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ON A PAPER OF ERŐD AND TURÁN-MARKOV INEQUALITIES FOR NON-FLAT CONVEX DOMAINS*

Szilárd Gy. Révész**

To the memory of János Erőd, 1916-1945

For a convex domain $K \subset \mathbb{C}$ the well-known general Markov inequality $\|p'\| \le c(K)n^2\|p\|$ holds for every polynomial p of degree n. At the same time for polynomials in general, $\|p'\|$ can be arbitrarily small as compared to $\|p\|$.

The situation changes when we assume that the polynomials have all their zeroes in the convex body K. This problem of lower bound for Markov factors was first investigated by Turán in 1939. Turán showed $\|p'\| \ge (n/2)\|p\|$ for the unit disk D and $\|p'\| \ge c\sqrt{n}\|p\|$ for the unit interval $I := [-1, 1]$. Soon after that, J. Erőd published a long article, discussing various extensions of the results and methods of Turán.

For decades, Erőd's paper was quoted only for the explicit calculation of the exact constant of the interval case. However, in recent years Levenberg and Poletsky, Erdélyi and also the author investigated Turán's problem for various sets – basically, convex domains. In this context the much richer content of Erőd's work is to be realized again.

Thus, the aim of the paper is twofold. On the one hand we give an account of the half-forgotten, old Hungarian article of Erőd, also commemorating its author. On the other hand we report on recent developments with particular emphasis on development of one of the key observations of Erőd, namely, the role of the *curvature* of the boundary curve in the estimation of the lower bound of Markov factors.

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1. Introduction

On the complex plane polynomials of degree n admit a Markov inequality $\|p'\|_K \le c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \ge c_K n^A \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials must belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n(K)} M(p) \qquad \text{with} \qquad M := M(p) := \frac{\|p'\|}{\|p\|} \ . \tag{1}$$

Theorem A (Turán, [15, p. 90]). *If $p \in \mathcal{P}_n(D)$, where D is the unit disk, then we have*

$$\|p'\|_D \ge \frac{n}{2} \|p\|_D \ . \tag{2}$$

Theorem B (Turán, [15, p. 91]). *If $p \in \mathcal{P}_n(I)$, where $I := [-1, 1]$, then we have*

$$\|p'\|_I \ge \frac{\sqrt{n}}{6} \|p\|_I \ . \tag{3}$$

Theorem A is best possible, as the example of $p(z) = 1 + z^n$ shows. This also highlights the fact that, in general, the order of the inverse Markov factor cannot be higher than n. On the other hand, a number of positive results, started with J. Erőd's work, exhibited convex domains having order n inverse Markov factors (like the disk). We come back to this after a moment.

Regarding Theorem B, Turán pointed out by the example of $(1 - x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [6], but the value of the constant is computed for all fixed n precisely in [4]. In fact, about two-third of the paper [4] is occupied by the rather lengthy and difficult calculation of these constants, which partly explains why later authors started to consider this achievement the only content of the paper. Our aim is to describe further ideas of Erőd, as presented in [4], and to describe development of these ideas to date.

As mentioned above, Erőd did not stop at calculation of $M_n(I)$. He then considered ellipse domains, which form a parametric family E_b naturally connecting the two sets I and D. Note that for the same sets E_b the best form of the Bernstein-Markov inequality was already investigated by Sewell, see [13].

Theorem C (Erőd, [4, p. 70]). *Let $0 < b < 1$ and let E_b denote the ellipse domain with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Then*

$$\|p'\| \geq \frac{b}{2} n \|p\| \tag{4}$$

for all polynomials p of degree n and having all zeroes in E_b.

Erőd himself provided two proofs, the first being a quite elegant one using elementary complex functions, while the second one fitting more in the frame of classical analytic geometry. In 2004 this theorem was rediscovered by J. Szabados, providing a testimony of the natural occurrence of the sets E_b in this context[1].

In fact, the key to Theorem A was the following observation, implicitly already in [15] and [4] and formulated explicitly in [6].

Lemma 1 (Turán, Levenberg-Poletsky). *Assume that $z \in \partial K$ and that there exists a disc D_R of radius R so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$|p'(z)| \geq \frac{n}{2R} |p(z)| \ . \tag{5}$$

So Levenberg and Poletsky [6] found it worthwhile to formally introduce the next definition.

Definition 1. A compact set $K \subset \mathbb{C}$ is called *R-circular*, if for any point $z \in \partial K$ there exists a disc D_R of radius R with $z \in \partial D_R$ and $K \subset D_R$.

With this they formulated various consequences. For our present purposes let us choose the following form, c.f. [6, Theorem 2.2].

