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EXTREMAL PROBLEMS AND A DUALITY PHENOMENON

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1. INTRODUCTION AND RESULTS

The extremal problem of calculating

$$\alpha(k):=\max\{a:\exists f\geq 0, f(x)=1+a\cos x+b\cos kx,\ a,b\in\mathbb{R}\}$$

arise in certain problems of analytic number theory. In another work [5] the calculation of

$$\gamma(k):=\sup\{a:\exists f\geq 0, f(x)=1+a\cos x+\sum_{n=k+1}^{N} a_n\cos nx, N\in\mathbb{N}, a_n\in\mathbb{R}\ (n\in\mathbb{N})\}$$

has an important role. The latter problem reminds of the following extremal problem of Fejér: determine

$$\lambda(k):=\max\{a_1:\exists f\geq 0, f(x)=1+\sum_{n=1}^{k} a_n\cos nx, a_n\in\mathbb{R}\ (n\leq k)\}\ .$$

Fejér proved /see [2] or [3] I pp. 869--870/

(1) $\lambda(k)=2\cos\frac{\pi}{k+2}$,

and the values of the other quantities are /cf. Proposition 1 and [4]/

$$\alpha(k)=\frac{1}{\cos\frac{\pi}{2k}}\ ,\quad \gamma(k)=\frac{1}{\cos\frac{\pi}{k+2}}\ .$$

As a common generalization we consider for any $H\subset\mathbb{N}_2:=\{2,3,\dots\}$

(2) $\beta(H):=\sup\{a:\exists f\geq 0, f\in T, f(x)=1+a\cos x+\sum_{n\in H} a_n\cos nx, a_n\in\mathbb{R}(n\in H)\}$,

where T denotes the set of ordinary trigonometric polynomials. Another extremal quantity of particular importance is

(3) $\Delta(k):=\frac{1}{2}\sup\{\beta(H):H\subset\mathbb{N}_2, |H|=k\}$.

We determine the order of magnitude of $\Delta(k)$ in Proposition 2. This result has an application in [5].
Let us denote $\overline{H}:=\mathbb{N}_2-H$. It is almost immediate that $\beta(H)\cdot\beta(\overline{H})\leq 2$. However, as the case of λ and γ suggests, even the following duality statement holds true.

THEOREM 1. Let $H\subset\mathbb{N}_2$ be arbitrary. We have

$$\beta(H)\cdot\beta(\overline{H})=2\ .$$

In these extremal problems, in particular in number theoretic applications, there are cases when we have to restrict ourselves to polynomials with nonnegative coefficients. More generally let us introduce for any $H,K\subset\mathbb{N}_2$

(4) $F(H,K):=\{f\in T, f\geq 0, f(x)=1+a\cos x+\sum_{n=2}^{\infty} a_n\cos nx, a_n\leq 0\ (n\notin H), a_n\geq 0\ (n\notin K)\}$.

The corresponding extremal quantity is

(5) $\beta(H,K):=\sup\{a:\exists f\in F(H,K)$ with $a=a_1=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx\}$.

This setting generalizes the preceeding problem as $\beta(H,H)=\beta(H)$. For positive definite polynomials we write simply $\beta(H,\emptyset)$. Now let us denote for a pair $H,K\subset\mathbb{N}_2$

(6) $H^*:=\overline{(H\cap(2\mathbb{N}+1))\cup(K\cap 2\mathbb{N})}$,

$K^*:=\overline{(H\cap 2\mathbb{N})\cup(K\cap(2\mathbb{N}+1))}$.

Note that both sets H^* and K^* depend on both sets H and K . With these notations we formulate an even more general duality result.

THEOREM 2. Let $H,K\subset\mathbb{N}_2$ be arbitrary and H^*,K^* be as above. We have $\beta(H,K)\cdot\beta(H^*,K^*)=2$.

Clearly, Theorem 1 follows easily from Theorem 2 since if $K=H$ then $H^*=K^*=\overline{H}$.

2. PROOF OF THEOREM 2.

Let us denote $S=\mathbb{R}/2\Pi\mathbb{Z}$ and $\int=\int_S$. We put $G:=\{f(x+\Pi):f\in F(H,K)\}$.

Now if $f\in F(H,K)$ has coefficients a_n and $g\in F(H^*,K^*)$ has coefficients b_n then we have according to $0\le f,g$

$$0\le\frac{1}{\Pi}\int f(x+\Pi)g(x)dx=2-a_1b_1+\sum_{n=2}^{\infty}(-1)^n a_nb_n\le$$

$$\le 2-a_1b_1 ,$$

since $(-1)^n a_nb_n\le 0$ $(n\in\mathbb{N}_2)$ as a quick reflection to (4) and (6) shows. Taking supremum according to (5), the above inequality entails

(7) $\beta(H,K)\cdot\beta(H^*,K^*)\le 2$.

