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ON MARKOV CONSTANTS OF HOMOGENEOUS POLYNOMIALS OVER REAL NORMED SPACES

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Let $P\colon X \to Y$ be a homogeneous polynomial of degree $\leq m$ between the normed spaces X and Y. In 1997 L. A. Harris proved that the Fréchet derivative of P satisfies the Markov inequality $\|\widehat{D}^k P\| \leq c_{m,k}\|P\|$, where the best constant $c_{m,k}$ can be obtained as a solution of an extremal problem for polynomials on the real line. He also gave upper and lower estimates to $c_{m,k}$ and computed exact values up to $m = 20$. Here we obtain improved estimates and thus we find that the exact order of magnitude of $c_{m,k}$ is the $(m \log m)^k$ order of the upper estimate of Harris for at least $k = 1, 2$. Our method relies on the technique of potential theory with external fields, what we apply with varying weights and at a border-case situation.

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1. Introduction

To estimate the norm of the derivatives of a polynomial under normalizing or further restrictions is a central field of interest. Applications range from approximation theory to the theory of holomorphic mappings over infinite dimensional spaces and from estimations of various polarization constants to characterization of particular Banach spaces.

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In 1997 L. A. Harris proved, that for homogeneous polynomials over real normed spaces, the optimal Markov constant can be identified with an extremal quantity which may be called "Bernstein-Markov constant of a weighted space of polynomials at $x = 0$". We shall call this quantity "the Harris-Markov constant". Let us recall its definition from [H1].

Definition. The Harris-Markov constant $c_{m,k}$, $0 \le k \le m$, $k, m \in \mathbb{N}$, is

$$c_{m,k} := \max\{|p^{(k)}(0)| \colon p \in \mathcal{P}_m, |p(t)| \le (1 + |t|)^m (t \in \mathbb{R})\}. \tag{1.1}$$

Here and throughout the paper, $\mathcal{P}_m$ denotes the set of real polynomials of degree not exceeding m. Polynomials from X to Y, where X and Y are real normed spaces, will be denoted by $\mathcal{P}_m(X, Y)$ with the understanding of m as the bound for the degree of polynomials in $\mathcal{P}_m(X, Y)$. For definitions and basic facts concerning polynomials between normed spaces we refer to the recent monographs [D, Chapter 1] or [H, Appendix A].

Harris wrote: "Although we are unable to obtain a general formula for the constants $c_{m,k}$, we do establish elementary upper and lower bounds and provide a good estimate on their asymptotic growth. We determine the value of some low dimensional cases and find associated extremal polynomials..." cf. [H1, p. 476]. Actually, Harris went on to compute $c_{m,k}$ for all $k, m \le 20$ in [H2]. In general, the best estimates he obtained were the following.

Theorem A (Harris). *For all* $0 \le k \le m$*,*

$$\frac{m^m}{k^k (m-k)^{m-k}} \le \frac{c_{m,k}}{k!} \le \binom{m}{k} \frac{m^{m/2}}{k^{k/2} (m-k)^{(m-k)/2}}. \tag{1.2}$$

Moreover, there exists an absolute constant M *such that*

$$c_{m,k} \le (M \ m \ \log m)^k \tag{1.3}$$

for all $0 \le k \le m$*,* $m \ge 1$*.*

Harris also investigated the analogous questions when in (1.1) the bound $(1+|t|)^m$ is changed to $\phi(t)^m$ with some positive, continuous and even function satisfying some of the conditions

$$\begin{aligned} &\text{(i)} && \phi(x) \ge \phi(0), \\ &\text{(ii)} && \lim_{x\to\infty} \phi(x)/x = A \qquad (A \ne 0), \\ &\text{(iii)} && \phi(x) \ge A|x| \qquad (A \ne 0), \\ &\text{(iv)} && |x|\phi(1/x) = \phi(x) \qquad (x \ne 0). \end{aligned} \tag{1.4}$$

Note that all these conditions hold if (i) and (iv) hold true. A particular example of importance is when $\phi(x) = (1 + |x|^p)^{1/p}$, which satisfies (i)–(iv) (with $A = 1$) when $0 < p < \infty$.

The importance of such questions lies in the fact that they are closely related to estimations of norms of derivatives of polynomials over normed spaces. We quote [H1, Theorem 2] here.

Theorem B (Harris.) *Let X, Y be real normed spaces and ϕ be a non-negative function satisfying*

$$\phi(x+y) \le \phi(x) + \|y\| \qquad (x, y \in X) \tag{1.5}$$

and let $P\colon X \to Y$ be a polynomial satisfying

$$\|P(x)\| \le \phi(x)^m \qquad (x \in X). \tag{1.6}$$

Then we have

$$\|\widehat{D}^k P(x)\| \le c_{m,k}\, \phi(x)^{m-k} \qquad (x \in X,\ 0 \le k \le m). \tag{1.7}$$

Moreover, if $X = \ell^1(\mathbb{R}^2)$, $Y = \mathbb{R}$ and $\phi(x) = \|x\|$, then for each m and k there exists a homogeneous polynomial $P \in \mathcal{P}_m(X, Y)$, depending only on k and m, such that equality holds in (1.7) *for some x with $\phi(x)$ any given non-negative number.*

In this theorem $\widehat{D}^k P(x)$ denotes the homogeneous polynomial associated with the k-th order Fréchet derivative $D^k P(x)$ and is given by

$$\widehat{D}^k P(x)y = \frac{d^k}{dt^k} P(x+ty)\big|_{t=0}. \tag{1.8}$$

The extremal polynomial of the above theorem is constructed from the extremal polynomial $p = p_{m,k}(t)$ in (1.1) the following way:

$$P(x, y) = x^m p\left(\frac{y}{x}\right) \qquad (x \ne 0)\ .$$

Clearly, (1.1) constitutes the most important special case of all possible weighted variants with ϕ defined according to (1.4), since the weight $(1+|t|)^m$ can be used to obtain estimates in the full generality of real normed spaces. Other weights are of interest, however, if we consider some special normed spaces. The sharpness statement of Theorem B suggests that the worst case is (as in many extremal problems and inequalities for real normed spaces), the ℓ^1 space case. For Hilbert spaces, e.g., one can work with $(1+t^2)^{m/2}$, see [H3, Corollary 5].

Let us point out that analogous statements hold for complex normed spaces but there the corresponding Markov constants were found by Harris cf. [H1, Theorem 2], complex part, with the exact values being $c^{\mathbb{C}}_{m,k} = \frac{m^m\ k!}{k^k (m-k)^{m-k}}$. Hence, we restrict ourselves to the real case. Our goal in this work is to sharpen Harris' estimates on the value of the constants (1.1), finding the correct order of magnitude and even the probable asymptotic value of $c_{m,k}$ when m increases to infinity. We accomplish this by an application of a powerful technique in recent approximation theory, namely, potential theory. The application is of interest even in itself, being a delicate border-case matter. Actually only a slight reduction of the degree, provided by the property (iv) of (1.4), can help us out to achieve the crucial technical necessity of $p(x)w(x) \to 0$ $(x \to \infty)$. Since all our estimates are worked out for varying weights and spaces, a sufficiently precise calculation and a number of theoretical information is needed to obtain the desired conclusion. Our first result is

Theorem 1. *We have, with some absolute constant $c_1 > 0$,*

$$c_1 m \log m \ \le c_{m,1} \le 3m \log m. \tag{1.9}$$

Further results, in particular for $c_{m,k}$ with $k \ge 2$, follow in the sequel. However, let us observe right here that iteration of Theorem B yields $c_{m,k} \le \prod_{j=1}^{\ell} c_{m_j,k_j}$ with any system of natural numbers $k_1, \dots, k_\ell$ and $m_1 := m$, $m_j := m_{j-1} - k_{j-1} \ge 0$ for $j = 2, \dots \ell$. It follows that we also have $c_{m,k} \le \prod_{j=1}^{\ell} c_{m,k_j}$ for $k_1 + \dots + k_\ell \le m$. This iterative property of the Harris-Markov constants $c_{m,k}$ is not seen from the definition of $c_{m,k}$ in (1.1).

Harris has obtained a couple of appealing properties of the weighted extremal polynomials and the underlying space, such as oscillation points, existence of simultaneously extremal polynomials P and Q (for even and odd differences $m-k$) etc., see [H2], [H4]. We do not rely on these, which is understandable as nowadays potential theory essentially comprises these properties (although in disguised forms sometimes).

A recent work of Kilgore [K] aims at the determination of the Bernstein-Markov constants of weighted polynomial spaces with similar weights. In this work both the background and the method are different from ours, but the results partly overlap. In these instances our estimates are even sharper, due to the strength of the potential theoretic approach.

Another connection to previous work in approximation theory was pointed out to us after we published the first version of this work as *Preprint* # **4**/2001 *of the Mathematical Institute of the Hungarian Academy of Sciences*. It was recognized after the work of Harris that earlier investigation of the Bernstein-Markov factors of the so-called *self-reciprocal polynomial spaces* by Kroó and Szabados [KS] and also by Borwein and Erdélyi, (see p. 339 of [BE]), fits into this context. In fact, the upper estimate (1.3) of Harris with some unspecified, large constant M obtains directly from these works (see e.g. Lemma 4 of [KS]). Our estimate is a bit sharper providing an explicit constant. On the other hand, the lower estimates (and the proofs) of [KS] and [BE] involved the norm of the derivative of polynomial over an interval. The $c_1 m \log m$ lower estimate, and thus the precise order of magnitude of $c_{m,1}$, is obtained in Theorem 1 for the first time.

