J. Korean Math. Soc. **41** (2004), No. 1, pp. 157–174

# PLANK PROBLEMS, POLARIZATION AND CHEBYSHEV CONSTANTS

Szilárd Gy. Révész and Yannis Sarantopoulos

Abstract. In this work we discuss "plank problems" for *complex* Banach spaces and in particular for the classical $L_p(\mu)$ spaces. In the case $1 \leq p \leq 2$ we obtain optimal results and for finite dimensional complex Banach spaces, in a special case, we have improved an early result by K. Ball [3]. By using these results, in some cases we are able to find best possible lower bounds for the norms of homogeneous polynomials which are products of linear forms. In particular, we give an estimate in the case of a *real* Hilbert space which seems to be a difficult problem. We have also obtained some results on the so-called $n$-th (linear) polarization constant of a Banach space which is an isometric property of the space. Finally, known polynomial inequalities have been derived as simple consequences of various results related to plank problems.

## 1. Introduction

Recall that a *convex body* in $\mathbb{R}^n$ is a compact convex set that has a non-empty interior. A *plank* in $\mathbb{R}^n$ is the region between two parallel hyperplanes. In 1930 Tarski posed the *plank problem:*

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Received November 30, 2002.

2000 Mathematics Subject Classification: Primary 46G25; Secondary 52A40, 46B07.

Key words and phrases: Plank problem, homogeneous polynomials over normed spaces, linear polarization constants, quasi-monotonous sequences, Banach-Mazur distance, characterization of Banach spaces, local theory of Banach spaces, complexification of Banach spaces, weak-star convergent subsequences.

This joint research has been supported by a "Marie Curie Research Fellowship" of the European Community programme "Improving Human Potential" under contract #HPMF-CT-2000-00670.

The first author was supported in part by the Hungarian National Foundation for Scientific Research, Grant # T 034531 and T 032872.

Tarski's conjecture. *If a convex body of minimum width 1 is covered by a collection of planks in $\mathbb{R}^n$, then the sum of the widths of these planks is at least* 1.

Tarski himself proved this if the body is an Euclidean ball in 2 or 3 dimensions. The problem was solved in general by T. Bang [5] in 1951.

Given a convex body $K$, the *relative width* of a plank $S$ is the width of $S$ divided by the width of $K$ in the direction perpendicular to $S$. T. Bang asked a more general question:

Question 1. *If a convex body is covered by a union of planks, must the relative widths of the planks add up to at least* 1?

The general case of this affine plank problem is still open. However, in the special case in which the convex body is *centrally symmetric* the problem was solved by K. Ball in [3]. Observe that if $K$ is a centrally symmetric convex body, then it may be regarded as the unit ball of some finite-dimensional Banach space. In the next section we discuss K.Ball's results on the plank problem. By using the local theory of Banach spaces we also state and prove a plank-type problem for $L_p$-spaces. Finally, in the last section we use the plank problems in order to get some lower estimates for the norm of the simplest and, in many cases, the most useful polynomials on Banach spaces which are products of linear forms. In particular, we derive known inequalities, see [2] and [6], as simple consequences of the plank problems and we prove some new results.

Throughout this paper $X$ will be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ is the real or complex field and $X^*$ will denote the dual space. The closed unit ball and the unit sphere will be denoted by $B_X$ and $S_X$ respectively. A function $P : X \to \mathbb{K}$ is a continuous *$n$-homogeneous polynomial* if there is a continuous symmetric $n$-linear form $L : X^n \to \mathbb{K}$ for which $P(x) = L(x, \dots, x)$, for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. We define

$$\|P\| = \sup\{|P(x)| : x \in B_X\}.$$

We let $\mathcal{P}(^nX)$ denote the Banach space of all continuous $n$-homogeneous polynomials on $X$. For general background on polynomials, we refer to [7].

## 2. Plank problems

A plank in a Banach space $X$ is a set of the form

$$\{x \in X : |\phi(x) - m| \leq w\}$$

for some continuous linear functional $\phi$ on the space. If the norm of $\phi$ is 1, then the relative width of the plank is $w$ and in this case $w$ is said to be the *half-width* of the plank.

THEOREM A (K. BALL [3]). *If $(\phi_k)$ is a sequence of linear functionals, of norm 1, on a Banach space $X$, $(m_k)$ is a sequence of real numbers and $(w_k)$ is a sequence of positive numbers satisfying*

$$\sum_{k=1}^{\infty} w_k < 1,$$

*then there is a point $x \in B_X$ such that for every $k$*

$$|\phi_k(x) - m_k| > w_k.$$

If $X$ is a finite-dimensional space it may be assumed that there are only finitely many $(\phi_k)_{k=1}^{n}$ and it suffices to prove that if

$$\sum_{k=1}^{n} w_k = 1,$$

then there is a point $x \in B_X$ such that for every $k$

$$|\phi_k(x) - m_k| \geq w_k.$$

By slicing the planks into thin "sheets" it may also be assumed that all the $w_k$ are the same, i.e. that each is equal to $1/n$. In the special case in which all $m_k$ are zero, under the assumption that the width of each plank is $2/n$, the previous Theorem can be stated as follows.