Theorem D (Erőd, Levenberg-Poletsky). *If K is an R-circular set and $p \in \mathcal{P}_n(K)$, then*

$$\|p'\| \geq \frac{n}{2R} \|p\| \ . \tag{6}$$

Note that here it is not assumed that K be convex; a circular arc, or a union of disjoint circular arcs with proper points of join, satisfy the criteria. However, other curves, like e.g. the interval itself, do not admit such inequalities; as said above, the order of magnitude can be as low as $\sqrt{n}$ in general.

Erőd did not formulate the result that way; however, he was clearly aware of that. This can be concluded from his various argumentations, in particular for the next result.

[1] After learning about the overlap with Erőd's work, the result was not published.

Theorem E (Erőd, [4, p. 77]). *If K is a C^2-smooth convex domain with the curvature of the boundary curve staying above a fixed positive constant $\kappa > 0$, and if $p \in \mathcal{P}_n(K)$, then we have*

$$\|p'\| \geq c(K)n\|p\|. \tag{7}$$

From Erőd's argument one can not easily conclude that the constant is $c(K) = \kappa/2$; on the other hand, his statement is more general than that. Although the proof is slightly incomplete, let us briefly describe the idea. We shall return to this and provide a somewhat different, complete proof giving also the value $c(K) = \kappa/2$ of the constant later in §4.

Proof. The norm of p is attained at some point of the boundary, so it suffices to prove that $|p'(z)|/|p(z)| \geq cn$ for all $z \in \partial K$. But the usual form of the logarithmic derivative and the information that all the n zeroes $z_1, \dots, z_n$ of p are located in K allows us to draw this conclusion once we have for a fixed direction $\varphi := \varphi(z)$ the estimate

$$\Re\left(e^{i\varphi}\frac{1}{z - z_k}\right) \geq c > 0 \qquad (k = 1, \dots, n). \tag{8}$$

Choosing φ the (outer) normal direction of the convex curve ∂K at $z \in \partial K$, and taking into consideration that z_k are placed in $K \setminus \{z\}$ arbitrarily, we end up with the requirement that

$$\Re\left(e^{i\varphi}\frac{1}{z - w}\right) = \frac{\cos\alpha}{|z - w|} \geq c \qquad (w \in K \setminus \{z\},\ \alpha := \varphi - \arg(z - w))\ . \tag{9}$$

Now if K is strictly convex, then for $z \neq w$ we do not have $\cos\alpha = 0$, a necessary condition for keeping the ratio off zero. It remains to see if $|z - w|/\cos\alpha$ stays bounded when $z \in \partial K$ and $w \in K \setminus \{z\}$, or, as is easy to see, if only $w \in \partial K \setminus \{z\}$. Observe that $F(z, w) := |z - w|/\cos\alpha$ is a two-variate function on ∂K^2, which is not defined for the diagonal $w = z$, but under certain conditions can be extended continuously. Namely, for given z the limit, when $w \to z$, is the well-known geometric quantity $2\rho(z)$, where $\rho(z)$ is the radius of the osculating circle (i.e., the reciprocal of the curvature $\kappa(z)$). (Note here a gap in the argument for not taking into consideration also $(z', w') \to (z, z)$, which can be removed by showing uniformity of the limit.) Hence, for smooth ∂K with strictly positive curvature bounded away from 0, we can define $F(z, z) := 2/\kappa(z) = 2\rho(z)$. This makes F a continuous function all over ∂K^2, hence it stays bounded, and we are done.

To show the uniformity of the limit when $z', w' \to z$, let us fix the arc length parametrization of $\gamma := \partial K$, and assume that $z = \gamma(s)$ and $w = \gamma(t)$: similarly $z' = \gamma(s')$ and $w' = \gamma(t')$. Now $w - z = \int_s^t \dot\gamma(u)du =$

$\int_s^t \int_s^u \ddot{\gamma}(v)dvdu + (t-s)\dot{\gamma}(s)$,

$$F(z,w) = \frac{\langle w-z; \ddot{\gamma}(s)\rangle}{|w-z|^2 |\ddot{\gamma}(s)|} \tag{10}$$

and so in view of $\langle \dot{\gamma}(s); \ddot{\gamma}(s)\rangle = 0$ we are led to

$$F(z,w) = \frac{\int_s^t \int_s^u \langle \ddot{\gamma}(v); \ddot{\gamma}(s)\rangle dvdu}{\langle \int_s^t \int_s^u \ddot{\gamma}(v)dvdu + (t-s)\dot{\gamma}(s); \int_s^t \int_s^u \ddot{\gamma}(v)dvdu + (t-s)\dot{\gamma}(s)\rangle\, |\ddot{\gamma}(s)|}.$$