We are deeply indebted to Professor V. Totik for detecting a technical gap in the proof of the first version and giving hints to fill the gap. This concerns the unpleasant logarithmic singularity of the density function $v_w(x)$ of the equilibrium measure at the origin (see the discussion in §2). In the earlier preprint version we simply overlooked this singularity and assumed that v_w is continuous at 0 (which is not true for the deviation of the right-hand side and left-hand side derivatives of the weight function $w(x)$ at 0). Fortunately, very recent stage of development of potential theory allows us to incorporate such singularities into the theoretical scheme used, as will be seen below.

2. Application of potential theory

In this section we consider a *weight function*

$$w(x) = e^{-Q(|x|)} \tag{2.1}$$

with the following properties for $Q\colon [0,\infty) \to \mathbb{R}$:

$$\begin{aligned} &\text{i)} \quad Q \ge 0, \quad \text{and} \quad Q(0) = 0;\\ &\text{ii)} \quad Q \in C^2[0,\infty), \quad \text{i.e. } Q \text{ is twice continuously differentiable;}\\ &\text{iii)} \quad xQ'(x) \quad \text{is strictly monotonically increasing;}\\ &\text{iv)} \quad Q(x) - \log x \to +\infty \quad (x \to +\infty). \end{aligned} \tag{2.2}$$

Note that most of the potential theoretic apparatus has been worked out for much more general weights as well, but condition iv) is essential. Our restrictive presentation is aimed at keeping a good balance between generality and transparency, as for functional analytic applications (2.2) is more than sufficiently general.

Our standard reference for potential theoretic matters will be [ST] where we find the following notation and facts. The weighted *energy integral* and the (weighted) *potential* are

$$\begin{aligned} I_w(\mu) &:= \int\int \log[|z-t|w(z)w(t)]^{-1} d\mu(z)\, d\mu(t)\\ &= \int\int \log\frac{1}{|z-t|} d\mu(z)\, d\mu(t) + 2\int Q\, d\mu \end{aligned} \tag{2.3}$$

and

$$U^\mu(z) := \int \log\frac{1}{|z-t|}\, d\mu(t) \tag{2.4}$$

with respect to any $\mu \in \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the set of the normalized non-negative regular Borel measures on $\mathbb{R}$. The basic facts of weighted potential theory starts with the assertion that there exists a unique element $\mu_w \in \mathcal{M}(\mathbb{R})$, called the (weighted) *equilibrium measure*, which minimizes the energy integral (2.3), where the resulting (weighted) *capacity*

$$V_w := \inf\{I_w(\mu) \colon \mu \in \mathcal{M}(\mathbb{R})\} = I_w(\mu_w) \tag{2.5}$$

is finite, and μ_w has finite logarithmic energy

$$-\infty < \int \int \log\left(\frac{1}{|z-t|}\right) d\mu_w(z)\, d\mu_w(t) < \infty \ . \tag{2.6}$$

Moreover, the support S_w of μ_w is a compact set of positive capacity, and defining the *modified Robin constant*

$$F_w := V_w - \int Q\, d\,\mu_w, \tag{2.7}$$

we have

$$U^{\mu_w}(z) + Q(z) \leq F_w \qquad (\forall\ z \in S_w) \tag{2.8}$$

and

$$U^{\mu_w}(z) + Q(z) \geq F_w \qquad (\text{q.e. } z \in \mathbb{R}). \tag{2.9}$$

Here and everywhere in the paper we use the standard term *quasi everywhere*, in the notation q.e., to denote that the exceptional set is of zero capacity.

Here comes our set of restrictive conditions ii) and iii) to our help first. With a reference to Corollary IV. 1. 12 on p. 203 of [ST], or, more generally, to [B], we can now assert that S_w is a compact, symmetric (with respect to the origin) interval with the endpoint a_w being the unique solution of a certain equation, namely

$$S_w = [-a, a], \quad \frac{2}{\pi}\int_0^1 \frac{at\, Q'(at)}{\sqrt{1-t^2}}\, dt = 1. \tag{2.10}$$

Note that $a = a_w$ is (essentially) the celebrated Mhaskar-Rahmanov-Saff number [MS] in weighted approximation. Now, taking into account Theorem IV. 3. 3 on p. 226 of [ST], we find that μ_w is absolutely continuous, and its density function $v := v_w$ can be expressed by a certain integral equation (given in (2.20) below).

But here is a second convenient yield of the conditions (2.2), namely, that in view of (2.2) ii), and Theorem IV. 2. 5 on p. 216 of [ST], we even have $\mu_w' \in C((-a,0) \cup (0,a))$. Moreover (see in Lemma 2), even the potential function U^{μ_w} of μ_w (as defined in (2.4)) has to be continuous, even on the whole of $\mathbb{C}$, which, in turn, implies now that (2.9) must hold all over the support S_w of μ_w.

The w-weighted norm (for some weight $w > 0$) is defined as

$$\|f\|_w := \sup\{|f(x)w(x)| \colon x \in \mathbb{R}\}; \tag{2.11}$$

the continuous real functions of finite norm (2.11) form the weighted space C_w. The *weighted Chebyshev problem*, see (1.5) of [MS] or p. 162–163 of [ST], is the determination of the weighted Chebyshev constants $E_n(w)$ and the weighted Chebyshev polynomials $T_n(x, w)$, where

$$\begin{aligned} E_n(w) &:= \inf\{\|[w(x)]^n\{x^n - p_{n-1}(x)\}\| \colon p_{n-1} \in \mathcal{P}_{n-1}\} \\ &= \|T_n(\cdot, w)\|_{w^n}. \end{aligned} \tag{2.12}$$

One of the most important results of weighted potential theory (and one good reason for being developed) is the following. We have

$$E_n(w)^{1/n} \geq \exp(-F_w) = \lim_{n\to\infty} E_n(w)^{1/n}, \tag{2.13}$$

see [MS], (2.3) and (2.5), or Theorem III. 3. 1. on p. 163 of [ST].

Once the extremal problem is translated to a weighted Chebyshev problem of the type (2.13), with some appropriate weight (2.1)–(2.2), we calculate the value of F_w. To that, we can determine $S_w = [-a_w, a_w]$ using (2.10), and having found a_w, we can struggle through the computation of $v_w := \mu_w'$ and μ_w, the potential (2.4) of μ_w, and finally the Robin constant (2.7). However, instead of this laborious method, we can take advantage of a number of shortcuts furnished by powerful general results of potential theory and our convenient choice of the weights in (2.2). Namely, following (1.1) of p. 194 of [ST] or (1.3) of [MS], we can introduce the so called *F-functional*

$$F(K) := \log(\operatorname{cap}(K)) - \int Q\, d\nu_K, \tag{2.14}$$

where K runs over compact sets of $\mathbb{R}$ and ν_K is the (unweighted) equilibrium measure of K, while $\operatorname{cap}(K)$ is (unweighted) capacity of K. So the point is here: $F(K)$ is defined by "tame" objects, without recourse to the weight or its weighted generalized objects defined above. On the other hand, F is closely related to the Robin constant:

$$\max\{F(K) \colon K \subset \mathbb{R} \text{ compact}\} = F(S_w) = -F_w, \tag{2.15}$$

see [ST], Theorem IV. 1. 5 on p. 194.

In our case we already know that $S_w = [-a, a]$ with some a, hence to find $F(S_w)$ it suffices to maximize the F-functional (2.14) over intervals symmetric with respect to the origin. As for the unit interval $I = [-1, 1]$, the equilibrium measure $\nu_I = \nu_1$ is the well-known arc-sine or Chebyshev distribution, the dilated interval $[-a, a]$ will have the equilibrium distribution

$$d\nu_a(x) = \frac{2}{\pi}\frac{dx}{\sqrt{a^2 - x^2}} \qquad (x \in (-a, a)). \tag{2.16}$$

Since $\operatorname{cap}([-a, a]) = \frac{a}{2}$, we now have

$$\begin{aligned} \varphi(a) := F([-a, a]) &= \log\frac{a}{2} - \frac{2}{\pi}\int_0^a \frac{Q(x)}{\sqrt{a^2 - x^2}}\,dx \\ &= \log\frac{a}{2} - \frac{2}{\pi}\int_0^1 \frac{Q(at)}{\sqrt{1 - t^2}}\,dt, \end{aligned} \tag{2.17}$$

and the only thing to do is to find the maximum of (2.17). Let us now summarize what we shall apply.

Lemma 1. *If $w = e^{-Q}$ is a weight satisfying* (2.2)*, then in the weighted Chebyshev problem* (2.12) *the following estimate holds true:*

$$E_n(w) \ge \exp\left(n \max_{a>0} \varphi(a)\right), \tag{2.18}$$

where $\varphi(a)$ is given in (2.17).