THEOREM B (K. BALL [3]). *If $X$ is a finite-dimensional Banach space and $\phi_k \in S_{X^*}$, $1 \leq k \leq n$, then there is a point $x \in B_X$ such that for every $k$*

$$|\phi_k(x)| \geq \frac{1}{n}.$$

The condition in Theorem A that the $w_k$ add up to 1 is sharp. If $X$ is the space $\ell_1$, take $\phi_k$ to be the standard basis vectors of the dual $\ell_\infty$. However, for a Hilbert space one might expect to be able to improve upon this condition.

Theorem C (K. Ball [4]). *Let $(x_k)_{k=1}^n$ be a sequence of norm 1 vectors in a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $(t_k)_{k=1}^n$ be a sequence of non-negative numbers satisfying*

$$\sum_{k=1}^n t_k^2 = 1.$$

*Then there is a unit vector $x$ for which*

$$|\langle x, x_k\rangle| \geq t_k\,,$$

*for every $k$.*

For the proof of the previous result one can take $t_k = 1/\sqrt{n}$. Small modifications are needed to handle the general case, see [4].

It is an interesting fact that the proof of Theorem B for finite-dimensional complex Banach spaces is an easy consequence of the previous complex plank problem for a Hilbert space.

*Proof of Theorem B.* We may assume without loss of generality that $(X, \|\cdot\|)$ is an $n$-dimensional normed space. In 1948 Fritz John [10] proved that there exists a unique ellipsoid of maximal volume, denoted by $D_X^{\max}$, so that

$$D_X^{\max} \subset B_X \subset \sqrt{n} D_X^{\max}.$$

Therefore, see also theorem 5.6 in [9], there is an inner product $\langle\cdot,\cdot\rangle$ on $X$ giving a norm $\|\cdot\|_2$ such that for all $x \in X$

$$\|x\| \leq \|x\|_2 \leq \sqrt{n}\cdot\|x\|. \tag{1}$$

Let $x_k \in S_{X^*}$, $1 \leq k \leq n$, that is

$$\|x_k\|_{X^*} = \max\{|\langle x, x_k\rangle| : \|x\| \leq 1\} = 1.$$

By using (1) we have

$$\begin{aligned} 1 &= \|x_k\|_{X^*} = \max_x |\langle x/\|x\|, x_k\rangle| \\ &\leq \max_x \|x/\|x\|\|_2 \cdot \|x_k\|_2 \leq \sqrt{n}\cdot\|x_k\|_2\,. \end{aligned}$$

Hence

$$\|x_k\|_2 \geq 1/\sqrt{n}\,, \tag{2}$$

$k = 1, \ldots, n$. But from Theorem C there exists an $x \in X$, $\|x\|_2 = 1$, so that

$$|\langle x,\, x_k/\|x_k\|_2\rangle| \geq 1/\sqrt{n}\,,$$

$k = 1, \dots, n$ and if we apply (2) we get

$$|\langle x,\, x_k\rangle| \geq \frac{1}{\sqrt{n}} \cdot \|x_k\|_2 \geq \frac{1}{n}.$$

Consequently, by using (1) the previous inequality implies

$$\left|\langle \frac{x}{\|x\|},\, x_k\rangle\right| \geq \frac{1}{\|x\|} \cdot \frac{1}{n} \geq \frac{1}{\|x\|_2} \cdot \frac{1}{n} = \frac{1}{n}.$$

For $x_k \in S_{X^*}$, $1 \leq k \leq n$, we have shown that there exists an $x_0 \in S_X$, $x_0 := x/\|x\|$, such that

$$|\langle x_0, x_k\rangle| \geq 1/n.$$

□

If we use the complex plank problem for a Hilbert space we can prove a plank problem for complex $L_p(\mu)$ spaces. For this we need a well known result due to D. Lewis [13] (see also [16] or [33]):

Let $E$ be an $n$-dimensional subspace of $L_p(\mu)$, $1 \leq p \leq \infty$. For the Banach- Mazur distance $d(E, \ell_2^n)$ we have

$$d(E, \ell_2^n) \leq n^{\left|\frac{1}{2}-\frac{1}{p}\right|}. \tag{3}$$

In particular, for any $n$-dimensional Banach space $E$ we have

$$d(E, \ell_2^n) \leq \sqrt{n}\,, \tag{4}$$

which is due to F. John [10]. Recall that the Banach-Mazur distance between two isomorphic Banach spaces $X$ and $Y$, denoted by $d(X,Y)$, is defined as

$$d(X,Y) := \inf\{\|T\| \cdot \|T^{-1}\| :\ T : X \to Y \text{ is an isomorphism}\}.$$

PROPOSITION 1. *Let $X$ be a complex $L_p(\mu)$ space. If $f_k \in (L_p(\mu))^*$, $1 \leq k \leq n$, are unit vectors, then there is a point $x \in B_{L_p(\mu)}$ such that for every $k$*

$$|f_k(x)| \geq \begin{cases} n^{-1/p} & \text{if } 1 \leq p \leq 2 \\ n^{-1/p'} & \text{if } 2 \leq p \leq \infty, \end{cases} \tag{5}$$