In the numerator we can apply $\ddot{\gamma} \in C(\mathbb{R}/L\mathbb{Z})$ (where L is the arc length of γ) and also $0 < \kappa \le |\ddot{\gamma}| \le \lambda < \infty$, say. Moreover, for w in a δ-neighborhood of z (in arc length distance), we even have $|\ddot{\gamma}(v) - \ddot{\gamma}(s)| < \epsilon$ and hence the numerator can be reformulated as

$$\frac{1}{2}(t-s)^2 |\ddot{\gamma}(s)|^2 + \int_s^t \int_s^u \delta(s,v)dvdu = \frac{1}{2}(t-s)^2 (|\ddot{\gamma}(s)|^2 + \eta(s,t))$$
$$(|\delta(s,v)|, |\eta(s,t)| < \epsilon).$$

On the other hand already $(0 <)|\ddot{\gamma}| \le \lambda$ suffices (using also the fact that $|\dot{\gamma}(s)| = 1$ in arc length parametrization) to get for the denominator that it is

$$(t-s)^2\,(1+\theta(s,t))\,|\ddot{\gamma}(s)| \qquad \left(|\theta(s,t)| < \delta\lambda + \frac{1}{4}\delta^2\lambda^2\right).$$

Summing up, we are led to

$$F(z,w) = \frac{\frac{1}{2}(t-s)^2 (|\ddot{\gamma}(s)|^2 (1+o(1))}{(t-s)^2\,(1+o(1))\,|\ddot{\gamma}(s)|} = \frac{1+o(1)}{2} |\ddot{\gamma}(s)| \qquad (w \to z).$$

Note that by the uniform continuity of $\ddot{\gamma}$, the $o(1)$ is uniform (in arc length parameter); therefore for the pair of points (z', w') we similarly obtain

$$F(z', w') = (1+o(1))\frac{1}{2}|\ddot{\gamma}(s')| = (1+o(1))\frac{1}{2}|\ddot{\gamma}(s)|$$

if $(z', w') \to z$, as needed. □

From this argument it can be seen that whenever we have the property (9) for all given boundary points $z \in \partial K$, then we also conclude the statement. This explains why Erőd could allow even vertices, relaxing the conditions of the above statement to hold only piecewise on smooth Jordan arcs, joining at vertices. However, to have a fixed bound, either the number of vertices has to be bounded, or some additional condition must be imposed on them. Erőd did not elaborate further on this direction.

Convex domains (or sets) *not* satisfying the R-circularity criteria with any fixed positive value of R are termed to be *flat*. Clearly, the interval is flat, like any polygon or any convex domain which is not strictly convex. From this definition it is not easy to tell if a domain is flat, or if it is circular, and if so, then with what (best) radius R. We shall deal with the issue in this work, aiming at finding a large class of domains having cn order of the inverse Markov factor with some information on the arising constant as well.

On the other hand a lower estimate of the inverse Markov factor of the same order as for the interval was obtained in full generality in 2002, see [6, Theorem 3.2].

Theorem F (Levenberg-Poletsky). *If $K \subset \mathbb{C}$ is a compact, convex set, $d := \operatorname{diam} K$ is the diameter of K and $p \in \mathcal{P}_n(K)$, then we have*

$$\|p'\| \ge \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\| \ . \tag{11}$$

Clearly, we can have no better order, for the case of the interval the $\sqrt{n}$ order is sharp. Nevertheless, already Erőd [4, p. 74] addressed the question: "For what kind of domains does the method of Turán apply?" Clearly, by "applies" he meant that it provides cn order of oscillation for the derivative.

The most general domains with $M(K) \gg n$, found by Erőd, were described on p. 77 of [4]. Although the description is a bit vague, and the proof shows slightly less, we can safely claim that he has proved the following result.

Theorem G (Erőd). *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles $< \pi$, with all the arcs being C^2-smooth and being either straight lines of length $\ell < \Delta(K)/4$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from 0 by a fixed constant. Then there is a constant $c(K)$, such that $M_n(K) \ge c(K)n$ for all $n \in \mathbb{N}$.*

To deal with the flat case of straight line boundary arcs, Erőd involved another approach, cf. [4, p. 76], appearing later to be essential for obtaining a general answer. Namely, he quoted Faber [5] for the following fundamental result going back to Chebyshev.

Lemma 2 (Chebyshev). *Let $J = [u, v]$ be any interval on the complex plane with $u \ne v$ and let $J \subset R \subset \mathbb{C}$ be any set containing J. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in R} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \ge 2 \left(\frac{|J|}{4} \right)^k \ . \tag{12}$$

The relevance of Chebyshev's Lemma is that it provides a quantitative way to handle contribution of zero factors at some properly selected set J. One uses this for comparison: if $|p(\zeta)|$ is maximal at $\zeta \in \partial K$, then the maximum on some J can not be larger. Roughly speaking, combining this with geometry we arrive at an effective estimate of the contribution, hence even on the location of the zeroes. For more in this direction see [4, 10].

In his recent work [3], Erdélyi considered various special domains. Apart from further results for polynomials of some special form (e.g. even or real polynomials), he obtained the following.

