we are done. Finally, if (3.23) holds with inequality, then we take $r := \frac{-2}{ab_1} \in (0, 1)$ and consider $\sigma = \mu * P_r$. Plainly σ is even, nonnegative, and normalized, and $\sigma \in \mathcal{M}(a, \overline{M}, \overline{L})$ according to (3.22), (3.23), the choice of r, (3.18), and the definitions (3.1) and (1.18). The proof is complete. □

3.5. Proposition

i. $\mathcal{D}(\omega(\ ,\overline{M}, \overline{L})) \subseteq [A(M, L), B(M, L)]$ $(= \mathrm{cl}\mathcal{D}(\alpha(\ , M, L)))$.

ii. *For* $a \notin \mathcal{D}(\omega(\ ,\overline{M}, \overline{L}))$, *in particular for* $a < A(M, L)$ *or* $a > B(M, L)$ *we have* $\omega(a, \overline{M}, \overline{L}) = +\infty$.

Proof. The proof of the proposition follows from (3.10), Lemma 3.3, and Lemma 3.4.

3.6. Theorem

For arbitrary $M, L \subset \mathbb{N}_2$ *and* $A(M, L) < a < B(M, L)$ *we have*

$$\alpha(a, M, L) = \omega(a, \overline{M}, \overline{L}). \tag{3.24}$$

Proof. We are entitled to prove the converse inequality of (3.10). Since M and L are fixed, let us omit them and use α, $\mathcal{F}$ and ω, $\mathcal{M}$ in place of $\alpha(a, M, L)$, $\mathcal{F}(a, M, L)$ and $\omega(a, \overline{M}, \overline{L})$, $\mathcal{M}(a, \overline{M}, \overline{L})$, respectively.

Let us first consider the easier case $|a| \leq 1$ and $L = \emptyset$. In this case any $f \in \mathcal{F}$ has a Fourier expansion with nonnegative coefficients and $f_0(x) = 1 + a\cos x$ has the minimal possible value at

0 among all $f \in \mathcal{F}$. (As $|a| \le 1$, we have $f_0 \ge 0$ and hence $f_0 \in \mathcal{F}$.) Thus $\alpha = 1 + a$. On the other hand, the measure

$$\tau_0(x) := (1+a)\lambda - \delta \sim a - 2\sum_{k=1}^{\infty} \cos kx \tag{3.25}$$

belongs to $\mathcal{M}(a, \emptyset, \mathbb{N}_2)$ (with $b = a$ if $a \ne 0$,) and $\emptyset \subset \overline{M}$, $\overline{L} = \overline{\emptyset} = \mathbb{N}_2$ shows $\tau_0 \in \mathcal{M}(a, \overline{M}, \overline{L})$. It is immediate that $t(\tau_0) = 1 + a = \alpha$, whence $\omega \ge \alpha$ and this case is settled.

For the remaining cases our proof will be similar to the proof of Lemma 3.4 with some additional and more precise calculation. We again consider the sets (3.15) and (3.16), but here $\mathcal{F} \ne \emptyset$ and $\mathcal{G} \cap \mathcal{P}$ can be nonvoid as well. However, for any $f \in \mathcal{G} \cap \mathcal{P}$ we have from Lemma 2.10 that $f(0) > \alpha$ save the case $|a| < 1$ and $L = \emptyset$, which has been settled above. Hence with

$$\mathcal{H} := \{g \in \mathcal{G} : g(0) \le \alpha\} \tag{3.26}$$

we have $\mathcal{P} \cap \mathcal{H} = \emptyset$ while $\mathcal{H}$ is convex and closed just like $\mathcal{G}$. It remains to check $\mathcal{H} \ne \emptyset$. Now if $\alpha \ge 1 + a$, then $h(x) = 1 + a\cos x \in \mathcal{H}$. On the other hand, if $\alpha < 1 + a$, then certainly $L \ne \emptyset$ and, with any $l \in L$, the function $h(x) = 1 + a\cos x - (1 + a - \alpha)\cos lx \in \mathcal{H}$. Whence $\mathcal{H}$ is nonvoid, and we can apply the same machinery with the application of the separation theorem of convex sets and the Riesz representation theorem similarly to the proof of Lemma 3.4, but now for $\mathcal{P}$ and $\mathcal{H}$ in place of $\mathcal{P}$ and $\mathcal{G}$. That leads to the linear functional I and the corresponding measure $\mu \in \mathbf{BM}(\mathbb{T})$ with

$$I = \langle\ , d\mu\rangle, \qquad I\mathcal{H} \le 0 < I\mathcal{P}, \qquad d\mu(x) \sim 1 + \sum_{k=1}^{\infty} b_k \cos kx. \tag{3.27}$$