*where $\frac{1}{p} + \frac{1}{p'} = 1$.*

*Proof.* Because of Theorem A, is enough to consider the case $1 < p < \infty$. But in this case the $L_p(\mu)$ spaces are reflexive and hence there exist $x_k \in B_{L_p(\mu)}$ with $|f_k(x_k)| = \|f_k\| = 1$, $k = 1, \dots, n$. If $E := \operatorname{span}\{x_1, \dots, x_n\}$, then $E$ is a subspace of $L_p(\mu)$ of dimension $d \leq n$. In view of (3), there exists an isomorphism $T : \ell_2^d \to E$ such that $\|T\| = 1$

and $\|T^{-1}\| \le d^{|1/2-1/p|}$. If $g_k = f_k|_E$, then $\|g_k\| = 1$ for $k = 1, \dots, n$. But from Theorem C, the complex plank problem for Hilbert spaces, there exists an $x \in \ell_2^d$, $\|x\|_2 = 1$, so that $|(g_k \circ T)(x)|/\|g_k \circ T\|_2 \ge n^{-1/2}$. Therefore,

$$(6) \qquad |g_k(Tx)| \ge \frac{1}{\sqrt{n}} \cdot \|g_k \circ T\|_2 \ge \frac{1}{\sqrt{n}} \cdot \frac{\|g_k\|}{\|T^{-1}\|} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\|T^{-1}\|}.$$

Since $\|Tx\| \le \|T\| \|x\|_2 = 1$, we have shown that for some $x_0 := Tx \in B_{L_p(\mu)}$ we have

$$|f_k(x_0)| = |g_k(x_0)| \ge \frac{1}{n^{\frac{1}{2}} \cdot d^{|\frac{1}{2}-\frac{1}{p}|}}.$$

The above inequality proves (5). Observe that the proof is similar for a $d$-dimensional $L_p(\mu)$ space, $d < n$. □

If we use the distance estimate (4) and work exactly as in the proof of Proposition 1, then we can prove that Theorem C implies Theorem B. In fact, for a $d$-dimensional complex Banach spaces, $d < n$, we can improve the estimate given in Theorem B.

COROLLARY 2. *Let $X$ be a $d$-dimensional complex Banach space, $d < n$ and let $f_k \in S_{X^*}$, $1 \le k \le n$. Then, there exists a point $x \in B_X$ such that for every $k$*

$$(7) \qquad |f_k(x)| \ge (n \cdot d)^{-1/2}.$$

*Moreover, if $X$ is a $d$-dimensional complex $L_p(\mu)$ space then*

$$(8) \qquad |f_k(x)| \ge n^{-1/2} \cdot d^{-|1/2-1/p|}.$$

REMARK 1. If $(x_k)_{k=1}^n$ is a sequence of norm 1 vectors in the complex Hilbert space $\ell_2^d$, with $d < n$, it is plausible that for some unit vector $x$ the estimate $|\langle x, x_k\rangle| \ge 1/\sqrt{n}$, $1 \le k \le n$, in Theorem C can be improved. Then of course inequalities (7) and (8) can be substantially improved.

The following example shows that in general the estimates in (5) cannot be improved.

EXAMPLE 1. Consider the space $\ell_p^n$ and the coordinate functionals $e_k, 1 \le k \le n$. If $1 \le p \le 2$ and $x = n^{-1/p} \sum_{k=1}^n e_k$, then $\|x\|_p = 1$ and $e_k(x) = n^{-1/p}, 1 \le k \le n$. On the other hand, if $2 < p \le \infty$ take $x = n^{-1/p'} \sum_{k=1}^n e_k$. Then $\|x\|_p < 1$ and $e_k(x) = n^{-1/p'}, 1 \le k \le n$.

Although the estimate in Theorem B is true for any real or complex Banach space, we are not aware of any improvement upon this estimate for *real* $L_p(\mu)$ spaces. However, we cannot have the same estimates as in (5).

EXAMPLE 2. Let $x_1, \dots, x_{2n}$ be unit vectors in the Euclidean space $\mathbb{R}^2$ distributed uniformly around the circle and let $x$ be any vector in the real space $\ell_p^2$ , with $\|x\|_p \leq 1$. As it has been observed by Pádraig Kirwan, see p.706 in [27], for the arbitrary unit vector $x/\|x\|_2$ in the plane there is an $x_k$, $1 \leq k \leq 2n$, for which

$$|\langle x_k, x/\|x\|_2\rangle| \leq \sin(\pi/2n) \leq \pi/2n.$$

Hence, there is a unit vector $x_k/\|x_k\|_{p'}$ among the unit vectors $x_1/\|x_1\|_{p'}$, $\dots, x_{2n}/\|x_{2n}\|_{p'}$ in the dual space $\ell_{p'}^2$ of $\ell_p^2$, such that

$$|\langle x_k/\|x_k\|_{p'}, x\rangle| \leq (\pi/2n)(\|x\|_2/\|x_k\|_{p'}) \leq (\pi/2n)2^{|p-2|/p}.$$

Now, this last inequality shows that in general (5) or (8) cannot be true on any *real* $L_p(\mu)$ space for $n$ large enough.

## 3. Polarization and Chebyshev constants

If $f_1, f_2, \dots, f_n$ are bounded linear functionals on a Banach space $X$, then the product $(f_1 f_2 \cdots f_n)(x) := f_1(x)f_2(x)\cdots f_n(x)$ is a continuous $n$-homogeneous polynomial on $X$. If $\|f_1 f_2 \cdots f_n\| = \sup_{\|x\|=1} |f_1(x)f_2(x) \cdots f_n(x)|$, there exists $C_n > 0$ such that

$$\|f_1\|\|f_2\|\cdots\|f_n\| \leq C_n \|f_1 f_2 \cdots f_n\|.$$

This inequality was derived by R.A. Ryan and B. Turett (theorem 9 in [27]) in their study of the strongly exposed points of the predual of the space of continuous 2-homogeneous polynomials. In [6] it was proved, in the case of *complex* Banach spaces, that $C_n \leq n^n$ and the constant $n^n$ is best possible. Since it is possible to improve this estimate for specific spaces we introduce the following definition (see also [6]).

