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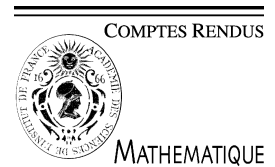


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Harmonic Analysis

Failure of Wiener's property for positive definite periodic functions

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Abstract

We say that Wiener's property holds for the exponent $p > 0$ whenever a positive definite function f , which belongs to $L^p(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, necessarily belongs to $L^p(\mathbb{T})$, too. This holds true for $p \in 2\mathbb{N}$ by a classical result of Wiener. Recently various concentration results were proved for idempotents and positive definite functions on measurable sets on the torus. They enable us to prove a sharp version of the failure of Wiener's property for $p \notin 2\mathbb{N}$, strengthening results of Wainger and Shapiro. **To cite this article:** A. Bonami, S.Gy. Révész, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Résumé

Contre-exemples à la propriété de Wiener pour les fonctions périodiques définies-positives. On dit que l'exposant p possède la propriété de Wiener si toute fonction périodique définie-positive qui est de puissance p -ième intégrable au voisinage de 0 l'est sur un intervalle de période. C'est le cas des entiers pairs, d'après un résultat classique de Wiener. Nous avons récemment obtenu des phénomènes de concentration des polynômes idempotents ou définis-positifs sur un ensemble mesurable du tore qui nous permettent de donner une version forte du fait que les exposants $p \notin 2\mathbb{N}$ n'ont pas la propriété de Wiener, améliorant ainsi les résultats de Wainger et Shapiro. **Pour citer cet article :** A. Bonami, S.Gy. Révész, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Notre point de départ est la propriété suivante :

Théorème 1 (Wiener, Shapiro). Si p est un entier pair et si $0 < a < 1/2$, alors tout polynôme trigonométrique 1-périodique défini-positif satisfait à l'inégalité

$$\frac{1}{2a} \int_{-a}^{+a} |f|^p \geq \frac{1}{2} \int_{-1/2}^{+1/2} |f|^p. \quad (1)$$

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De plus la constante $1/2$ est critique, même si on se restreint aux idempotents, c'est-à-dire aux polynômes trigonométriques à coefficients 0 ou 1.

Nous renvoyons à [11] pour la démonstration de l'inégalité (1). Le fait que la constante $1/2$ est critique est conséquence de la propriété suivante, que nous allons démontrer :

Proposition 2. Pour tout entier $k > 2$, si $a < 1/k$ et $b > 1/k$, il existe un idempotent f tel que

$$\int_{-a}^{+a} |f|^p \leq b \times \int_{-1/2}^{+1/2} |f|^p.$$

Notre exemple est inspiré de [3]. On prend $f := D_n * \mu_k$, où D_n est le noyau de Dirichlet

$$D_n(x) := \sum_{v=0}^{n-1} e(vx) = e^{\pi i(n-1)x/2} \frac{\sin(\pi nx)}{\sin(\pi x)}, \quad (2)$$

et μ_k est la moyenne des masses de Dirac aux racines k -ièmes de l'unité. Les deux ont pour coefficients de Fourier 0 ou 1, et f est un idempotent. Une seule racine de l'unité appartient à l'intervalle $(-a, +a)$, et le rapport entre $\int_{-a}^{+a} |f|^p$ et $\int_{-1/2}^{+1/2} |f|^p$ tend vers $1/k$ quand n tend vers l'infini.

Remarque 3. L'intervalle $(-a, +a)$ ne peut pas être remplacé par un ensemble E ayant 0 pour point de densité, même si $|E|$ est arbitrairement proche de 1. En effet, supposons que le complémentaire de E est la réunion (modulo 1) de tous les intervalles centrés en tous les rationnels j/l et de longueur $1/l^3$, avec $j \neq 0$ et $l > L$. La mesure de E est arbitrairement proche de 1 lorsque L tend vers ∞ , alors que, pour le même idempotent $f := D_n * \mu_k$, le rapport entre $\int_E |f|^p$ et $\int_{-1/2}^{+1/2} |f|^p$ tend vers 0 quand n et k tendent vers l'infini.