First of all, for any $m \in M, l \in L$, $K > 0$, and $h \in \mathcal{H}$, the function $h(x) + K\cos mx - K\cos lx$ belongs to $\mathcal{H}$; hence

$$0 \ge I\big(h(x) + K\cos mx - K\cos lx\big) = Ih + \frac{K}{2}b_m - \frac{K}{2}b_l, \tag{3.28}$$

and after a small calculation and letting $K \to +\infty$ we obtain

$$b_m \le b_l \qquad (\forall m \in M,\ \forall l \in L). \tag{3.29}$$

Now let us define the bounds

$$S := \sup_{m\in M} b_m, \qquad T := \inf_{l\in L} b_l. \tag{3.30}$$

As in (3.21), (3.22) we can deduce

$$T \geq 0 \tag{3.31}$$

making use of the functions $1 + a\cos x - K\cos lx$ for any $l \in L$ and $K \to +\infty$, as for $K > \max\{0, 1 + a - \alpha\}$ these functions belong to $\mathcal{H}$. Summing up, (3.29)–(3.31) means

$$b_m \leq S \leq T \leq b_l \ (\forall m \in M,\ \forall l \in L), \quad \text{and} \quad T \geq 0. \tag{3.32}$$

Now if $S \leq 0$, then (3.32) entails (3.22) and we get $\mu \in \mathcal{M}\left(\frac{-2}{b_1}, \overline{M}, \overline{L}\right)$, while μ is nonnegative and normalized. Hence Lemma 3.4 can be applied to show that $-2/b_1$ is either $\leq A(M, L)$ or $\geq B(M, L)$. The same reasoning applies to the case $T = 0$ as well. Now observe that it would follow $a \leq -\frac{2}{b_1} \leq A(M, L)$ (< 0) or $a \geq -2/b_1 \geq B(M, L)$ (> 0), contradicting to our hypothesis whenever we have had

$$2 + ab_1 \leq 0. \tag{3.33}$$

To prove (3.33) we can use $1 + a\cos x \in \mathcal{H}$ and (3.19) in case $\alpha \geq 1 + a$. On the other hand, if $\alpha < 1 + a$, then $L \neq \emptyset$ and for all $l \in L$, $1 + a\cos x - (1 + a - \alpha)\cos lx \in \mathcal{H}$; whence from (3.27)

$$0 \geq I\big(1 + a\cos x - (1 + a - \alpha)\cos lx\big) = 1 + \frac{ab_1}{2} - \frac{(1 + a - \alpha)b_l}{2},$$

and simple calculation yields

$$(1 + a - \alpha)T \geq 2 + ab_1. \tag{3.34}$$

Summing up, we see that for $a \in \big(A(M, L), B(M, L)\big)$ and $\alpha \geq 1 + a$ one cannot have $S \leq 0$ (as it would lead to a contradiction) and for $a \in \big(A(M, L), B(M, L)\big)$ and $\alpha < 1 + a$ one cannot have $T = 0$ as in this case (3.34) entails (3.33), and we are led to the same contradiction.

Now we have only two cases. The first case is when $\alpha \geq 1 + a$ and $S > 0$, and the second case is when $\alpha < 1 + a$ and $T > 0$. As we have already seen, we can use (3.33) in the first case and (3.34) in the second. Moreover, if $\alpha \geq 1 + a$, then $1 + a\cos x + (\alpha - 1 - a)\cos mx \in \mathcal{H}$ $(\forall m \in M)$ and

$$0 \geq I(1 + a\cos x + (\alpha - 1 - a)\cos mx) = 1 + \frac{ab_1}{2} + (\alpha - 1 - a)\frac{b_m}{2}$$

yields

$$(\alpha - 1 - a)S \leq -(2 + ab_1). \tag{3.35}$$

At this point we can unify the two cases introducing

$$C := \begin{cases} S & \text{if } \alpha \geq 1 + a, \\ T & \text{if } \alpha < 1 + a. \end{cases} \tag{3.36}$$