DEFINITION 1 (Benítez-Sarantopulos-Tonge [6]). The *nth* (*linear*) *polarization constant* of a Banach space $X$ is defined by

$$\begin{aligned} c_n(X): &= \inf\{M > 0 : \|f_1\|\cdots\|f_n\| \\ &\leq M \cdot \|f_1 \cdots f_n\|, \forall f_1, \dots, f_n \in X^*\} \\ &= 1/\inf_{f_1,\dots,f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x)\cdots f_n(x)|. \end{aligned}$$

The *(linear) polarization constant* of a Banach space $X$ is

$$c(X) := \limsup c_n(X)^{1/n}.$$

In contrast, the *"general" nth polarization constant* $\mathbb{K}(n, X)$ of a Banach space $X$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is defined by

$$\mathbb{K}(n, X) := \inf\{M : \|L\| \le M\|P\|, \quad \forall\, P \in \mathcal{P}(^nX)\}, \tag{9}$$

where $L$ is the continuous symmetric $n$-linear form associated to $P$ (in this definition we consider all continuous $n$-homogeneous polynomials, not only products of linear forms). This polarization constant has been studied in [28] and [29], see also section 1.3 and in particular definition 1.40 in [7].

We can easily see that $c_n(\cdot)$ and $\mathbb{K}(n, \cdot)$ are isometric properties of Banach spaces. It is useful in our work to record some results on the $n$th (linear) polarization constant. First we prove that in Definition 1 "$\limsup c_n(X)^{1/n}$" can be replaced by "$\lim_{n\to\infty} c_n(X)^{1/n}$". For this we need a known result on quasi-monotone sequences. Recall that a sequence $(a_n)$ is called *quasi-monotone* if for all $m, n \in \mathbb{N}$

$$(m + n)a_{m+n} \le ma_m + na_n. \tag{10}$$

It seems that the proof of the following result is due to M. Fekete who has used it in his classical paper [8] (see p.233). For a proof see also Part I, Chapter 3, Problem 98 in [26].

Lemma 3 (M. Fekete [8]). *Any quasi-monotone sequence $(a_n)$ either converges to its infimum or diverges to $-\infty$.*

Proposition 4. *For the polarization constant $c(X)$ of a Banach space $X$ we have*

$$c(X) = \lim_{n\to\infty} c_n(X)^{1/n},$$

*where the limit could also be $\infty$.*

*Proof.* Let $\epsilon > 0$ and let $m, n \in \mathbb{N}$. By Definition 1, there exist $f_j, g_k \in S_{X^*}$, $1 \le j \le m$, $1 \le k \le n$, so that

$$1 \ge (c_m(X) - \epsilon)\|f_1 \cdots f_m\|, \quad 1 \ge (c_n(X) - \epsilon)\|g_1 \cdots g_n\|.$$

But then, by using once more Definition 1 and the fact that the norm of the product of two homogeneous polynomials is always less or equal the product of their norms, we have

$$\begin{aligned}(c_{m+n}(X))^{-1} &\le \|f_1 \cdots f_m \cdot g_1 \cdots g_n\| \le \|f_1 \cdots f_m\| \cdot \|g_1 \cdots g_n\| \\ &\le ((c_m(X) - \epsilon)(c_n(X) - \epsilon))^{-1}.\end{aligned}$$

This last inequality is true for any $\epsilon > 0$ and therefore

$$c_{m+n}(X) \geq c_m(X)c_n(X).$$

By taking $a_k := -\ln c_k(X)^{1/k}$, we finally have

$$(m+n)a_{m+n} \leq ma_m + na_n\,,$$

and the proof follows by Lemma 3. □

If $\mathbb{K}(X) := \limsup \mathbb{K}(n,X)^{1/n}$ is the *"general" polarization constant* of a Banach space $X$, see also definition 1.40 in [7], in view of the previous result it is natural to ask.

QUESTION 2. *Is $\mathbb{K}(X) = \lim_{n\to\infty} \mathbb{K}(n,X)^{1/n}$, where $X$ is any Banach space?*

We can easily see that $c_n(X/M) \leq c_n(X)$, for any quotient $X/M$ of $X$. For subspaces of $X$ we have the following result, see lemma 11 in [6] which is similar to lemma 1.46 in [7].

LEMMA 5 (Benítez- Sarantopoulos- Tonge [6]). *If $Y$ is a closed subspace of a Banach space $X$ and $P$ is a bounded projection of $X$ onto $Y$, then*

$$c_n(Y) \leq \|P\|^n c_n(X).$$

*In particular, if $Y$ is 1-complemented subspace of $X$ then*

$$c_n(Y) \leq c_n(X).$$

The Banach-Mazur distance between two isomorphic Banach spaces can be used in order to get estimates for the $n$-th polarization constants (see lemma 12 in [6]).