Translation with Π leads to

(8) $\beta:=\beta(H,K)=\sup\{a_1:\exists f\in G,$

$$\frac{1}{\Pi}\int f(x)\cos x\,dx=-a_1\}.$$

Consider the Banach space $C(S)$ with the supremum norm. Denote

(9) $P:=\{f\in C(S):f>0\}$,

$$\Lambda=\{\sum_{n=2}^{\infty}a_n\cos nx\in T:(-1)^na_n\le 0$$

$$(n\notin H),(-1)^na_n\le 0\ (n\notin K)\},\ F=1-\beta\cos x+\Lambda.$$

Plainly Λ and P are convex cones, further P is open and F is the translation of Λ by the "vector" $1-\beta\cos x$. Moreover, $F\cap P=\emptyset$. Indeed, if $f\in F\cap P$ exists, then with $\delta:=\min f>0$ we find a $g=(f-\delta)/(1-\delta)\ge 0$, $g\in G$, and for this g a_1 would be $-\beta/(1-\delta)<-\beta$, a contradiction in view of (8).

Hence $F\cap P=\emptyset$ and we can apply the separation theorem of convex sets in the Banach space $C(S)$ for the sets F and P , cf. [1] 2.2.2. Corollary, p. 118.

We get a nontrivial linear functional $L\ne 0$ and a $w\in\mathbb{R}$ satisfying

(10) $LP\ge w\ge LF$.

Since P is a cone, so is LP , ie. $LP=\{0\}$ or $[0,\infty)$ or $(0,\infty)$ and so we can suppose $w=0$. We get also from $0\in\Lambda$

(11) $L(1-\beta\cos x)\le 0$,

and from $h\in H$ and $k\in K$ similarly we get for any $A\ge 0$

$$L(1-\beta\cos x+A(-1)^h\cos hx)\le 0\quad(h\in H)\ ,$$

$$L(1-\beta\cos x+A(-1)^{k+1}\cos kx)\le 0\ (k\in K)$$

so with $A\to+\infty$ we obtain

(12) $(-1)^hL(\cos hx)\le 0$ $(h\in H)$,

$(-1)^kL(\cos kx)\ge 0$ $(k\in K)$.

Now apply the Riesz Representation Theorem, cf. [1] 4.10.1. Theorem, p. 203: $L=\int d\mu$ with some regular Borel measure μ on S . Here $\mu\ne 0$ since $L\ne 0$, μ is nonnegative since $LP\ge 0$, and considering $\mu(x)+\mu(-x)$ we can also suppose that μ is even. Since $L(1)\ge 0$, and $L(1)=0$ would imply $LP=0$ and $L=0$, we find $L(1)=\int 1\,d\mu>0$, hence we can normalize supposing $\mu(S)=2\Pi$. In all, the Fourier series of the nonnegative Borel measure μ is

(13) $\mu\sim 1+\sum_{n=1}^{\infty}b_n\cos nx$,

where in view of (11) and (12)

(14) $1-\frac{1}{2}\beta b_1\le 0$, $(-1)^hb_h\ge 0$ $(h\in H)$,

$(-1)^kb_k\ge 0$ $(k\in K)$.

Consider F_N , the N^{th} Fejér kernel, and $\sigma_N:=F_N*\mu$ the N^{th} Fejér polynomial of μ . Since both F_N and μ are nonnegative, so is σ_N , and we have

(15) $0\le\sigma_N(\mu,x)=1+\sum_{n=1}^{N}(1-\frac{n}{N})\,b_n\cos nx$.

It is easy to check that (14) and (15) entails $\sigma_N\in F(H^*,K^*)$, and so

$$\beta(H^*,K^*)\ge(1-\frac{1}{N})\,b_1 .$$

According to the first part of (14) and letting $N\to\infty$ we obtain $\beta(H^*,K^*)\ge 2/\beta$, hence in view of (7) and (8) the proof is completed.

3. COMPUTATION OF SOME EXTREMAL QUANTITIES

<u>Proposition 1.</u> $\alpha(k)=1/\cos\frac{\Pi}{2k}$ and $\beta(\{k\})=2\cos\frac{\Pi}{2k}$.

<u>Proof.</u> In view of Theorem 1 it suffices to show the first part.

Let $f\in F(\{k\})$ ie. $0\le f(x)=1+a\cos x+$ $+b\cos kx$. Considering $x=\Pi+\frac{\Pi}{2k}$ we get $0\le 1-a\cos\frac{\Pi}{2k}$ hence $\alpha(k)\le 1/\cos\frac{\Pi}{2k}$.

On the other hand the nonnegativity of the polynomial

$$f(x)=1+\frac{1}{\cos\frac{\Pi}{2k}}\cos x+\frac{(-1)^k \operatorname{tg}\frac{\Pi}{2k}}{k}\cos kx$$

proves the assertion.

Proposition 2. For any $k\in \mathbb{N}$ we have

$$1-\frac{5}{(k+1)^2}\le\Delta(k)\le 1-\frac{0.5}{(k+1)^2}\ .$$

Proof. Taking $H=\{2,3,\dots,k+1\}$ Fejér's result (1) gives $\beta(H)=2\cos\frac{\Pi}{k+3}$ and some calculation yields the lower estimate. On the other hand for any $H\subset\mathbb{N}_2$, $|H|=k$ we can take $n\notin H$ with $2\le n\le k+2$, and using Proposition 1 we get $\beta(H)\le\beta(\overline{\{n\}})=2\cos\frac{\Pi}{2n}\le 2-(k+1)^{-2}$, whence the assertion.

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