Actually, (2.18) holds true for all admissible weights, even if S_w is not an interval, by the maximizing property of the F functional. But if $\max_a F([-a, a]) \ne F(S_w)$, then the resulting estimate is not sharp, and, because of the exponent n appearing in (2.18), this flaw can be serious. On the other hand, for w fixed and $n \to \infty$, (2.18) gives an asymptotically sharp estimate of $E_n(w)^{1/n}$ in view of the right-hand side of (2.13).

So far so good, but dealing with the Harris-Markov constants we have to realize, that even for $k = 1$ and $m \to \infty$ in the estimation of $c_{m,1}$ we encounter varying weights not only in the sense of (2.12), but also in the sense that even w is changing. When attempting the calculation of $c_{2,m}$, e.g., then this is even more so, so for any lower estimate of $c_{k,m}$, that is, upper estimate of $E_m(w)$, we can not just let $m \to \infty$.

Instead, a more refined "error estimate" of the deviation of the Chebyshev polynomial $T(\cdot, w)$ and the limiting case potential is necessary. To this, again, smoothness properties set forth in (2.2) ii)-iii) prove to be useful. Thanks to pioneering work of V. Totik, we need not dwell into calculations and constructions here, but we can make use of a result, general and sharp enough for a sound lower estimation. Namely, Theorem VI. 4. 2 on p. 330 of [ST] can (almost) be applied to the equilibrium measure μ_w supported on $S_w = [-a_w, a_w]$. In our case μ_w is absolutely continuous with some density function v_w continuous on $(-a_w, 0) \cup (0, a_w)$, see Theorem IV. 2. 5 on p. 216 of [ST], but to apply the above quoted result we also need some further properties of w, Q and μ_w to be shown.

First of all, let us normalize the support by the change of variable $t = a_w\, x$, that is, $\widetilde{Q}(x) = Q(t) = Q(a_w\, x)$, so that $g(x) = a_w\, v_w(a_w\, x)$ will now be the density function of $d\,\mu(x) = d\,\mu_w(a_w\, x)$, the equilibrium measure on $[-1,1]$ with weight $\widetilde{w} = \exp(-\widetilde{Q})$ over $\mathbb{R}$. Again, $f(x) := \widetilde{Q}(x)$ satisfies the conditions (2.2) i)-iv), and we can apply* Theorem IV. 3. 3 on p. 226 of [ST] to obtain that

$$\begin{aligned} U^\mu(x) &= \int_{-1}^{1} \log\frac{1}{|x-t|}\, g(t)\, dt \\ &= -\widetilde{Q}(x) + \frac{2}{\pi}\int_0^1 \frac{\widetilde{Q}(t)}{\sqrt{1-t^2}}\, dt + \log 2 \qquad \big(x \in (-1,1)\big) \end{aligned} \tag{2.19}$$

with the even function $g(x)$ satisfying

$$\begin{aligned} g(x) &= L[\widetilde{Q}\,'](x) \\ &= \frac{2}{\pi^2}\int_0^1 \sqrt{\frac{1-x^2}{1-y^2}} \cdot \frac{x\widetilde{Q}\,'(x) - y\widetilde{Q}\,'(y)}{(x^2-y^2)}\, dy \qquad \big(x \in (0,1)\big). \end{aligned} \tag{2.20}$$

Note that (2.19)–(2.20) are provided by Theorem IV. 3. 3 (and Theorem IV. 3. 1 quoted in its formulation) first for a.e. $x \in (0,1)$, but our a priori knowledge of continuity of g (and, moreover, also U^μ, see the proof of Lemma 2 below) then ensures the validity of these formulae for all x. Finally, let us observe that the integrand in equation (2.20) of g is always non-negative by (2.2) iii) (except at $y = x$), and so by continuity of the integrand guaranteed by (2.2) ii), $g(x)$ has to be > 0 as well. (In fact, this is the argument of (3.9), Part c) of Theorem IV.3.1. on p. 221 of [ST].)

So, now we show that at the endpoints of the support g has an asymptotics const. $\sqrt{1-x^2}$. As $\widetilde{Q}, \widetilde{W}$ are even by (2.2) i), so are μ and g, hence it suffices to deal with the right endpoint. By the smoothness condition assumed in (2.2) ii), $xQ'(x) - yQ'(y) = \big(Q''(\xi)\xi + Q'(\xi)\big)(x-y)$ with some $\xi \in (x,y)$, and for all $x \neq y$. Hence we also have by C^2 condition (2.2) ii)

$$0 \leq \frac{xQ'(x) - yQ'(y)}{(x-y)(x+y)} = \frac{\xi Q''(\xi) + Q'(\xi)}{x+y} \to \frac{xQ''(x) + Q'(x)}{2x} \qquad (y \to x)\ .$$

We find that

$$0 \leq \Phi(x,y) := \begin{cases} \frac{xQ'(x)-yQ'(y)}{x-y} & (x \neq y) \\ \frac{1}{2}Q''(x) + \frac{Q'(x)}{2x} & (x = y) \end{cases}$$

*Note that in Theorem IV. 3. 3 the minus sign appearing in $f(x) = -Q(x)$ is obviously a misprint.

is continuous on $(0,\infty)^2$, hence for $1/2 \le x \le 1$ we have by (2.20)

$$0 \le \frac{\pi^2 g(x)}{2\sqrt{1-x^2}} = \int_0^1 \frac{\Phi(x,y)}{x+y}\frac{dy}{\sqrt{1-y^2}} \to \int_0^1 \frac{\Phi(1,y)}{1+y}\frac{dy}{\sqrt{1-y^2}} \qquad (x \to 1),$$

where this last limit exists and is finite by the uniform continuity of Φ on $[1/2,1]\times[0,1]$. (Note that Φ is continuous even at $y=0$ for all $x \in [1/2,1]$.) So we have $g(x) \sim c\cdot\sqrt{1-x^2}$ $(x \to 1)$, and the conditions of Theorem VI. 4. 2 on p. 330 of [ST] are satisfied except at 0, where $g(x)$ may have a logarithmic singularity. Indeed, $w(x)$ is not necessarily differentiable at 0, but only of class Lipschitz 1, and it may have only one-sided derivatives. This is so because although $Q \in C^2[0,\infty)$ by (2.2) ii), but $Q'_+(0)$ is not necessarily zero, and as $w'_\pm(0) = \mp Q'(0)w(0)$, $w'_+(0) \ne w'_-(0)$ in this case. It is a standard calculus exercise to deduce from equation (2.20) and $Q \in C^2[0,\infty)$ that $g(x)/\log(1/|x|) \to const. \ne 0$ in case $w'_+(0) \ne w'_-(0)$. In fact we will only need that $g(x)$ is slowly varying (in the sense given in [T]). Thus we spare the reader the details of this calculation. Instead, we formulate now

Lemma 2. *If $w = e^{-Q}$ is a weight function satisfying* (2.2) i)-iv)*, then the support of the equilibrium measure is an interval $S_w = [-a_w, a_w]$, and $d\mu_w = v_w$ is continuous even on $\mathbb{R}$. The potential function*

$$U^{\mu_w}(x) := \int_{-a_w}^{a_w} \log\left(\frac{1}{|x-t|}\right) v_w(t)\,dt \tag{2.21}$$

is continuous, and

$$U^{\mu_w}(x) = -Q(x) + \frac{2}{\pi}\int_0^1 \frac{Q(a_w x)}{\sqrt{1-x^2}}\,dx + \log\frac{2}{a_w} \qquad \big(|x| \le a_w\big), \tag{2.22}$$

while

$$U^{\mu_w(x)} \le -Q(x) + \frac{2}{\pi}\int_0^1 \frac{Q(a_w x)}{\sqrt{1-x^2}}\,dx + \log\frac{2}{a_w} \qquad \big(x \in \mathbb{R}\big). \tag{2.23}$$

Moreover, there exists an absolute constant $0 < C < \infty$ so that for all n there is a monic polynomial $p_n \in \mathcal{P}_n$ satisfying

$$|p_n(x)| \le C \exp\big(-n\,U^{\mu_w}(x)\big) \qquad (x \in \mathbb{R})\,. \tag{2.24}$$

Proof. Substituting $x = a_w t$ we can apply the above given deduction of properties of μ_w and $g(x) := g_w(x) := a_w v_w(a_w x)$. Now (2.22) follows by substitution from (2.19), while (2.23) is the generally valid statement of (2.9), which extends from *q.e.* even to $\mathbb{R}$, once U^{μ_w} is shown to be continuous.

Here the general theory would yield continuity of U^{μ_w} from continuity of $g(x)$, what we do not have in full extent. However, with its zeros and discontinuities described by square root and logarithmic asymptotics at the endpoints and the midpoint, respectively, g lies certainly in $L^p[-1,1]$ for all $p>1$. Thus Lemma 3.1 of [S] provides $U^{\mu_w} \in C(\mathbb{C})$.

Finally, the conditions to Theorem VI. 4. 2 on p. 330 of [ST] have been checked already, with the only exception of a possible logarithmic singularity at 0. Thus it suffices to extend Theorem VI. 4. 2 of [ST] in case of finitely many logarithmic, or, even the better, slowly varying singularities allowed. However, this extension has been covered by a remark of Totik following the quotation of Theorem VI. 4. 2 as Theorem C on p. 276 in [T]. Thus, we are entitled to use the same result even for the given case of the possible logarithmic singularity. That leads to statement (2.24) concerning p_n, and thus concludes the proof of the Lemma.