Il est connu que l'inégalité (1) n'est pas satisfaite lorsque p n'est pas un entier pair, voir [11] et [12]. Plus précisément, en nous basant sur les familles d'idempotents construits dans [2], nous nous intéressons à l'existence de polynômes trigonométriques 1-périodiques définis-positifs qui satisfont à l'inégalité

$$\int_E |f|^p \leq \varepsilon \left(\int_{-1/2}^{+1/2} |f|^q \right)^{p/q}. \quad (3)$$

Ici E est un sous-ensemble mesurable de $(-1/2, +1/2)$, p n'est pas un entier pair, $q \leq p$ et ε peut être choisi arbitrairement petit. Nous allons voir que, dans certains cas, f peut être choisi à spectre ayant des lacunes. Nous utiliserons la notation suivante : si f a pour série de Fourier $\sum_k a_k e^{2i\pi n_k x}$, où n_k est une suite croissante et $a_k \neq 0$ pour tout k , alors

$$\text{Gap}(f) := \min_k (n_{k+1} - n_k). \quad (4)$$

Nous montrons les résultats suivants :

- (i) Si $p = q > 1$, on peut trouver f idempotent satisfaisant (3), avec $\text{Gap}(f)$ arbitrairement grand.
- (ii) Si $p = q \leq 1$, on peut trouver f satisfaisant (3), avec $\text{Gap}(f)$ arbitrairement grand. De plus f peut être choisi idempotent si E est un ouvert.
- (iii) Si $p < 2$ et $q < p$ quelconque, on peut trouver f satisfaisant (3).
- (iv) Si $p > 2$ il existe $q(p) < p$ tel qu'on puisse trouver f satisfaisant (3).

A partir de ces exemples, nous pouvons construire des fonctions f appartenant à $L^p({}^c E)$ et n'appartenant pas à L^q , c'est-à-dire des contre-exemples à la propriété de Wiener. En particulier, nous avons le théorème suivant :

Théorème 4. Soit $1 < p < \infty$, avec $p \notin 2\mathbb{N}$. Pour tout ensemble symétrique $E \subset (-1/2, +1/2)$ de mesure positive, il existe une fonction 1-périodique définie-positive intégrable f qui est dans $L^p({}^c E)$ tandis que $f \notin L^p((-1/2, +1/2))$. De plus f peut être choisi de la forme $\sum_{k \geq 0} a_k e^{2i\pi n_k x}$, avec $n_{k+1} - n_k$ tendant vers l'infini.

Démonstration. On choisit une suite E_k de sous ensembles mesurables disjoints de E de mesure positive, tels que $|E_k| < 2^{-\alpha k}$, où α sera fixé ultérieurement. Soit f_k une suite de polynômes 1-périodiques idempotents tels que

$$\int_{\mathbb{T} \setminus E_k} |f_k|^p \leq 2^{-kp} \int_{-1/2}^{+1/2} |f_k|^p. \quad (5)$$

L'existence de cette suite est assurée par le point (i). On peut supposer que $\text{Gap}(f_k) > k$. Grâce à l'inégalité de Hölder, on obtient que

$$\int_{-1/2}^{+1/2} |f_k| \leq 2^{-\alpha(1-1/p)k} \left(\int_{E_k} |f_k|^p \right)^{1/p} + \left(\int_{(-1/2, +1/2) \setminus E_k} |f_k|^p \right)^{1/p} \leq 2 \times 2^{-k} \left(\int_{-1/2}^{+1/2} |f_k|^p \right)^{1/p},$$

si $\alpha(1 - 1/p) = 1$. Si on normalise la suite f_k de sorte que $\int_{\mathbb{T}} |f_k|^p = 2^{\frac{k}{p}}$, alors