Indeed, (3.34) for the case $\alpha < 1 + a$ and (3.35) in case $\alpha \geq 1 + a$ together give for all cases

$$(\alpha - 1 - a)C \leq -(2 + ab_1), \tag{3.37}$$

while (3.32), (3.36), and the above specification of the nonvoid cases entail

$$b_m \leq C \leq b_l \ (\forall m \in M, \ \forall l \in L) \quad \text{and} \quad C > 0. \tag{3.38}$$

Now we introduce the measure

$$\rho := \frac{2}{C}\mu - \delta + \alpha \cdot \lambda, \qquad d\rho(x) \sim r_0 + \sum_{k=1}^{\infty} r_k \cos kx. \tag{3.39}$$

For the coefficients r_k with $k \in \mathbb{N}_2$, (3.38), (3.39) immediately gives the coefficient conditions

$$r_m \leq 0 \ (m \in M), \qquad r_l \geq 0 \ (l \in L). \tag{3.40}$$

Next we consider the first two coefficients

$$r_0 = \frac{2}{C} - 1 + \alpha, \qquad r_1 = \frac{2b_1}{C} - 2. \tag{3.41}$$

Making use of $C > 0$ and (3.37) we obtain

$$\left(-\frac{a}{2}\right) \cdot r_1 = \frac{-ab_1}{C} + a \geq \frac{2}{C} + \alpha - 1 = r_0, \tag{3.42}$$

which entails the nonnegativity of the parameter

$$\gamma := \frac{(-a/2) \cdot r_1 - r_0}{1 + a^2/2|a|} \geq 0. \tag{3.43}$$

Let us define

$$\begin{aligned} d\tau(x) &:= d\rho(x) + \gamma \cdot \left(1 + \frac{a}{|a|}\cos x\right) dx, \\ d\tau(x) &\sim b + c\cos x + \sum_{k=2}^{\infty} t_k \cos kx. \end{aligned} \tag{3.44}$$

Computing the first coefficients we get

$$b = r_0 + \gamma, \qquad c = r_1 + \frac{a}{|a|}\gamma, \tag{3.45}$$

whence

$$\begin{aligned} \left(-\frac{a}{2}\right)c &= -\frac{a}{2}r_1 + \frac{-a^2}{2|a|}\gamma = -\frac{a}{2}r_1 + \gamma - \left(1 + \frac{a^2}{2|a|}\right)\gamma \\ &= \gamma - \frac{a}{2}r_1 - \left[\left(-\frac{a}{2}\right)r_1 - r_0\right] = b. \end{aligned} \tag{3.46}$$

Since $t_k = r_k$ for all $k \in \mathbb{N}_2$ trivially, (3.40), (3.44), and (3.46) imply

$$\tau \in \mathcal{M}(a, \overline{M}, \overline{L}). \tag{3.47}$$

Plainly, (3.39) and (3.44) ensure

$$\tau + \delta = \frac{2}{C}\mu + \alpha\lambda + \gamma\left(1 + \frac{a}{|a|}\cos x\right) dx \geq \alpha \cdot \lambda \tag{3.48}$$

as γ, C, $(1 \pm \cos x)\, dx$, and μ are all nonnegative (cf. (3.43), (3.38), and (3.27)). Now (3.48) leads to $t(\tau) \geq \alpha$, and this implies $\omega \geq \alpha$ in view of (3.47). The theorem is proved. □

3.7.

The construction of the above proof also gave $\omega(a, \overline{M}, \overline{L}) = t(\tau)$ for some $\tau \in \mathcal{M}(a, \overline{M}, \overline{L})$, thus proving that the sup in the definition (3.2) of ω is actually a maximum at least for $A(M, L) < a < B(M, L)$. Naturally, for $a < A(M, L)$ or $a > B(M, L)$ the supremum is essential, as $\omega(a, \overline{M}, \overline{L}) = +\infty$ according to Proposition 3.5. The only question is whether for the border cases $A(M, L) \in \mathcal{D}\big(\omega(\ , \overline{M}, \overline{L})\big)$ or $B(M, L) \in \mathcal{D}\big(\omega(\ , \overline{M}, \overline{L})\big)$ the supremum is attained or not.

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