From this we arrive at an upper estimate of $E_n(w)$.

Lemma 3. *Let $w = e^{-Q}$ be a weight function satisfying* (2.2) i)-iv). *Then there exists an absolute constant $C>0$ such that*

$$E_n(w) \le C \cdot \exp\big(n \cdot \max_{a>0} F([-a,a])\big). \tag{2.25}$$

Proof. By its definition in (2.12), $E_n(w)$ is a minimum, hence any particular polynomial $p_n \in \mathcal{P}_n$, with leading term x^n, furnishes an upper estimate on its value. Take the polynomial (2.24) of Lemma 2; its weighted norm is

$$\begin{aligned} \|p_n\|_{w^n} &= \|p_n(x)\exp\big(-n\,Q(x)\big)\| \le C\big\|\exp\big(-n\big(U^{\mu_w}(x)+Q(x)\big)\big)\big\| \\ &\le C\exp(-n\,F_w) = C\exp\big(n\ \max_a F([-a,a])\big), \end{aligned} \tag{2.26}$$

taking into account (2.9) (with all over $\mathbb{R}$ by continuity), (2.15) and the fact that $S_w = [-a_w, a_w]$ is an interval in our case.

Finally, let us clarify the connection of Harris' problem and the weighted Chebyshev constants.

Lemma 4. *Let w be a weight function satisfying* (2.2). *Then we have*

$$\begin{aligned} \sup\Big\{\alpha_n \ :\ \exists P_n \in \mathcal{P}_n,\ P_n(x) = \sum_{j=0}^n \alpha_j\, x^j, \\ \text{satisfying} \quad \|P_n\|_{w^n} \le 1\Big\} = 1/E_n(w). \end{aligned} \tag{2.27}$$

Proof. It is obvious that $p_n := 1/\alpha_n \cdot P_n$ has leading coefficient 1, hence it is taken into account in (2.12). In view of the normalization $\|P_n\|_{w^n} = 1$, Lemma 4 follows from (2.12).

3. The weights and coefficients in Harris' problem

Harris applied the following simple reformulation to estimate $c_{m,1}$ (see [H2], [H3]).

Lemma 5 (Harris). *For the constants defined in* (1.1) *we have*

$$c_{m,1} = \max\left\{\alpha_{m-1} : P_{m-1}(x) = \sum_{j=0}^{m-1} \alpha_j x^j \in \mathcal{P}_{m-1}, \right. \\ \left. |P_{m-1}(x)| \le (1+|x|)^m \right\} . \tag{3.1}$$

Proof. Consider any polynomial

$$P(x) = \sum_{j=0}^{m-1} \alpha_j x^j \in \mathcal{P}_{m-1}, \quad |P(x)| \le (1+|x|)^m . \tag{3.2}$$

The "reciprocal" polynomial $Q(x) := x^m\, P\left(\frac{1}{x}\right)$ will satisfy the same estimate, has degree $\le m$, and $Q'(0) = \alpha_{m-1}$ is its derivative at zero. Thus we must have $|\alpha_{m-1}| \le c_{m,1}$.

On the other hand, consider now $p \in \mathcal{P}_m$ with expansion $p(x) = \sum_{j=0}^{m} \beta_j x^j$ satisfying $c_{m,1} = p'(0) = \beta_1$. Take first

$$p_1(x) := \frac{p(x) - p(-x)}{2} = \sum_{\ell=0}^{\left[\frac{m-1}{2}\right]} \beta_{2\ell+1}\, x^{2\ell+1},$$

and then

$$p_2(x) := x^m p_1\left(\frac{1}{x}\right) = \sum_{\ell=0}^{\left[\frac{m-1}{2}\right]} \beta_{2\ell+1}\, x^{m-2\ell-1}.$$

Since $|p(x)| \le (1+|x|)^m$, we also have $|p_1(x)| \le (1+|x|)^m$, and so even $|p_2(x)| \le (1+|x|)^m$. Thus $p_2 \in \mathcal{P}_{m-1}$ satisfies the conditions in (3.1), and as β_1 is the leading coefficient of p_2, we find $c_{m,1} = \beta_1 \le$ right-hand side of (3.1). Now collecting the above (3.1) follows.

Comparing Lemma 4 and Lemma 5 gives

Lemma 6 (Harris). *Let $m \in \mathbb{N}$ be arbitrary, and let us define*

$$w(x) := w_{m,1}(x) := (1+|x|)^{\frac{-m}{m-1}} . \tag{3.3}$$

Then we have

$$c_{m,1} = \frac{1}{E_{m-1}(w)} . \tag{3.4}$$

Continuing the process in Lemma 5, we may try to obtain direct estimates even to higher derivatives, i.e., Harris-Markov constants of higher indices.

Lemma 7. *Let $m \in \mathbb{N}$ be arbitrary, and let us define*

$$w(x) := w_{m,2}(x) := (|x|^m + (1+|x|)^m)^{\frac{-1}{m-2}}. \tag{3.5}$$

Then we have

$$c_{m,2} = \frac{1}{2E_{m-2}(w)} \ . \tag{3.6}$$

Proof. Let first $P \in \mathcal{P}_{m-2}$, $|P(x)| \le w(x)^{-(m-2)}$ with

$$P(x) = \sum_{j=0}^{m-2} \alpha_j \, x^j \ .$$

Similarly as above, it is easily seen that $\frac{1}{2}|\alpha_{m-2}| \le c_{m,2}$, if we consider the reciprocal polynomial $Q(x) = x^m P\left(\frac{1}{x}\right)$, and take into account $Q''(0) = \frac{1}{2}\alpha_{m-2}$. Then Lemma 4 yields $\frac{1}{2}/E_{m-2}(w) \le c_{m,2}$.

Conversely, take now any polynomial with expansion

$$p(x) = \sum_{j=0}^{m} \beta_j \, x^j \in \mathcal{P}_m.$$

Consider

$$p_1(x) = \frac{p(x)+p(-x)}{2} = \sum_{j=0}^{[m/2]} \beta_{2j} \, x^{2j},$$

and

$$p_2(x) = x^m p_1\left(\frac{1}{x}\right) = \beta_0 \, x^m + \beta_2 \, x^{m-2} + \cdots \ \in \mathcal{P}_m \ .$$

If $|p(x)| \le (1+|x|)^m$ was satisfied for all $x \in \mathbb{R}$, then the same must hold for $p_1(x)$ and $p_2(x)$ as well. But this condition also implies $|\beta_0| = |p(0)| \le 1$, hence from the estimate on p_2 it follows that for

$$q_2(x) := p_2(x) - \beta_0 \, x^m = \beta_2 \, x^{m-2} + \beta_4 \, x^{m-4} \cdots \in \mathcal{P}_{m-2}$$

we have

$$|q_2(x)| \le |x|^m + (1+|x|)^m = w^{-(m-2)}(x) \ .$$

Hence, by Lemma 4 again,

$$2p''(0) = \beta_2 \le 1/E_{m-2}(w) \ ,$$

and this being valid to all admissible p, even $c_{m,2} \leq \frac{1}{2}/E_{m-2}(w)$ follows. This concludes the proof of Lemma 7.

One may want to continue this process of exhibiting weighted estimates for $c_{m,k}$. However, the simple estimate $|\beta_0| \leq 1$ gets less nice for $|\beta_1|$, and so on. Finally we can obtain

Proposition 1. *Let $1 \leq k \leq m$, k, $m \in \mathbb{N}$ be arbitrary, and let us consider the weight*

$$\begin{aligned} w(x) &:= w_{m,k}(x) \\ &:= \left\{(1+|x|)^m + \sum_{j=1}^{[k/2]} (k-2j)!\, c_{m,k-2j} |x|^{m-k+2j}\right\}^{\frac{1}{m-k}} . \end{aligned} \tag{3.7}$$

Then we have

$$c_{m,k} \leq \frac{1}{k!\ E_{m-k}(w)} \ . \tag{3.8}$$

Moreover, for $k = 1, 2$ (3.8) holds with equality.

Proof. Similarly to the above.