LEMMA 6 (Benítez- Sarantopoulos- Tonge [6]). *If $X_1$ and $X_2$ are isomorphic Banach spaces, then*

$$c_n(X_1) \leq\ d^n(X_1,X_2)c_n(X_2).$$

For each $n \in \mathbb{N}$, infinite dimensional Hilbert spaces have the smallest $n$-th polarization constant. To see this we use the fundamental theorem of Dvoretzky which states that every infinite dimensional Banach space $X$ contains for each $n \in \mathbb{N}$ and $\epsilon > 0$ an $n$-dimensional subspace $E$ such that $d(E,\ell_2^n) \leq 1+\epsilon$. For a simplified proof we refer to [23] or [25]. By duality, see also p.42 in [25], a similar statement holds for quotient spaces of $X$:

LEMMA 7 (Dvoretzky's Theorem). *Every infinite dimensional Banach space $X$ contains for each $n \in \mathbb{N}$ and $\epsilon > 0$ an $n$-dimensional quotient space $E$ such that $d(E,\ell_2^n) \leq 1+\epsilon$.*

We are grateful to V. M. Kadets [11] for calling our attention on this dual version of Dvoretzky's Theorem and thus suggesting the right approach in giving a short proof of the following result.

PROPOSITION 8. *If $X$ is an infinite dimensional Banach space, then*

$$c_n(X) \geq c_n(\ell_2^n)\,, \forall n \in \mathbb{N}. \tag{11}$$

*Proof.* Fix $n \in \mathbb{N}$. By the previous Lemma, for every $\epsilon > 0$ there exists an $n$-dimensional quotient space $E$ of $X$ with $d(E, \ell_2^n) \leq 1+\epsilon$. Since $c_n(E) \leq c_n(X)$, an application of Lemma 6 yields

$$c_n(\ell_2^n) \leq (1+\epsilon)^n c_n(E) \leq (1+\epsilon)^n c_n(X).$$

But this is true for any $\epsilon > 0$ and the proof follows. □

The next theorem, for *complex* Banach spaces, is just a restatement of corollary 4 and proposition 6 in [6]. However, the first part of the theorem for real or complex Banach spaces follows immediately from Theorem A, too.

THEOREM 9. (a) *If $X$ is any Banach space, then $c_n(X) \leq n^n$. In particular, if* $\dim(L_1(\mu)) \geq n$ *then* $c_n(L_1(\mu)) = n^n$.

(b) *If $X$ is a complex Banach space with $c_n(X) = n^n$, then $X$ contains $(1+\epsilon)$-isomorphic copies of $\ell_1^n$ for all $\epsilon > 0$. Moreover, if $X$ has dimension $n$, then $c_n(X) = n^n$ if and only if $X$ is isometrically isomorphic to $\ell_1^n$. In this case, the only $f_k \in S_{X^*}$ $(1 \leq k \leq n)$ which satisfy*

$$\inf_{f_1,\ldots,f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)| = n^n$$

*are the coordinate functionals.*

REMARK 2. Part (a) of the theorem was proved in [6] for complex Banach spaces. However, in the real case the previous theorem was proved only in the special case $n = 2$ (see proposition 14 in [6]). It is an open question as to whether or not part (b) of the theorem holds true for *real* Banach spaces of dimension $n$, $n \geq 3$.

If $H$ is a Hilbert space, in [6] was conjectured that $c_n(H) = n^{n/2}$. By using the relation between complex gaussian variables and the permanent, J. Arias-de-Reyna [2] has proved this conjecture only for complex Hilbert spaces. We refer to [22] for a summary on the theory of permanents of square matrices, especially with regard to inequalities satisfied by the permanent. The key in J. Arias' proof is an inequality on permanents due to E. H. Lieb [14] (see also [15] for some other matrix inequalities). In turn, Lieb's inequality generalizes the permanent analogue of the Hadamard determinant theorem due to M. Marcus [18], see also [19] and [21]. It is interesting that the following theorem, due to J. Arias-de-Reyna [2], is a corollary of Theorem C, that is K. Ball's complex plank problem [4].

THEOREM 10 (J.Arias-de-Reyna [2]). *If $(H, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space, then $c_n(H) \le n^{n/2}$. In other words, if $x_k \in H$, $1 \le k \le n$, are unit vectors then*

$$\sup_{\|x\|=1} |\langle x, x_1\rangle \cdots \langle x, x_n\rangle| \ge n^{-n/2} .$$

*If $\dim(H) \ge n$, then $c_n(H) = n^{n/2}$.*

In view of Example 2, we don't have a similar result as in Theorem C for *real* Hilbert spaces. However, the previous result could be true for real Hilbert spaces as well. A first approach in trying to tackle this problem is to use the best possible estimate we have for complex Hilbert spaces. This way we can give an upper estimate for $c_n(H)$, where $H$ is a real Hilbert space, which is better than $n^n$. For this, consider the complex Hilbert space $\widetilde{H} = H \oplus iH$, with norm $\|x+iy\| := \sqrt{\|x\|^2 + \|y\|^2}$, which is the *natural complexification* of $H$. If $\widetilde{P} \in \mathcal{P}(^n\widetilde{H})$ is the unique complex extension of the $n$-homogeneous polynomial $P(x) := \langle x, x_1\rangle \cdots \langle x, x_n\rangle$ defined on $H$, it is known (see [24] or [30]) that $\|\widetilde{P}\| = \sup_{\|x+iy\|=1} |\widetilde{P}(x+iy)| \le 2^{(n-2)/2}\|P\|$. But since $\|\widetilde{P}\| \ge n^{-n/2}$, we finally have $\|P\| \ge 2(2n)^{-n/2}$ which is better than the estimate $(4n)^{-n/2}$ given in [2].