Theorem H (Erdélyi). *Let Q denote the square domain with diagonal $[-1,1]$. Then for all polynomials $p \in \mathcal{P}_n(Q)$ we have*

$$\|p'\| \geq C_0 n \|p\| \tag{13}$$

with a certain absolute constant C_0.

Note that the regular n-gon K_n is already covered by Erőd's Theorem G if $n \geq 26$, but not the square Q, since the side length h is larger than the quarter of the transfinite diameter Δ: actually, $\Delta(Q) \approx 0.59017\ldots h$, while

$$\Delta(K_n) = \frac{\Gamma(1/n)}{\sqrt{\pi}2^{1+2/n}\Gamma(1/2+1/n)} h > 4h \quad \text{iff} \qquad n \geq 26,$$

see [9, p. 135]. Erdélyi's proof is similar to Erőd's argument[2]: sacrificing generality gives the possibility for a better calculation for the particular choice of Q.

Returning to the question of the order in general, let us recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ *having non-empty interior*. Clearly, assuming boundedness is natural, since all polynomials of positive degree have $\|p\|_K = \infty$ when the set K is unbounded. Also, all convex sets with non-empty interior are *fat*, meaning that $\mathrm{cl}(K) = \mathrm{cl}(\mathrm{int} K)$. Hence taking the closure does not change the sup norm of polynomials under study. The only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his $c\sqrt{n}$ lower estimate is of the right order. Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order.

Theorem I (Halász and Révész, [10]). *Let $K \subset \mathbb{C}$ be any convex domain having minimal width $w(K)$ and diameter $d(K)$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\frac{\|p'\|}{\|p\|} \geq C(K)n \qquad \text{with} \qquad C(K) = 0.0003\frac{w(K)}{d^2(K)} \; . \tag{14}$$

In the proof of this result in [10], due to generality, the precision of constants could not be ascertained e.g. for the special ellipse domains considered

[2]Erdélyi was apparently not aware of the full content of [4] when presenting his rather similar argument.

in [4]. Thus it seems that the general results are not capable to fully cover e.g. Theorem C.

Our aim here is to show that even that is possible for a quite general class of convex domains with order n inverse Markov factors and a different estimate of the arising constants. This will be achieved working more in the direction of Erőd's first observation, i.e. utilizing information on curvature. Since these results need some technical explanations, in particular for the geometric terms we use, formulation of these will be postponed until §4. Before that, the next section is dedicated to the life and work of János Erőd, and in §3 we start with describing the underlying geometry.

2. A few words about the life and work of János Erőd

With this paper we would like to call the attention of the approximation theory community to the rich content of the original paper [4]. It is necessary since out of the dozen or so references in the literature to [4], none of these works mention – and, actually, very few people are aware of the fact – that Erőd's work covered a lot more than the mere calculation of $M_n(I)$. The paper was written and published in Hungarian, back in the eve of World War II, and in spite of the fact that both the Mathematical Reviews and the Zentralblatt reviews mention the general features of the paper, that aspect seems to be forgotten. A particular aim of our paper is to commemorate János Erőd, the person, too.

János Erőd was born to the Ehrlich family in Gyöngyös, a city some 80 km East-East-North of Budapest, on 30 November 1916, during World War I. The Jewish family had three children: János was born second, between his two sisters Carmen and Márta. Sometimes in the 1920's the family converted to the protestant church; on this occasion the family name was changed to the Hungarian name "Erőd", although the parents kept the name "Ehrlich". János did very well at school and graduated with an excellent grade; furthermore, he was a successful problem solver of the legendary "KöMaL, Középiskolai Matematikai és Fizikai Lapok" ("Secondary School Mathematics and Physics Journal"). Therefore, he continued studies in mathematics and physics at the Budapest University of Sciences. He was only 23 when he received his PhD in mathematics in 1939: his thesis is just the reprint of the only paper [4] he wrote. Although the topic is a continuation of the work of Paul Turán, it can not be seen from the dry quotations how close personal contacts they might have had. Nevertheless, Turán and Erőd mutually refer to each other in [15] and [4], so at least they knew about each other.

Because of the Jewish laws already in effect, he could not hope for a university employment. However, he registered to the Reformed Church Theology

College in Pápa, another city about 120 km to the West from Budapest. Also there he graduated with excellent grade after completing the four year curriculum in the three years 1939-1942. He passed his first and second clergyman exams in 1943 and 1944, again with excellence. Becoming a reformed church priest, he could serve his church at various locations including the vicinity of Pápa and Győr. For a while he became the director of the church's orphan boys' house in Komárom, some 80 km's West-North-West of Budapest (now belonging to Slovakia).