4. Estimation of $c_{m,1}$

In view of Lemma 1, Lemma 3 and Lemma 6, to estimate $c_{m,1}$ we are entitled to estimate $F([-a,a])$, defined in (2.17). By differentiation (or reflecting back to (2.10) and to the fact that here $S_w = [-a_w, a_w]$ is an interval), we are to solve the equation (2.10) for the value of $a = a_w$. Note that here

$$Q(x) = \frac{m}{m-1} \log(1+|x|) \tag{4.1}$$

satisfies all the conditions (2.2) i)-iv). So we must have

$$\begin{aligned} 1 &= \frac{2}{\pi} \int_0^1 \frac{at \cdot \frac{m}{m-1}}{1+at} \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{2m}{\pi(m-1)} \int_0^1 \left(1 - \frac{1}{1+at}\right) \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{2m}{\pi(m-1)} \left\{\frac{\pi}{2} - \int_0^1 \frac{dt}{(1+at)\sqrt{1-t^2}}\right\}, \end{aligned} \tag{4.2}$$

that is

$$\frac{\pi}{2m} = \int_0^1 \frac{dt}{(1+at)\sqrt{1-t^2}} =: L(a) \ . \tag{4.3}$$

Obviously, $L(a)$ is strictly decreasing. By calculus, for $a>1$ we have

$$L(a)=\frac{\log(a+\sqrt{a^2-1})}{\sqrt{a^2-1}}=\frac{\operatorname{arch}(a)}{\sqrt{a^2-1}} \qquad (a>1)\ , \tag{4.4}$$

hence for $m\ge 2$, i.e., $\frac{\pi}{2m}<1=L(1)$ we can apply (4.4). Observe that $L(a)$ decreases monotonically to 0 when $a\to\infty$, hence for all m there exists a unique solution of equation (4.3). If L^{-1} is the inverse function of L, then actually we have

$$a_w=L^{-1}\left(\frac{\pi}{2m}\right) \qquad (m\ge 2)\ . \tag{4.5}$$

Lemma 8. *For the weight in* (3.3) *the support interval is* $S_w=[-a_w,a_w]$ *with*

$$\frac{2}{\pi}m\log\big(\frac{4}{\pi}m\log m\big)<a_w<\frac{2}{\pi}m\left\{\log\big(\frac{4}{\pi}m\log m\big)+\frac{3\log\log m}{\log m}\right\}$$

for all $m>20$.

Proof. Since the function $L(a)$, given by (4.3)–(4.4), is strictly decreasing, it suffices to prove that for

$$\begin{aligned} a_w^- &:= \frac{2}{\pi}m\log\left(\frac{4}{\pi}m\log m\right), \\ a_w^+ &:= \frac{2}{\pi}m\left\{\log\left(\frac{4}{\pi}m\log m\right)+\frac{3\log\log m}{\log m}\right\} \end{aligned} \tag{4.6}$$

the function $L(a)$ attains values

$$L(a_w^-)>\frac{\pi}{2m}>L(a_w^+)\ . \tag{4.7}$$

First we estimate $L(a)$ from above:

$$\begin{aligned} L(a) &= \frac{\log(a+\sqrt{a^2-1})}{\sqrt{a^2-1}}=\frac{\log(2a)+\log\left(1-\frac{a-\sqrt{a^2-1}}{2a}\right)}{a\sqrt{1-\frac{1}{a^2}}} \\ &< \frac{\log(2a)}{a}\cdot\frac{1}{\sqrt{1-\frac{1}{a^2}}}<\frac{\log(2a)}{a}\left(1+\frac{1}{a^2}\right) \qquad (a\ge 2)\ . \end{aligned} \tag{4.8}$$

On the other hand,

$$\begin{aligned} L(a) &= \frac{\log(2a)}{a}\left\{1+\frac{1-\sqrt{1-\frac{1}{a^2}}}{\sqrt{1-\frac{1}{a^2}}}\right\}+\frac{\log\left(1-\frac{1-\sqrt{1-\frac{1}{a^2}}}{2}\right)}{a\sqrt{1-\frac{1}{a^2}}} \\ &= \frac{\log(2a)}{a}\left\{1+\frac{1}{\sqrt{1-\frac{1}{a^2}}}\left[2b+\frac{\log(1-b)}{\log a}\right]\right\}, \end{aligned} \tag{4.9}$$

where

$$b := \frac{1-\sqrt{1-\frac{1}{a^2}}}{2} \in \left(0, \frac{1}{2}\right) .$$

Now we can use the concavity of the log function in the form of the inequality

$$\log(1-b) > \frac{\log\left(\frac{1}{2}\right)}{\frac{1}{2}} b = -2\log 2 \cdot b \qquad \left(0 < b < \frac{1}{2}\right) ,$$

to deduce from (4.9) the lower estimate

$$L(a) > \frac{\log(2a)}{a} \qquad (a \geq 2) , \tag{4.10}$$

and thus, in view of (4.8), also

$$\frac{\log(2a)}{a} < L(a) < \frac{\log(2a)}{a}\left(1+\frac{1}{a^2}\right) \qquad (a \geq 2). \tag{4.11}$$

Let us apply first the lower estimate (4.10) to obtain

$$\begin{aligned} L(a_w^-) \quad &> \quad \frac{\log(2a_w^-)}{a_w^-} = \frac{\log\left(\frac{4}{\pi}\, m \log\left(\frac{4}{\pi}\, m\log m\right)\right)}{\frac{2}{\pi}\, m \log\left(\frac{4}{\pi}\, m \log m\right)} \\ &= \quad \frac{\pi}{2m}\frac{\log\left(\frac{4}{\pi}\, m \left(\log m + \log\left(\frac{4}{\pi}\log m\right)\right)\right)}{\log\left(\frac{4}{\pi}\, m\log m\right)} > \frac{\pi}{2m} . \end{aligned} \tag{4.12}$$

Next we use the inequalities $\log\frac{4}{\pi} < \frac{1}{4}$, $\log\log m/\log m < 0.4$ $(m \geq 21)$ and $\log(1+x) < x \quad (x > 0)$ to obtain

$$\begin{aligned} \frac{\log(2a_w^+)}{a_w^+} &= \frac{\log\left\{\frac{4}{\pi}\, m\left[\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}\right]\right\}}{\frac{2}{\pi}\, m\left\{\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}\right\}} \\ &< \frac{\pi}{2m}\ \frac{\log\left(\frac{4}{\pi}\, m\left[\log m + \log\log m + \frac{3}{2}\right]\right)}{\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}} \\ &= \frac{\pi}{2m}\left\{1 + \frac{\log\left(1 + \frac{\log\log m + 3/2}{\log m}\right) - \frac{3\log\log m}{\log m}}{\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}}\right\} \\ &< \frac{\pi}{2m}\left\{1 + \frac{\frac{\log\log m + 3/2}{\log m} - \frac{3\log\log m}{\log m}}{\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}}\right\} \\ &= \frac{\pi}{2m}\left\{1 + \frac{3 - 4\log\log m}{2\log m\left(\log\left(\frac{4}{\pi}\, m\log m\right) + \frac{3\log\log m}{\log m}\right)}\right\} . \end{aligned}$$

Since here the numerator is negative, we can estimate further and get

$$\frac{\log(2a_w^+)}{a_w^+} < \frac{\pi}{2m}\left\{1 - \frac{4\log\log m - 3}{12\log^2 m}\right\} < \frac{\pi}{2m}\left(1 - \frac{1}{12\log^2 m}\right) . \tag{4.13}$$

Next we combine the trivial estimate $a_w^+ > \frac{2}{\pi} m \log m > 10 \log m$ with (4.8) and (4.13) to obtain

$$L(a_w^+) < \frac{\pi}{2m}\left(1 - \frac{1}{12\log^2 m}\right)\left(1 + \frac{1}{a_w^{+2}}\right) < \frac{\pi}{2m} . \tag{4.14}$$

Comparing (4.12) and (4.14) with (4.7), we conclude the proof of Lemma 8.

Lemma 9. *For any parameter value $a > 3$ we have*

$$I(a) := -\int_0^1 \frac{\log\left(\frac{at}{1+at}\right)}{\sqrt{1-t^2}}\, dt = L(a) + \frac{1}{a + \vartheta(a)} , \tag{4.15}$$

where $\vartheta(a)$ depends only on a and satisfies $0 < \vartheta(a) < 1$.

Proof. After differentiating under the integral sign we get

$$\frac{dI(a)}{da} = -\int_0^1 \frac{1}{a(1+at)\sqrt{1-t^2}}\, dt = -\frac{1}{a} L(a) . \tag{4.16}$$

Moreover, since $I(\infty) = 0$, we find

$$I(a) = \int_a^\infty \frac{L(\alpha)}{\alpha}\, d\alpha . \tag{4.17}$$

For a lower estimate we make use of (4.11), which yields

$$\begin{aligned} I(a) &> \int_a^\infty \frac{\log(2\alpha)}{\alpha}\, d\alpha = \frac{\log(2a)}{a} + \frac{1}{a} \\ &= \frac{\log(2a)}{a}\left(1 + \frac{1}{a^2}\right) + \frac{1}{a} - \frac{\log 2a}{a^3} > L(a) + \frac{1}{a+1} \end{aligned} \tag{4.18}$$

since for $a > 3$,

$$\log 2a < \log 2 + \frac{\log 3}{2}(a-1) < a - 1 < \frac{a^2}{a+1}.$$

To derive an upper estimate, we first insert the exact value (4.4) of $L(a)$ to obtain by substitution in the integrals

$$\begin{aligned} I(a) &= \int_a^\infty \frac{\operatorname{arch}(\alpha)}{\alpha\sqrt{\alpha^2 - 1}}\, d\alpha = \int_{\operatorname{arch}(a)} \frac{u}{\cosh\ u}\, du \\ &= \int_{\log b}^\infty \frac{2u\, e^u}{e^{2u} + 1}\, du \\ &= 2\int_b^\infty \frac{\log t}{t^2 + 1}\, dt \qquad (b := a + \sqrt{a^2 - 1}) . \end{aligned} \tag{4.19}$$

By the trivial estimate $(t^2+1)^{-1} < t^{-2}$ and integration,

$$\int_b^\infty \frac{\log t}{t^2+1}\,dt < \int_b^\infty \frac{\log t}{t^2}\,dt = \frac{\log b}{b} + \frac{1}{b}, \tag{4.20}$$

and comparing (4.19) and (4.20) we can estimate further:

$$\begin{aligned} I(a) &< \frac{2\log b}{b} + \frac{2}{b} = \frac{\log b}{\sqrt{a^2-1}}\,\frac{2\sqrt{a^2-1}}{b} + \frac{2a}{b}\cdot\frac{1}{a} \\ &= L(a)\left(1 - \frac{a-\sqrt{a^2-1}}{b}\right) + \frac{1}{a}\left(1 + \frac{a-\sqrt{a^2-1}}{b}\right) \\ &< L(a) + \frac{1}{a} \end{aligned} \tag{4.21}$$

Now (4.21) together with (4.18) yields (4.15).