COROLLARY 11. *For the $n$-th polarization constant of a real Hilbert space $H$ we have the estimate*

$$c_n(H) \le 2^{n/2-1} \cdot n^{n/2}.$$

*If $H$ is any real or complex Hilbert space of dimension at least $n$, then*

$$c_n(H) \ge n^{n/2}.$$

For the lower estimate in the previous inequality simply we have to take $n$ orthonormal vectors in the Hilbert space. Now we are in a position to give a first account on the order of magnitude of the polarization constants.

THEOREM 12. *Let $X$ be a real or complex Banach space. Then $c(X) = \infty$ if and only if $\dim(X) = \infty$.*

*Proof.* If $\dim(X) = \infty$, an application of Proposition 8 and Corollary 11 yields $c_n(X)^{1/n} \geq c_n(\ell_2^n)^{1/n} \geq n^{1/2}$, for all $n \in \mathbb{N}$. Therefore, $c(X) = \infty$.

Now we have to show that if $X$ is a finite dimensional Banach space, say $\dim(X) = m$, then $X$ must have finite polarization constant $c(X)$. For this, first of all observe that if $d(X, \ell_2^m) = d$, with $d \leq \sqrt{m}$, Lemma 6 can be used to obtain

$$c_n(X) \leq d^n c_n(\ell_2^m).$$

Hence, it suffices to show that $c_n(\ell_2^m)^{1/n}$ remains bounded as $n \to \infty$. Let $x_1, \ldots, x_n$ be $n$ arbitrary unit vectors in $\ell_2^m$. Since we need to find a lower estimate for $\max_{\|x\|_2=1} \sum_{k=1}^n \ln |\langle x, x_k\rangle|$, we shall use averaging over the unit sphere of $\ell_2^m$, that is we shall integrate with respect to the normalized surface measure $d\sigma$ of $S_{\ell_2^m}$. Notice that

$$C(m) := \int_{S_{\ell_2^m}} \ln |\langle x, x_k\rangle| \, d\sigma(x) \,,$$

is independent of the choice of the unit vector $x_k$. Therefore,

$$C(m) = \int_{S_{\ell_2^m}} \ln |\langle x, e_1\rangle| \, d\sigma(x).$$

We have

$$\max_{\|x\|_2=1} \sum_{k=1}^n \ln |\langle x, x_k\rangle| \geq \int_{S_{\ell_2^m}} \sum_{k=1}^n \ln |\langle x, x_k\rangle| \, d\sigma(x) = nC(m) \,,$$

which implies $\max_{\|x\|_2=1} \prod_{k=1}^n |\langle x, x_k\rangle| \geq e^{nC(m)}$. We have shown that $c_n(\ell_2^m) \leq e^{-nC(m)}$ and hence $c(\ell_2^m) \leq e^{-C(m)} < \infty$. This concludes our proof. □

Remark 3. In fact, the above proof can be elaborated to show $c(H) = e^{-C(m)}$, where $H$ is an $m$-dimensional Hilbert space. To this and other related results we shall return later.

Here is a different approach in trying to estimate the $n$-th polarization constant of an arbitrary *real* Hilbert space. First of all a perturbation argument, which was communicated to us by A. M. Tonge [31], yields the following result. We spare the reader the details.

Lemma 13. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $a_k \in H$, $1 \leq k \leq n$, be unit vectors. If for some unit vector $\xi \in H$ we have $\sup_{\|x\|=1} |\langle x, a_1\rangle$*

$\cdots\langle x, a_n\rangle| = |\langle \xi, a_1\rangle \cdots \langle \xi, a_n\rangle|$, *then*

$$n\xi = \frac{a_1}{\langle \xi, a_1\rangle} + \cdots + \frac{a_n}{\langle \xi, a_n\rangle} .$$

To find a lower estimate for the $\sup_{\|x\|=1} |\langle x, a_1\rangle \cdots \langle x, a_n\rangle| = |\langle \xi, a_1\rangle \cdots \langle \xi, a_n\rangle|$, without loss of generality we can take $a_1, \dots, a_n$ to be linearly independent in the Euclidean space $H = \mathbb{R}^n$. If $\xi = \xi_1 a_1 + \cdots + \xi_n a_n$, then we have $\|\xi_1 a_1 + \cdots + \xi_n a_n\|_2 = 1$ and from the previous Lemma

$$\xi_1\langle \xi, a_1\rangle = \cdots = \xi_n\langle \xi, a_n\rangle = 1/n.$$

Hence

$$|\langle \xi, a_1\rangle \cdots \langle \xi, a_n\rangle| = \frac{1}{n^n} \cdot \frac{1}{|\xi_1 \cdots \xi_n|} \geq \frac{1}{n^{n/2}} \cdot \frac{1}{(\xi_1^2 + \cdots + \xi_n^2)^{n/2}} . \tag{12}$$