In 1944 his parents and his younger sister Márta were deported. They were taken to Auschwitz – none of them returned. In February, 1945 János decided to return to Pápa to his fiance, Jolán Nemes. He could stay unnoticed only for a very short time. He was arrested together with Jolánka. The cause formally was not that he was a Jew, but some (rather unrealistic) accusations of treachery by establishing a radio contact with the advancing Soviet troops. The young couple was interrogated in the military base of Pápa. Dezső Trócsányi, János Erőd's theology professor of the College, protested against the brutal torture of János, but to no avail. The young couple was killed, very likely in the barracks. Their remnants were not found. Neither their grave, nor the exact date of their death is known.

However, the mathematical achievements of János were not lost, even if somewhat forgotten. It is in order to commemorate also its martyr author, when reflecting back to the rich content of this pioneering work.

3. Some geometrical notions

Recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C} \cong \mathbb{R}^2$ having non-empty interior. For a convex domain K any interior point z defines a parametrization $\gamma(\varphi)$ of the boundary ∂K, taking the unique point $\{z + te^{i\varphi} : t \in (0, \infty)\} \cap \partial K$ for the definition of $\gamma(\varphi)$. This defines the closed Jordan curve $\Gamma = \partial K$ and its parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$. By convexity, at any boundary point $\zeta = \gamma(\theta) \in \partial K$, the chords to boundary points in some small vicinity of ζ with parameter $< \theta$ or with $> \theta$ have arguments below and above the argument of the direction of any tangential (supporting) line at ζ. Thus the tangent direction or argument function $\alpha_-(\theta)$ can be defined as e.g. the supremum (or $\limsup$) of arguments of chords from the left; similarly, $\alpha_+(\theta) := \inf\{\arg(z - \zeta) \ : \ z = \gamma(\varphi), \ \varphi > \theta\}$, and any line $\zeta + e^{i\beta}\mathbb{R}$ with $\alpha_-(\theta) \leq \beta \leq \alpha_+(\theta)$ is a supporting line to K at $\zeta = \gamma(\theta) \in \partial K$. In particular the curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with the unique value of $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$. It is clear that interpreting $\alpha_\pm$ as functions on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function. In other words, we are allowed to change parameterizations to arc length, say, when in case of $|\Gamma| = a$ with $\Gamma = \partial K$ the functions $\alpha_\pm$ map from $[0, a]$ to $[0, 2\pi]$.

Observe that $\alpha_\pm$ are non-decreasing functions with total variation $\mathrm{Var}\,[\alpha_\pm] = 2\pi$, and that they have a common value precisely at continuity points, which occur exactly at points where the supporting line to K is unique. At points of discontinuity $\alpha_\pm$ is the left-, resp. right continuous extension of the same function. For convenience, and for better matching with [2], we may even define the function $\alpha := (\alpha_+ + \alpha_-)/2$ all over the parameter interval.

For obvious geometric reasons we call the jump function $\beta := \alpha_+ - \alpha_-$ the *supplementary angle* function. In fact, β and the usual Lebesgue decomposition of the non-decreasing function α_+ to $\alpha_+ = \sigma + \alpha_* + \alpha_0$, consisting of the pure jump function σ, the non-decreasing singular component α_*, and the absolute continuous part α_0, are closely related. By monotonicity there are at most countable many points where $\beta(x) > 0$, and in view of bounded variation we even have $\sum_x \beta(x) \le 2\pi$, hence the definition $\mu := \sum_x \beta(x)\delta_x$ defines a bounded, positive Borel measure. Now it is clear that $\sigma(x) = \mu([0, x])$, while $\alpha_*' = 0$ a.e., and α_0 is absolutely continuous. In particular, α or α_+ is differentiable at x exactly when $\beta(x) = 0$ and x is not in the exceptional set of non-differentiable points with respect to α_*. That is, we have differentiability almost everywhere, and

$$
\begin{aligned}
\int_x^y \alpha'(t)dt =& \alpha_0(y) - \alpha_0(x) \\
=& [\alpha_+(y) - \sigma(y) - \alpha_*(y))] - [\alpha_+(x) - \sigma(x) - \alpha_*(x)] \\
\le& \alpha_-(y) - \alpha_+(x) \ .
\end{aligned}
\tag{15}
$$

It follows that we have the criteria

$$
\alpha'(t) \ge \lambda \qquad \text{a.e.} \quad t \in [0, a]
\tag{16}
$$

if and only if

$$
\alpha_\pm(y) - \alpha_\pm(x) \ge \lambda(y - x) \qquad \forall x, y \in [0, a] \ .
\tag{17}
$$

Here we reserved to the arc length parametrization. Recall that one of the most important geometric quantities, curvature, is just $\kappa(s) := \alpha'(s)$, whenever parametrization is by arc length s.