$$f(x) := \sum_{k \geq 1} e^{2i\pi m_k x} f_k(x), \quad (6)$$

est de la forme souhaitée. La série converge normalement dans L^1 et $L^p({}^c E)$. Comme les E_j sont disjoints,

$$\|f\|_p \geq \|f\|_{L^p(E_k)} \geq \|f_k\|_p - \sum_j \|f_j\|_{L^p({}^c E_j)} \geq 2^{\frac{k}{p}} - \sum_{j>0} 2^{-\frac{j}{p}},$$

ce qui permet de conclure que f n'est pas dans L^p . $\square$

En utilisant la propriété (ii), il est possible de donner des exemples du même type lorsque $p \leq 1$, l'intégrabilité étant remplacée par l'appartenance à un espace de Hardy. On peut également utiliser (iii) et (iv) pour construire d'autres exemples.

Dans la direction opposée, rappelons que si n_k est une suite lacunaire à la Hadamard et s'il existe E de mesure positive tel que $f(x) = \sum_k a_k e^{2i\pi n_k x}$ est dans $L^p(E)$, alors la fonction f est dans L^p , voir [14]. Cette propriété a été généralisée par Miheev [9] à tous les ensembles $\Lambda(p)$ pour $p > 2$.

1. Introduction

Let f be a periodic integrable function which is positive definite, that is, has non negative Fourier coefficients. Assume that it is bounded (in $\|\cdot\|_\infty$) in a neighborhood of 0, then it necessarily belongs to $L_\infty(\mathbb{T})$, too. In fact, its maximum is obtained at 0 and, as $f(0) = \sum_k \hat{f}(k)$, f has an absolutely convergent Fourier series.

The same question can be formulated in any L^p space. Actually, the following question was posed by Wiener in a lecture, after he proved the L^2 case:

Wiener's Problem. Let $1 \leq p < \infty$. Is it true, that if for some $\varepsilon > 0$ a positive definite function $f \in L^p(-\varepsilon, \varepsilon)$, then we necessarily have $f \in L^p(\mathbb{T})$, too?

We refer to [11] for the story of this conjecture, see also [7] and [13]. The observation that the answer is positive if $p \in 2\mathbb{N}$ has been given by Wainger [12], as well as by Erdős and Fuchs [4]. The main step of the proof is the fact that all 1-periodic positive definite trigonometric polynomials satisfy (1), see [11]. For optimality of constants, see [7,8]. Generalizations in higher dimension may be found in [6] for instance. It was shown by Shapiro [11] and Wainger [12] that the answer to Wiener's problem is to the negative for all other values of p . Negative results were obtained for other groups in e.g. [5] and [7].

There is even more evidence that the Wiener property must hold when $p = 2$ and we prescribe large gaps in the Fourier series of f . Indeed, in this case by well-known results of Wiener and Ingham, see e.g. [13,14], we necessarily have an essentially uniform distribution of the L^2 norm on intervals longer than the reciprocal of the gap, even without the assumption that f be positive definite. As Zygmund pointed out, see the Notes to Chapter V §9, page 380 in [14], Ingham type theorems were not known for $p \neq 2$, nevertheless, one would feel that prescribing large gaps in the Fourier series should lead to better control of the global behavior by means of having control on some subset like e.g. $(-\varepsilon, \varepsilon)$. So the analogous Wiener question can be posed restricting to positive definite functions having gaps tending to ∞ . However, we answer negatively as well. In this strong form the question, to the best of our knowledge, has not been dealt with yet. Also we are able to replace the interval $(-\varepsilon, +\varepsilon)$ by any measurable symmetric subset E of the torus of measure $|E| < 1$. Neither extension can be obtained by a straightforward use of the methods of Shapiro and Wainger.