Consider the quadratic form

$$q(x) := \|x_1 a_1 + \cdots + x_n a_n\|_2^2 = \sum_{i,j=1}^{n} \langle a_i, a_j\rangle x_i x_j , \tag{13}$$

where $x = x_1 a_1 + \cdots + x_n a_n$ is chosen so that $q(x) = 1$. (Hence the quadratic form is symmetric and positive definite.) To find the extreme values of $f(x) := x_1^2 + \cdots + x_n^2$ subject to the side condition $g(x) = 0$, where $g(x) = q(x) - 1$, we use the method of Lagrange multipliers. Since $f$ and $q$ are homogeneous polynomials of degree 2, we can apply Euler's identity to obtain

$$t\nabla f(x) - \nabla q(x) = 0 , \tag{14}$$

where $t = 1/f(x)$. The vector equation (14) then leads to the characteristic equation of the quadratic form (13). Let $t_1, \dots, t_n$ be the roots of the characteristic equation, i.e. the eigenvalues of the Gram matrix $A = [\langle a_i, a_j\rangle], i, j = 1, 2, \dots, n$. Note that $t_1, \dots, t_n$ must be real and positive. If $t_1$ is the smallest eigenvalue, then $f(x) \leq 1/t_1$. Therefore, by using (12) we finally get

$$\sup_{\|x\|_2=1} |\langle x, a_1\rangle \cdots \langle x, a_n\rangle| = |\langle \xi, a_1\rangle \cdots \langle \xi, a_n\rangle| \geq (t_1/n)^{n/2}.$$

Thus, we have shown the following result which was communicated to us by M. Marcus [20] in 1996.

PROPOSITION 14 (M. Marcus [20]). *If $a_1, a_2, \dots, a_n$ are unit vectors in the Euclidean space $\mathbb{R}^n$, then*

$$\sup_{\|x\|_2=1} |\langle x, a_1\rangle \cdots \langle x, a_n\rangle| \geq (t_1/n)^{n/2} ,$$

*where $t_1$ is the smallest eigenvalue of the Gram matrix $A = [\langle a_i, a_j\rangle]$, $i, j = 1, 2, \dots, n$.*

It is an easy but a tedious calculus exercise to show that for small values of $n$ in the Euclidean space $\mathbb{R}^n$ we have the same $n$th polarization constant "$n^{n/2}$" as in the case of complex Hilbert spaces.

PROPOSITION 15. *Let $a_1, \dots, a_n$ be unit vectors in the Euclidean space $\mathbb{R}^n$, $n = 2, 3, 4$. Then*

$$\sup_{\|x\|_2=1} |\langle x, a_1\rangle \cdots \langle x, a_n\rangle| \geq n^{-n/2}.$$

*Moreover, equality occurs if and only if the vectors $a_1, \dots, a_n$ are orthonormal.*

CONJECTURE. Let $a_1, \dots, a_n$ be (linearly independent) unit vectors in $\mathbb{R}^n$. Then,

$$\sup_{\|x\|_2=1} |\langle x, a_1\rangle \cdots \langle x, a_n\rangle| \geq n^{-n/2},$$

with equality if and only if the vectors are orthonormal. That is, for the *real* space $\ell_2^n$ we have

$$c_n(\ell_2^n) = n^{n/2}.$$

Concerning the $n$-th polarization constant of complex $L_p(\mu)$ spaces, the first part of the following result is a direct consequence of inequalities (5) in Proposition 1.

PROPOSITION 16. *For a complex $L_p(\mu)$ space we have*

$$c_n(L_p(\mu)) \leq \begin{cases} n^{n/p} & \text{if } 1 \leq p \leq 2 \\ n^{n/p'} & \text{if } 2 \leq p \leq \infty, \end{cases} \tag{15}$$

*where $\frac{1}{p} + \frac{1}{p'} = 1$. If $\dim(\mathrm{L_p}(\mu)) \geq \mathrm{n}$, then for $1 \leq p \leq 2$ the constant "$n^{n/p}$" in (15) is best possible.*

*Proof.* We need to prove the last part and for this is enough to consider the space $\ell_p^m$, with $m \geq n$. If $x = (x_j)_{j=1}^m \in \ell_p^m$ and $e_k$, $1 \leq k \leq n$, are the first $n$ coordinate functionals, then

$$\begin{aligned} |e_1(x) \cdots e_n(x)| &= |x_1 \cdots x_n| = (|x_1|^p \cdots |x_n|^p)^{1/p} \\ &\leq \left(\frac{|x_1|^p + \cdots + |x_n|^p}{n}\right)^{n/p} \leq n^{-n/p} \|x\|_p^n, \end{aligned}$$

by the arithmetic-geometric mean inequality. Thus $\|e_1 \cdots e_n\| \leq n^{-n/p}$ and since by Proposition 1 the reverse inequality holds the proof follows. □

Observe that for the *complex* space $\ell_p^d$ , $d < n$, inequality (8) implies

$$c_n(\ell_p^d) \leq n^{n/2} \cdot d^{|n/2-n/p|}. \tag{16}$$

REMARK 4. The upper estimate in (15) for $2 \leq p \leq \infty$ can be improved . For instance consider the complex space $\ell_\infty^2$. As we shall see in Proposition 18, we have $c_n(\ell_\infty^2) = 2^{n-1}$. In particular $c_2(\ell_\infty^2) = 2$, while by (15) or (16) we have the estimate $c_2(\ell_\infty^2) \leq 2^2 = 4$.