Thus we can rewrite (16) as

$$
\kappa(t) \ge \lambda \qquad \text{a.e.} \quad t \in [0, a] \ ,
\tag{18}
$$

or, with radius of curvature $\rho(t) := 1/\kappa(t)$ introduced,

$$
\rho(t) \le \frac{1}{\lambda} \qquad \text{a.e.} \quad t \in [0, a] \ .
\tag{19}
$$

Again, ρ is a parametrization-invariant quantity (describing the radius of the osculating circle). Actually, it is easy to translate all these conditions to arbitrary parametrization of the tangent angle function α. Since also curvature and curvature radius are parametrization-invariant quantities, all the above hold for any parametrization.

Moreover, with a general parametrization let $|\Gamma(\eta,\zeta)|$ stand for the arc length of the rectifiable Jordan arc $\Gamma(\eta,\zeta)$ of the curve Γ between the two points $\zeta,\eta\in\Gamma=\partial K$. We can then say that the curve satisfies a Lipschitz-type increase or *subdifferential condition* whenever

$$|\alpha_\pm(\eta)-\alpha_\pm(\zeta)|\geq\lambda|\Gamma(\eta,\zeta)|\qquad(\forall\zeta,\eta\in\Gamma)\ . \tag{20}$$

Clearly, the above considerations show that all the above are equivalent.

In the paper we use the notation α (and also $\alpha_\pm$) for the tangent angle, κ for the curvature, and ρ for the curvature radius. These notations we shall use basically in function of the arc length parametrization s, but with a slight abuse of notation also $\alpha_-(\varphi)$, $\kappa(\zeta)$ etc. may occur with the obvious meaning.

4. Results for non-flat domains

The above Theorem C was formulated with very precise constants. In particular, it gives a good description of the "inverse Markov factor"

$$M(E_b):=\inf_{p\in\mathbb{P}_n(E_b)}M(p),$$

when n is fixed and $b\to0$. In this section we aim at a precise generalization of Theorem C using appropriate geometric notions. Our argument stems out of the notion of "circular sets", used in [6] and going back to Turán's work. This approach can indeed cover the full content of Theorem C. Moreover, the geometric observation and criteria we present will cover a good deal of different, not necessarily smooth domains. First let us have a recourse to Theorem E.

Theorem 1. *Let $K\subset\mathbb{C}$ be any convex domain with C^2-smooth boundary curve $\partial K=\Gamma$ having curvature $\kappa(\zeta)\geq\kappa$ with a certain constant $\kappa>0$ and for all points $\zeta\in\Gamma$. Then $M(K)\geq(\kappa/2)n$.*

Proof. As in [10], our proof hinges upon geometry in a large extent. For this smooth case we use the following result, which is well-known as Blaschke's Rolling Ball Theorem, cf. [1, p. 116].

Lemma 3 (Blaschke). *Assume that the convex domain K has C^2 boundary $\Gamma=\partial K$ and that there exists a positive constant $\kappa>0$ such that the curvature satisfies $\kappa(\zeta)\geq\kappa$ at all boundary points $\zeta\in\Gamma$. Then to each boundary* *points $\zeta \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa$, such that $\zeta \in \partial D_R$, and $K \subset D_R$.*

That is, if the curvature of the boundary curve of a twice differentiable convex body exceeds $1/R$, then the convex body is R-circular. From this an application of Theorem D yields the assertion. □

So now it is worthy to calculate the curvature of ∂E_b.

Lemma 4. *Let E_b be the ellipse with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Consider its boundary curve Γ_b. Then at any point of the curve the curvature is between b and $1/b^2$.*

Proof. Now we depart from arc length parameterization and use for $\Gamma_b := \partial E_b$ the parameterization $\gamma(\varphi) := (\cos(\varphi), b\sin(\varphi))$. Then we have

$$\kappa(\gamma(\varphi)) = \frac{|\dot\gamma(\varphi) \times \ddot\gamma(\varphi)|}{|\dot\gamma(\varphi)|^3} ,$$

that is,

$$\begin{aligned}
\kappa(\gamma(\varphi)) &= \frac{|(-\sin\varphi, b\cos\varphi) \times (-\cos\varphi, -b\sin\varphi)|}{|(-\sin\varphi, b\cos\varphi)|^3} \\
&= \frac{b\sin^2\varphi + b\cos^2\varphi}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} \\
&= \frac{b}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} .
\end{aligned}$$

Clearly, the denominator falls between $(b^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2} = b^3$ and $(\sin^2\varphi + \cos^2\varphi)^{3/2} = 1$, and these bounds are attained, hence $\kappa(\gamma(\varphi)) \in [b, 1/b^2]$ whenever $b \le 1$. □

Proof of Theorem C. The curvature of Γ_b at any of its points is at least b according to Lemma 4. Hence $M(E_b) \ge (b/2)n$ in view of Theorem 1, and Theorem C follows. □

However, not only smooth convex domains can be proved to be circular. E.g. it is easy to see that if a domain is the intersection of finitely many R-circular domains, then it is also R-circular. The next generalization is not that simple, but is still true.