Lemma 10. *Let w and Q be as in* (3.3) *and* (4.1)*, let $\varphi(a)$ be defined according to* (2.17)*, and let a_w be the unique root of the equation in* (2.10)*, i.e., $S_w = [-a_w, a_w]$ is the support of the equilibrium measure μ_w of w. Suppose that $m > 2$ is a natural number, and denote L and I as in* (4.3) *and* (4.15)*, respectively. Then for $a > 3$ we have*

$$\varphi(a) = \frac{-1}{m-1}\log\left(\frac{a}{2}\right) - \frac{2}{\pi}\,\frac{m}{m-1}\left(L(a) + \frac{1}{1+a} + \frac{\vartheta(a)}{a^2}\right), \qquad 0 < \vartheta(a) < 1.4, \tag{4.22}$$

and in particular

$$\varphi(a_w) = \frac{-1}{m-1}\log\left(\frac{e\,a_w}{2}\right) - \frac{2}{\pi}\,\frac{m}{m-1}\left(\frac{1}{1+a_w} + \frac{\vartheta_w}{a_w^2}\right), \qquad 0 < \vartheta_w < 1.4\,. \tag{4.23}$$

Proof. Starting from (2.17) we write

$$\begin{aligned} \varphi(a) &= \log\left(\frac{a}{2}\right) - \frac{2}{\pi}\int_0^1 \frac{\frac{m}{m-1}\log(1+at)}{\sqrt{1-t^2}}\,dt \\ &= \log\frac{a}{2} - \frac{2}{\pi}\,\frac{m}{m-1}\int_0^1 \frac{\log(at)}{\sqrt{1-t^2}}\,dt + \frac{2}{\pi}\,\frac{m}{m-1}\int_0^1 \frac{\log\left(\frac{at}{1+at}\right)}{\sqrt{1-t^2}}\,dt \\ &= \left(1 - \frac{m}{m-1}\right)\log\left(\frac{a}{2}\right) - \frac{2}{\pi}\,\frac{m}{m-1}\,I(a)\ , \end{aligned}$$

which yields (4.22) by an application of Lemma 9. Moreover, a_w is defined to solve (2.10), that is (4.3), hence substituting $L(a_w) = \frac{\pi}{2m}$ into (4.22) leads to (4.23).

Theorem 2. *Let $m > 20$ and w, Q be as in* (3.3), (4.1). *For this weight the modified Robin constant* (2.7) *satisfies*

$$F_w = \frac{1}{m-1}\left\{\log\left(\frac{e}{\pi}m\log m\right) + \frac{13}{4}\vartheta_3\frac{\log\log m}{\log m}\right\} \quad (0 \le \vartheta_3 \le 1), \tag{4.24}$$

and we have with some absolute constant c_0

$$\frac{1}{3}\,\frac{1}{m\log m} < \frac{\pi e^{-\frac{13\log\log m}{4\log m}}}{em\log m} \le E_{m-1}(w) \le \frac{c_0}{m\log m}\ . \tag{4.25}$$

Proof. As established above, with $\varphi(a)$ defined in (2.17) and a_w being the solution of (2.10) or, equivalently, (4.3), we have

$$-F_w = F(S_w) = F([-a_w, a_w]) = \varphi(a_w)\ . \tag{4.26}$$

Here we can apply (4.23), the inequalities $m \ge 21$, $a_w > \frac{2}{\pi}m\log m$ and Lemma 8, to obtain

$$\begin{aligned} 0 &> \varphi(a_w) + \frac{1}{m-1}\left(\log\left(\frac{e}{2}\right) + \log a_w\right) \\ &> -\frac{2}{\pi}\,\frac{m}{m-1}\,\frac{1.04}{\left(\frac{2}{\pi}\,m\log m\right)} = -\frac{1.04}{(m-1)\log m}\ . \end{aligned} \tag{4.27}$$

To compute the main term $\log a_w$ here we can use Lemma 8 again with $m \ge 21$ and the monotonicity of $\log\log m$, $\log m$ to derive

$$\begin{aligned} &\log\left(\tfrac{2}{\pi}\,m\log m\right) \\ &\quad< \log a_w < \log\left(\frac{2}{\pi}\,m\left(\log\left(\frac{4}{\pi}\,m\log m\right) + \frac{3\log\log m}{\log m}\right)\right) \\ &\quad= \log\left(\frac{2}{\pi}\,m\log m\right) + \log\left(1 + \frac{\log\left(\frac{4}{\pi}\log m\right)}{\log m} + \frac{3\log\log m}{\log^2 m}\right) \\ &\quad< \log\left(\frac{2}{\pi}\,m\log m\right) + \frac{\log\frac{4}{\pi} + \log\log m}{\log m} + \frac{3\log\log m}{\log^2 m} \\ &\quad< \log\left(\frac{2}{\pi}\,m\log m\right) + \frac{9\log\log m}{4\log m}\ . \end{aligned} \tag{4.28}$$

Collecting (4.26), (4.27) and (4.28), we obtain for $m > 20$ and with some appropriate $0 < \vartheta_1, \vartheta_2, \vartheta_3 < 1$ the relation

$$\begin{aligned} -F_w &= \frac{-1}{m-1}\left\{\log\left(\frac{e}{\pi}\,m\log m\right) + \frac{9}{4}\vartheta_1\frac{\log\log m}{\log m} + \frac{1.04\vartheta_2}{\log m}\right\} \\ &= \frac{-1}{m-1}\left\{\log\left(\frac{e}{\pi}\,m\log m\right) + \frac{13}{4}\vartheta_3\frac{\log\log m}{\log m}\right\}\ , \end{aligned} \tag{4.29}$$

and (4.24) is seen to follow. Now (4.26), (4.24) and Lemma 3 imply

$$E_{m-1}(w) \le C \exp\big(-(m-1)F_w\big) \le \frac{c_0}{m\log m} \ ,$$

as it is stated on the right side of (4.25), while applying Lemma 1 we obtain

$$E_{m-1}(w) \ge \frac{\pi}{e\, m\log m} \cdot e^{-\frac{13}{4}\frac{\log\log m}{\log m}} > \frac{1}{3}\,\frac{1}{m\log m} \ .$$

Proof of Theorem 1. On applying Lemma 6, the estimates in (4.25) entail

$$c_1 m\log m < c_{m,1} < 3\, m\log m$$

for all $m > 20$. From the computed values of $c_{m,1}$, see Table I and Table II in [H2], the same inequalities follow directly for $2 \le m \le 20$, too.

Corollary 1. *We have $c_{m,k} \le (3m\log m)^k$ for all $m, k \in \mathbb{N}$.*

Corollary 2. *We have*

$$\overline{\lim_{m\to\infty}} \frac{c_{m,1}}{(m\log m)} \le \frac{e}{\pi} \ ,$$

and for k fixed or increasing slowly in the sense $k = o\left(\frac{\log m}{\log\log m}\right)$ we have

$$\overline{\lim_{m\to\infty}} \frac{c_{m,k}}{(m\log m)^k} \le \left(\frac{e}{\pi}\right)^k \ .$$

Corollary 3. *There exists an absolute constant C_0 such that*

$$c_{m,k} \le C_0 m^k \log^k m.$$

Proof. Corollaries 1 and 2 follow from (4.25) and iteration for $k > 1$. Corollary 3 follows for $m > m_0$ from Corollary 2 and iteration, because the constant e/π is less than 1. On the other hand, for $m < m_0$ we also have $k < m_0$, and Corollary 1 suffices with $C_0 = 3^{m_0}$.