Up to now we have estimated the $n$-th polarization constant of any real or complex $L_1(\mu)$ space and any complex $L_p(\mu)$ space, $1 \leq p \leq 2$, in the case where $\dim(\mathrm{L_p}(\mu)) \geq \mathrm{n}$. In the rest of this work we find the $n$-th polarization constant of the *complex* space $\ell_1^d$, even in the case $n > d$, the two dimensional *complex* $C(K)$ space and the two dimensional real or complex Hilbert space. To find $c_n(\ell_1^d)$ we need the following inequality for norms of products of polynomials on complex Banach spaces (see theorem 3 in [6]):

Let $P_j$ be continuous homogeneous polynomials of degree $k_j$ on a complex Banach space $X$, $1 \leq j \leq n$. Then

$$\|P_1\| \cdots \|P_n\| \leq \frac{(k_1 + \cdots + k_n)^{k_1+\cdots+k_n}}{k_1^{k_1} \cdots k_n^{k_n}} \|P_1 \cdots P_n\|. \tag{17}$$

If $f_k \in (\ell_1^d)^* = \ell_\infty^d$, $1 \leq k \leq n$, then $\|f_k\| = |f_k(e_i)|$ for some $i$, $1 \leq i \leq d$. But then the previous inequality and example 1 in [6] yield the following result.

PROPOSITION 17. *The nth polarization constant of the complex space $\ell_1^d$ is*

$$c_n(\ell_1^d) = \max_{\substack{k_1+\cdots+k_d=n \\ k_i \geq 0}} \frac{n^n}{k_1^{k_1} \cdots k_d^{k_d}} = \prod_{l=0}^{d-1} \left( \frac{n}{[\frac{n+l}{d}]} \right)^{[\frac{n+l}{d}]} .$$

*In particular, if $n = m \cdot d$, then $c_n(\ell_1^d) = d^n$. Consequently, $c(\ell_1^d) = d$.*

As a result of inequality (7), observe that for any $d$-dimensional *complex* Banach space $X$ we have

$$c_n(X) \leq (d \cdot n)^{n/2} ,$$

where $d < n$.

In the classical case of polynomials $P_1, \ldots, P_m$ of one complex variable, A. Kroó and I. Pritsker [12] have improved early results of A. O. Gel'fond and K. Mahler [17] and they have shown the following sharp inequality

$$\|P_1\|_\infty \cdots \|P_m\|_\infty \leq 2^{n-1} \cdot \|P_1 \cdots P_m\|_\infty ,$$

where $\deg(P_1 \cdots P_m) = n$ and $\|\cdot\|_\infty$ denotes sup-norm. In particular, if $m = n$ and $P_k(z) = a_k z + b_k$, $a_k, b_k \in \mathbb{C}$, then

$$\prod_{k=1}^{n} \max_{|z|=1} |a_k z + b_k| \le 2^{n-1} \cdot \max_{|z|=1} \prod_{k=1}^{n} |a_k z + b_k|. \tag{18}$$

If $f_k \in (\ell_\infty^2)^* = \ell_1^2$, $1 \le k \le n$, where $\ell_\infty^2$ is the space $\mathbb{C}^2$ with sup-norm, by using the maximum modulus principle the previous inequality implies

$$\|f_1\| \cdots \|f_n\| \le 2^{n-1} \cdot \|f_1 \cdots f_n\|.$$

The constant "$2^{n-1}$" is best possible. Hence

PROPOSITION 18. *The $n$th polarization constant of the complex space $\ell_\infty^2$ is*

$$c_n(\ell_\infty^2) = 2^{n-1}.$$

The problem of finding the $n$th polarization constants of the 2-dimensional real and complex Hilbert spaces is related to the $n$th Chebyshev constants of the spheres of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

DEFINITION 2. The *$n$th Chebyshev constant* of a compact set $K$ on a normed space $(X, \|\cdot\|)$ is defined by

$$M_n(K) := \inf_{y_1,\dots,y_n \in K} \sup_{y \in K} \|y - y_1\|\|y - y_2\| \cdots \|y - y_n\|.$$

The fact that $M_n(S^1) = 2$, where $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, is well-known and easy to obtain. On the other hand, the explicit value of $M_n(S^2)$, where $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, is not known. However, G. Wagner [32] has proved that there exist constants $c_1, c_2 > 0$, so that

$$c_1 \le \log M_n(S^2) - (n/2)\log(4/e) \le c_2.$$

Here is a summary of the estimates for the polarization constants of $\ell_2^2$ which have been derived in [1].

PROPOSITION 19 (V. Anagnostopoulos- Sz. Révész [1]). *For the real space $\ell_2^2$ we have*

$$c_n(\ell_2^2) = \frac{2^n}{M_n(S^1)} = 2^{n-1},$$

*and so $c(\ell_2^2) = 2$.*

PROPOSITION 20 (V. Anagnostopoulos- Sz. Révész [1]). *For the complex space $\ell_2^2$ we have*

$$c_n(\ell_2^2) = \frac{2^n}{M_n(S^2)}.$$

*Hence, for the complex space $\ell_2^2$ there exist absolute constants $c$ and $C$, $0 < c < C < \infty$, so that*

$$c(\sqrt{e})^n \leq c_n(\ell_2^2) \leq C(\sqrt{e})^n.$$

*Therefore, $c(\ell_2^2) = \sqrt{e}$.*

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Szilárd Gy. Révész
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O.B. 127, 1364 Hungary
*E-mail*: revesz@renyi.hu

Yannis Sarantopoulos
National Technical University
School of Applied Mathematical and Physical Sciences
Department of Mathematics
Zografou Campus 157 80, Athens, Greece
*E-mail*: ysarant@math.ntua.gr