Lemma 5 (Stranzen). *Let the convex domain K have boundary $\Gamma = \partial K$ with angle function $\alpha_\pm$ and let $\kappa > 0$ be a fixed constant. Assume that $\alpha_\pm$ satisfies the curvature condition $\kappa(s) = \alpha'(s) \ge \kappa$ almost everywhere. Then to each boundary point $\zeta \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa$, such that $\zeta \in \partial D_R$, and $K \subset D_R$. That is, K is $R = 1/\kappa$-circular.*

Proof. This result is essentially the far-reaching, relatively recent generalization of Blaschke's Rolling Ball Theorem by Stranzen. A reference for it is Lemma 9.11 on p. 83 of [2]. Note that the proof of this lemma starts with establishing Condition (i) on p. 83 of [2], which is equivalent to the subdifferential condition (20). We could as well choose any of the equivalent formulations in (15)-(20). The only slight alteration from the formulation, suppressed in the above quotations, is that Stranzen's version assumes $\kappa(t) \geq \kappa$ wherever the curvature $\kappa(t) = \alpha'(t)$ exists (so almost everywhere for sure), while above we stated the same thing for almost everywhere, but not necessarily at every points of existence. This can be overcome by reference to the subdifferential version, too. Also, there is an even more recent proof, which provides this version directly, see [11]. □

Theorem 2. *Assume that the convex domain K has boundary $\Gamma = \partial K$ and that the a.e. existing curvature of Γ exceeds κ almost everywhere, or, equivalently, assume the subdifferential condition* (20) *(or any of the equivalent formulations in* (15)-(20)*) with $\lambda = \kappa$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\| \geq \frac{\kappa}{2} n \|p\| \ . \tag{21}$$

Proof. The proof follows from a combination of Theorem D and Lemma 5. □

Let us illustrate the strengths and weaknesses of the above results on the following instructive examples, suggested to us by J. Szabados (personal communication). Consider for any $1 < p < \infty$ the ℓ_p unit ball

$$\begin{array}{rcl} B^p & := & \{(x,y) \,:\, |x|^p + |y|^p \leq 1\}, \\ \Gamma^p & := & \partial B^p = \{(x,y) \,:\, |x|^p + |y|^p = 1\}. \end{array} \tag{22}$$

Also, let us consider for any parameter $0 < b \leq 1$ the affine image ("ℓ_p-ellipse")

$$\begin{array}{rcl} B^p_b & := & \{(x,y) \,:\, |x|^p + |y/b|^p \leq 1\}, \\ \Gamma^p_b & := & \partial B^p_b = \{(x,y) \,:\, |x|^p + |y/b|^p = 1\}. \end{array} \tag{23}$$

By symmetry, it suffices to analyze the boundary curve $\Gamma := \Gamma^p_b$ in the positive quadrant. Here it has a parametrization $\Gamma(x) := (x, y(x))$, where $y(x) = b\,(1-x^p)^{1/p}$. As above, the curvature of the general point of the arc in the positive quadrant can be calculated and we get

$$\kappa(x) = \frac{(p-1)bx^{p-2}(1-x^p)^{1/p-2}}{\left(1 + b^2 x^{2p-2}(1-x^p)^{2/p-2}\right)^{3/2}} \tag{24}$$

For $p > 2$, the curvature is continuous, but it does not stay off 0: e.g. at the upper point $x = 0$ it vanishes. Therefore, neither Theorem 1 nor Theorem 2 can provide any bound, while Theorem I provides an estimate, even if with a small constant: here $d(B) = 2$, $w(B) = 2b$, and we get $M(B) \geq 0.00015bn$.

When $p = 2$, we get back the disk and the ellipses: the curvature is minimal at $\pm ib$, and its value is b there, hence $M(B) \geq (b/2)n$, as already seen in Theorem C. On the other hand Theorem I yields only $M(B) \geq 0.00015bn$ also here.

For $1 < p < 2$ the situation changes: the curvature becomes infinite at the "vertices" at $\pm ib$ and ± 1, and the curvature has a positive minimum over the curve Γ. When $b = 1$, it is possible to explicitly calculate it, since the role of x and y is symmetric in this case and it is natural to conjecture that minimal curvature occurs at $y = x$; using geometric-arithmetic mean and also the inequality between power means (i.e. Cauchy-Schwartz), it is not hard to compute $\min \kappa(x, y) = (p-1)2^{1/p-1/2}$, (which is the value attained at $y = x$). Hence Theorem 2 (but not Theorem 1, which assumes C^2-smoothness, violated here at the vertices!) provides $M(B^p) \geq (p-1)2^{1/p-3/2}n$, while Theorem I provides, in view of $w(B^p) = 2^{3/2-1/p}$, something like $M(B^p) \geq 0.0003\, 2^{-1/2-1/p}n \geq 0.0001n$, which is much smaller until p comes down very close to 1.