5. A direct approach to $c_{m,2}$

Instead of iterating the result for the first derivative, here we apply a direct approach to calculate $c_{m,2}$. That is, we use Lemma 7 and we deal with the weight

(5.1) $$w(x) := w_{m,2}(x) := (|x|^m + (1+|x|^m))^{\frac{-1}{m-2}} \ .$$

Correspondingly, the negative logarithm of the weight is

$$Q(x) = \frac{1}{m-2} \log\left(|x|^m + (1+|x|)^m\right) \ . \tag{5.2}$$

This is a function satisfying conditions (2.2) i)-iv), hence §2 applies and we are to solve (2.10). This becomes in our case

$$\begin{aligned} 1 &= \frac{2}{\pi}\int_0^1 \frac{at}{\sqrt{1-t^2}} \frac{\frac{m}{m-2}\left((at)^{m-1} + (1+at)^{m-1}\right)}{(at)^m + (1+at)^m}\, dt \\ &= \frac{m}{m-2}\left\{1 - \frac{2}{\pi}\int_0^1 \frac{(1+at)^{m-1}}{(at)^m + (1+at)^m} \frac{dt}{\sqrt{1-t^2}}\right\} \ , \end{aligned}$$

that is,

$$\frac{\pi}{m} = \int_0^1 \frac{(1+at)^{m-1}}{(at)^m + (1+at)^m} \frac{dt}{\sqrt{1-t^2}} =: K_m(a) \ . \tag{5.3}$$

First we compute an asymptotically precise value of $K_m(a)$.

Lemma 11. *With the function $K_m(a)$ defined in (5.3), and for any $m \in \mathbb{N}$, $2 < m < a$, we have*

$$\frac{-0.3}{a} < K_m(a) - \frac{\log(am)}{2a} < \frac{1.3}{a} \ . \tag{5.4}$$

Proof. Let us denote

$$f(x) := f_m(x) := \frac{(1+x)^{m-1}}{x^m + (1+x)^m} = \frac{1}{1+x} \frac{1}{1 + \left(\frac{x}{1+x}\right)^m} \ . \tag{5.5}$$

It is immediate that f is a decreasing function on $[0, \infty)$. First we apply partial integration and the inequality

$$0 \le \arcsin t - t \le (\pi/2 - 1)t \quad (0 \le t \le 1)$$

to get

$$\begin{aligned} K_m(a) &= \int_0^1 \frac{f(at)\, dt}{\sqrt{1-t^2}} = \left[f(at) \arcsin t\right]_0^1 - a \cdot \int_0^1 f'(at) \arcsin t\, dt \\ &= \frac{\pi}{2} f(a) - a\int_0^1 t \cdot f'(at)\, dt - a\int_0^1 \left(\frac{\pi}{2} - 1\right)\theta(t) \cdot t^3 f'(at)\, dt \\ &= \frac{\pi}{2} f(a) - f(a) + \int_0^1 f(at)\, dt - \theta_4 \cdot \left(\frac{\pi}{2} - 1\right) \cdot a \int_0^1 t^3 f'(at)\, dt \ , \end{aligned} \tag{5.6}$$

where $0 \le \theta(t) \le 1$ depends only on t and $0 \le \theta_4 \le 1$ is a quantity depending on m and a. The last term is easy to handle by partial integration. Namely,

$$\begin{aligned} a\int_0^1 t^3 f'(at)\,dt &= \left[t^3 f(at)\right]_0^1 - \int_0^1 3t^2 f(at)\,dt \\ &= f(a) - 3\int_0^1 t^2 f(at)\,dt \end{aligned} \tag{5.7}$$

and combining (5.6) and (5.7) leads to

$$K_m(a) - \int_0^1 f(at)\,dt = \left(\frac{\pi}{2} - 1\right)\left\{(1-\theta_4)f(a) + 3\theta_4 \int_0^1 t^2 f(at)\,dt\right\} . \tag{5.8}$$

Obviously this expression is positive. Hence using $f(x) < \frac{1}{1+x}$ and also

$$\int_0^1 t^2 f(at)\,dt < \frac{1}{a^3}\int_0^a \frac{u^2}{1+u}du = \frac{a^2 + 2\log(a+1) - 2a}{2a^3} < \frac{1}{2(a+1)} \quad (a \ge 3)$$

we obtain

$$0 \le K_m(a) - \int_0^1 f(at)dt < \left(\frac{\pi}{2} - 1\right)\frac{3}{2(a+1)} < \frac{0.9}{a+1} . \tag{5.9}$$

The main term for evaluation of $K_m(a)$ will be the integral $\int_0^1 f(at)\,dt$, i.e., $\frac{1}{a}\int_0^a f(x)dx$. To compute this, we first prove

Lemma 12. With the function (5.5) and for $m \in \mathbb{N}$, $2 < m < a$, we have

$$-0.3 < \int_0^a f_m(x)dx - \frac{1}{2}\log\big((a+1)m\big) < 0.35 .$$

Proof. First we make a few substitutions in the integral. Namely $t = 1 + x$, $\tau = \log t$, and $u = \tau - \log m$ lead to

$$\begin{aligned} J &:= J_m(a) := \int_1^{a+1} \frac{1}{t}\,\frac{dt}{1 + \left(1 - \frac{1}{t}\right)^m} \\ &= \int_0^{\log(a+1)} \frac{d\tau}{1 + (1 - e^{-\tau})^m} \\ &= \int_{-\log m}^{\log\left(\frac{a+1}{m}\right)} \frac{du}{1 + (1 - e^{-u-\log m})^m} \\ &= \int_0^{\log m} \frac{dv}{1 + \left(1 - \frac{e^v}{m}\right)^m} + \int_0^{\log\left(\frac{a+1}{m}\right)} \frac{du}{1 + \left(1 - \frac{e^{-u}}{m}\right)^m} \\ &=: J' + J'' , \end{aligned} \tag{5.10}$$

say. For the first integrand we use $\log(1-y) < -y$ $(y := e^v/m < 1)$ to get

$$1 > \frac{1}{1+\left(1-\frac{e^v}{m}\right)^m} > \frac{1}{1+\exp\left(-e^v\right)} > 1-\exp(-e^v). \tag{5.11}$$

Applying (5.11) in J' leads to

$$\begin{aligned} \log m \quad &> \quad J' > \log m - \int_0^{\log m} \exp(-e^v)\,dv \\ &> \quad \log m - \int_0^{\infty} \exp(-e^v)\,dv, \end{aligned} \tag{5.12}$$

and the last integral is easily estimated:

$$\int_0^\infty \exp(-e^v)dv = \int_1^\infty e^{-s}\frac{ds}{s} \le \sqrt{\int_1^\infty \frac{ds}{s^2}\int_1^\infty e^{-2s}ds} = \frac{1}{\sqrt{2}\cdot e}\ . \tag{5.13}$$

Next we apply Bernoulli's Inequality $q^m \ge 1+m(q-1) \quad (m\in\mathbb{N},\ q>0)$ with $q := 1-e^{-u}/m$ to obtain

$$\frac{1}{2} < \frac{1}{1+\left(1-\frac{e^{-u}}{m}\right)^m} < \frac{1}{1+(1-e^{-u})} = \frac{1}{2} + \frac{e^{-u}}{2(2-e^{-u})}\ . \tag{5.14}$$

Using (5.14) in J'' and extending the second integral to ∞ yields

$$\begin{aligned} \frac{1}{2}\log\left(\frac{a+1}{m}\right) \quad &< \quad J'' < \int_0^{\log\left(\frac{a+1}{m}\right)} \frac{du}{2} + \int_0^\infty \frac{e^{-u}\,du}{2(2-e^{-u})} \\ &= \quad \frac{1}{2}\log\left(\frac{a+1}{m}\right) + \frac{\log 2}{2}\ . \end{aligned} \tag{5.15}$$

Collecting (5.10), (5.12), (5.13) and (5.15) we get

$$-0.3 < -\frac{1}{\sqrt{2}\,e} < J - \frac{1}{2}\log\big((a+1)m\big) < \frac{\log 2}{2} < 0.35\ ,$$

proving Lemma 12.

Now, to finish the proof of Lemma 11 we compare (5.9), Lemma 12 and the elementary inequality $0 \le \log(a+1) - \log a < \frac{1}{a}$ and deduce (5.4) with an upper error

$$\frac{1}{2a^2} + \frac{0.35}{a} + \frac{0.9}{a+1} < \frac{1.3}{a}.$$

Next we estimate the solution a_w of (5.3).