For general $0 < b < 1$ we obviously have $d(B) = 2$, $(\sqrt{2}b <)2b/\sqrt{1+b^2} < w(B) < 2b$, and Theorem I yields $M(B) \geq 0.0001bn$ independently of the value of p.

Now $\min \kappa$ can be estimated within a constant factor (actually, when $b \to 0$, even asymptotically precisely) the following way. On the one hand, taking $x_0 := 2^{-1/p}$ leads to $\kappa(x_0) = (p-1)b2^{1+1/p}/(1+b^2)^{3/2} < b(p-1)2^{1+1/p}$, hence $\min \kappa(x) < b(p-1)2^{1+1/p}$. Note that when $b \to 0$, we have asymptotically $\kappa(x_0) \sim b(p-1)2^{1+1/p}$. On the other hand denoting $\xi := x^p$ and $\beta := 2/p - 1 \in (0, 1)$, from (24) we get

$$\begin{aligned} \frac{(p-1)b}{\kappa(x)} &= [\xi(1-\xi)]^\beta \left[\xi^{1-\beta} + b^2(1-\xi)^{1-\beta}\right]^{3/2} \\ &\leq 2^{-2\beta}\left[(\xi + (1-\xi))^{1-\beta}(1 + (b^2)^{1/\beta})^\beta\right]^{3/2}, \end{aligned}$$

with an application of geometric-arithmetic mean inequality in the first and Hölder inequality in the second factor. In general we can just use $b < 1$ and get

$$\kappa(x) \geq (p-1)b2^{2\beta}\left[1 + b^{2/\beta}\right]^{-3\beta/2} \geq (p-1)b2^{\beta/2} = (p-1)b2^{1/p-1/2},$$

within a factor $2^{3/2}$ of the upper estimate for $\min \kappa$.

Therefore, inserting this into Theorem 2 as above, we derive $M(B_b^p) \geq (p-1)b2^{1/p-3/2}n$.

In all, we see that Theorems 1 (essentially due to Erőd) and 2 usually (but not always, c.f. the case $p \approx 1$ above !)) give better constants, when they apply. However, in cases the curvature is not bounded away from 0, we can retreat to application to the fully general Theorem I, which, even if with a small absolute constant factor, but still gives a precise estimate even regarding dependence of the constant on geometric features of the convex domain. This latter phenomenon is not just an observation on some particular examples, but is a general result, also proved in [10], valid even for not necessarily convex domains.

Theorem J. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$ there exists a polynomial $p \in \mathcal{P}_n(K)$ of degree exactly n satisfying*

$$\|p'\| \leq C'(K)\, n\, \|p\| \qquad \text{with} \qquad C'(K) := 600\, \frac{w(K)}{d^2(K)}\ . \tag{25}$$

5. Further remarks and problems

In the case of the unit interval also Turán type L^p estimates were studied, see [16] and the references therein. It would be interesting to consider the analogous question for convex domains on the plane. Note that already Turán remarked, see the footnote in [15, p.141], that on D an L^p version holds, too. Also note that for domains there are two possibilities for taking integral norms, one being on the boundary curve and another one of integrating with respect to area. It seems that the latter is less appropriate and convenient here.

In the above we described a more or less satisfactory answer of the problem of inverse Markov factors for convex domains. However, Levenberg and Poletsky showed that starshaped domains already do not admit similar inverse Markov factors. A question, posed by V. Totik, is to determine exact order of the inverse Markov factor for the "cross" $C := [-1, 1] \cup [-i, i]$; clearly, the point is not in the answer for the cross itself, but in the description of the inverse Markov factor for some more general classes of sets.

Another question, still open, stems from the Szegő extension of the Markov inequality, see [14], to domains with sector condition on their boundary. More precisely, at $z \in \partial K$ K satisfies the *outer sector condition* with $0 < \beta < 2$, if there exists a small neighborhood of z where some sector $\{\zeta\ :\ \arg(\zeta - z) \in (\theta, \beta\pi + \theta)\}$ is disjoint from K. Szegő proved, that if for a domain K, bounded by finitely many smooth (analytic) Jordan arcs, the supremum of β-values satisfying outer sector conditions at some boundary point is $\alpha < 2$, then $\|P'\| \ll n^\alpha \|P\|$ on K. Then Turán writes: "Es ist sehr wahrscheinlich, daß auch den Szegőschen Bereichen $M(p) \geq cn^{1/\alpha}$...", that is, he finds it rather likely that the natural converse inequality, suggested by the known cases of the disk and the interval (and now also by any other convex domain) holds also for general domains with outer sector conditions.

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Szilárd Gy. Révész
Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences,
Reáltanoda utca 13-15., 1054 Budapest, HUNGARY
E-mail: revesz@renyi.hu
and
Institut Henri Poincaré,
11 rue Pierre et Marie Curie, 75005 Paris, FRANCE
E-mail: Szilard.Revesz@ihp.jussieu.fr