Lemma 13. *Let a_w be the unique positive root of* (5.3). *We then have, for any $m \geq 21$,*

$$\frac{1}{\pi} m \log\left(\frac{m}{\log m}\right) < a_w < \frac{1}{\pi} m \log(m \log^2 m) \ . \tag{5.16}$$

Proof. On comparing the first formula of (5.6) and the expression (5.5) we easily see that $f_m(x)$, and hence also $K_m(a)$, is strictly decreasing. Thus (5.3) has exactly one solution in view of

$$K_m(0) = \frac{\pi}{2} \geq \frac{\pi}{m} > 0 = \lim_{a\to\infty} K_m(a).$$

Thus, similarly to the argument in the proof of Lemma 8, we are to prove

$$K_m(a_w^-) > \frac{\pi}{m} > K_m(a_w^+) \tag{5.17}$$

with

$$a_w^- = \frac{1}{\pi} m \log\left(\frac{m}{\log m}\right), \quad a_w^+ = \frac{1}{\pi} m \log(m \log^2 m) \ . \tag{5.18}$$

Since $m \geq 21$, from the first inequality of (5.4) in Lemma 11 we obtain

$$\begin{aligned} &K_m(a_w^-) \\ &> \frac{\log\left(\frac{1}{\pi} m^2 \log\left(\frac{m}{\log m}\right)\right) - 0.6}{2\frac{1}{\pi} m \log\left(\frac{m}{\log m}\right)} \\ &= \frac{\pi}{2m}\left\{2 + \frac{2\log\log m + \log\log\left(\frac{m}{\log m}\right) - \log\pi - 0.6}{\log\left(\frac{m}{\log m}\right)}\right\} > \frac{\pi}{m} \ . \end{aligned} \tag{5.19}$$

On the other hand, the second inequality of (5.4) in Lemma 11 yields

$$\begin{aligned} K_m(a_w^+) &< \frac{\log\left(\frac{1}{\pi} m^2 \log(m\log^2 m)\right) + 2.6}{2\frac{1}{\pi} m \log(m \log^2 m)} \\ &= \frac{\pi}{2m}\left\{2 + \frac{-4\log\log m + \log\log(m\log^2 m) + 2.6 - \log\pi}{\log(m\log^2 m)}\right\} \\ &< \frac{\pi}{2m}\left\{2 + \frac{2.6 + \log\left(\frac{3}{\pi}\right) - 3\log\log m}{\log(m\log^2 m)}\right\} < \frac{\pi}{m} \ . \end{aligned} \tag{5.20}$$

Collecting (5.19) and (5.20) gives (5.17) with a_w^- and a_w^+ defined in (5.18), and thus Lemma 13 is seen to follow.

Lemma 14. *Let w and Q be as in* (5.1)–(5.2), $20 < m \in \mathbb{N}$ *and $a > m$ be arbitrary, let the function $\varphi_w = \varphi$ be as in* (2.17) *and let a_w be the unique* *root of the equation* (2.10)*, or, equivalently,* (5.3)*. Denote by I the expression in* (4.15)*. Then we have*

$$\varphi(a) = \frac{-2}{m-2}\left\{ \log\frac{a}{2} + \frac{m}{\pi} I(a) + \frac{1}{\pi}\int_0^1 \log\left(1+\left(\frac{at}{1+at}\right)^m\right)\frac{dt}{\sqrt{1-t^2}}\right\}. \tag{5.21}$$

Moreover, for the maximum of φ we have

$$\max\varphi = \varphi(a_w) = \frac{-2}{m-2}\left\{\log\left(\frac{e}{\sqrt{2}\,\pi} m\log m\right) + o(1)\right\} \tag{5.22}$$

with $o(1) \to 0 \quad (m\to\infty)$.

Proof. By definitions (2.17), (4.15) and (5.2) we can write

$$\begin{aligned}
\varphi(a) &= \log\frac{a}{2} - \frac{2}{\pi}\int_0^1 \frac{1}{m-2}\log[(at)^m + (1+at)^m]\frac{dt}{\sqrt{1-t^2}} \\
&= \log\frac{a}{2} - \frac{2}{\pi(m-2)} \\
&\quad\times \int_0^1 \frac{m\big[\log(at) - \log\left(\frac{at}{1+at}\right)\big] + \log\left[1+\left(\frac{at}{1+at}\right)^m\right]}{\sqrt{1-t^2}}\,dt \\
&= \log\frac{a}{2} - \frac{2m}{\pi(m-2)}\left\{\frac{\pi}{2}\log\left(\frac{a}{2}\right) + I(a)\right\} \\
&\quad - \frac{2}{\pi(m-2)}\int_0^1 \frac{\log\left(1+\left(\frac{at}{1+at}\right)^m\right)}{\sqrt{1-t^2}}\,dt,
\end{aligned}$$

and (5.21) follows. By definition, ϕ_w achieves its maximum at a_w. Moreover, we have already computed the approximate value of a_w in Lemma 13 and the value of $I(a)$ in Lemma 9 and (4.11). Note that here a_w is the maximum point for w in (5.1), not in (3.3). Thus, in contrast to (4.3), here we do not have the precise value of $L(a_w)$. But the above quoted estimates together with (5.3) and (5.4), and Lemma 11 are sufficient to obtain

$$\begin{aligned}
\log a_w + \frac{m}{\pi}I(a_w) &= \log\left(\frac{1}{\pi}m\log m\right) + o(1) + \frac{m}{\pi}L(a_w) + o(1) \\
&= \log\left(\frac{1}{\pi}m\log m\right) + \frac{m}{\pi}\frac{\log(a_w)}{a_w} + o(1)
\end{aligned} \tag{5.23}$$

and

$$\frac{m}{\pi}\frac{\log a_w}{a_w} = \frac{m}{\pi}\frac{\log(a_w\, m)}{2a_w} + o(1) = \frac{m}{\pi}K_m(a_w) + o(1) = 1 + o(1)\,. \tag{5.24}$$

Let us choose now a parameter $\mu = \mu(m) = o(1)$ $(m \to \infty)$ so that $m/a_w < \mu$. Then we have

$$\begin{aligned} \frac{\pi}{2}\log 2 \quad &> \quad \int_0^1 \log\left(1 + \left(\frac{a_w t}{1 + a_w\, t}\right)^m\right) \frac{dt}{\sqrt{1-t^2}} > \int_\mu^1 \\ &> \quad \log\left(1 + \left(\frac{a_w\, \mu}{1 + a_w\, \mu}\right)^m\right) \cdot \left(\frac{\pi}{2} - \arcsin \mu\right) \end{aligned} \tag{5.25}$$

and by Bernoulli's Inequality,

$$\begin{aligned} \log\left(1 + \left(\frac{a_w\, \mu}{1 + a_w\, \mu}\right)^m\right) \quad &> \quad \log\left(1 + \left(1 - \frac{m}{a_w\, \mu}\right)\right) \\ &= \quad \log\left(2 - \frac{m}{a_w\, \mu}\right) . \end{aligned} \tag{5.26}$$

Choosing e.g. $\mu := 1/\sqrt{\log m}$,

$$\mu(m) = o(1) \quad \text{and} \quad \frac{m}{a_w\, \mu} = o(1)$$

are satisfied simultaneously, and collecting (5.25) and (5.26) gives

$$\int_0^1 \log\left(1 + \left(\frac{a_w\, t}{1 + a_w\, t}\right)^m\right) \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}\log 2 - o(1) \ . \tag{5.27}$$

Combining (5.21), (5.23), (5.24) and (5.27) now leads to (5.22).

Theorem 3. *Let w and Q be as in* (5.1)–(5.2). *For this weight the modified Robin constant* (2.7) *satisfies*

$$F_w = \frac{2}{m-2}\left\{\log\left(\frac{e}{\sqrt{2}\,\pi}\, m \log m\right) + o(1)\right\} \qquad (m \to \infty) \ . \tag{5.28}$$

Moreover, with some absolute constant c_2 we have

$$\frac{1}{18 m^2 \log^2 m} \le E_{m-2} \le \frac{c_2}{m^2 \log^2 m} \ . \tag{5.29}$$

Proof. Referring to the general formulae (2.10) and (2.17), we again obtain similarly to (4.26) that $-F_w = \varphi(a_w)$ with the particular weight given by (5.1)–(5.2). Hence (5.22) of Lemma 14 implies (5.28). Now an application of Lemma 3 implies the upper estimate in (5.29), while the lower estimate follows by Lemma 1 and (5.28) at least for $m > m_0$. The same inequality holds for $m \le 20$ as well, according to the numerical values given by Harris [H2]. Finally, a detailed calculation, very much like the one for $c_{m,1}$ in particular in Theorem 2, can cover the case $m > 20$ fully.

However, (5.29) does not need these detailed computations, not even for the lower estimate, if we take into account the iterative property of $c_{m,k}$. As noted following Theorem 1, we have $c_{m,k} \le c_{m,1}^k$ and thus, by Lemmas 6 and 7, we obtain $2E_{m-2}(w) = 1/c_{m,2} \ge c_{m,1}^{-2}$. Hence, recalling (1.9) of Theorem 1 concludes the proof of (5.29).

Similarly to Corollary 2, we now conclude

Corollary 4. *We have*

$$\overline{\lim_{m\to\infty}} \frac{c_{m,2}}{m^2 \log^2 m} \le \left(\frac{e}{2\pi}\right)^2 . \tag{5.30}$$

Moreover, for k fixed or increasing slowly enough, we also have

$$\overline{\lim_{m\to\infty}} \frac{c_{m,k}}{m^k \log^k m} \le \frac{1}{4^{\left[\frac{k}{2}\right]}} \left(\frac{e}{\pi}\right)^k . \tag{5.31}$$

Note the improvement $4^{-\left[\frac{k}{2}\right]}$ compared to Corollary 2.

Proof. The relation (5.30) of Corollary 4 is a consequence of the first inequality of (2.13) (with $n = m-2$) and (5.28) from Theorem 3, in view of Lemma 7, (3.6). As above, iteration yields (5.31) if here we apply $\left[\frac{k}{2}\right]$ iterations of (5.30) and $k-2\left[\frac{k}{2}\right]$ (i.e., 1 or 0, according to the parity) iteration of (1.9) (with $k=1$).

Moreover, from the upper estimate in (5.29) we also get

Corollary 5. *With some $c_3 > 0$ we have $c_{m,2} \ge c_3 m^2 \log^2 m \quad (m \in \mathbb{N})$.*

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