We use the notation $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the torus. Then $e(t) := e^{2\pi it}$ is the usual exponential function adjusted to interval length 1, and we denote e_h the function $e(hx)$. $\mathcal{T}^+$ will stand for the set of *positive definite trigonometrical polynomials*, that is, 1-periodic trigonometric polynomials with non negative Fourier coefficients. Polynomials with coefficients 0 or 1 are called *idempotents* and are denoted as $\mathcal{P}$. For f with Fourier series $\sum_k a_k e^{2i\pi n_k x}$, where n_k is increasing and $a_k \neq 0$ pour tout k , we define $\text{Gap}(f)$ by (4). We say that f has gaps tending to ∞ when $\text{Gap}(f - S_N(f))$ tends to ∞ . Here $S_N(f)$ denote the partial sums of the Fourier series of f .

2. Concentration of integrals

Let us now consider the p -concentration problem, which comes from the following definition:

Definition 5. Let $p > 0$, and $\mathcal{F}$ be a class of functions on $\mathbb{T}$. We say that for the class $\mathcal{F}$ there is p -concentration if there exists a constant $c > 0$ so that for any symmetric measurable set E of positive measure one can find an idempotent $f \in \mathcal{F}$ with

$$\int_E |f|^p \geq c \int_{\mathbb{T}} |f|^p. \quad (7)$$

Note that for a positive definite function the function $|f|$ is necessarily even. This is why we consider 0-symmetric (or, just symmetric for short) intervals or sets.

The problem of p -concentration on the torus for idempotent polynomials has been considered in [1,3]. It was essentially solved recently in [2]. Also, the weaker question of concentration of p th integrals of positive definite functions has been dealt with, starting with the work [3] see [2, Theorem 48]:

Theorem 6. For all $0 < p < \infty$, p not an even integer; whenever a 0-symmetric measurable set E of positive measure $|E| > 0$ is given, then for all $\varepsilon > 0$ there exists $f \in \mathcal{T}^+$ so that

$$\int_{^c E} |f|^p \leq \varepsilon \int_{\mathbb{T}} |f|^p. \quad (8)$$

Moreover, f can be taken such that $\text{Gap}(f)$ is arbitrarily large. When E is an open set or when $p > 1$, then f can be chosen an idempotent.

Theorem 6 allows us to see immediately that there is no inequality like (1) for p not an even integer. What is new, compared to the results of Shapiro and Wainger, is the fact that this is also the case if f has arbitrarily large gaps, and that we can replace intervals $(-a, +a)$ by arbitrary measurable sets of measure less than 1.

The previous theorem can be strengthened for open sets, using an improvement of the methods of Shapiro in [11]. The construction is closely related to the failure of Hardy–Littlewood majorant property.

Theorem 7. For all $0 < q \leq p < 2$, whenever a 0-symmetric open set E of positive measure $|E| > 0$ is given, then for all $\varepsilon > 0$ there exists $f \in \mathcal{T}^+$ which satisfies (3).

The same is valid for $q < p$ with p not an even integer, provided that q is sufficiently close to p , that is $q > q(p)$, where $q(p) < p$.

Proof. Let us first assume that $p < 2$. Then, for D_n the Dirichlet kernel defined by (2), with n sufficiently large depending on ε , there exists a choice of $\eta_k = \pm 1$ such that

$$\|D_n\|_p \leq \varepsilon \left\| \sum_{k=0}^n \eta_k e_k \right\|_q.$$

Indeed, if it was not the case, taking the q -th power, integrating on all possible signs and using Khintchine's Inequality, we would find that $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq Cn^{1-\frac{1}{p}}$ ($p > 1$), $c\varepsilon\sqrt{n} \leq \|D_n\|_1 \leq C \log n$ and $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq C$ ($0 < p < 1$) which leads to a contradiction. We note $g(t) := \sum_{k=0}^n \eta_k e_k(t)$ and $G(t) := D_n(t)$.

We assume that E contains $I \cup (-I)$, where $I := (\frac{k}{N}, \frac{k+1}{N})$. Let Δ be a triangular function based on the interval $(-\frac{1}{2N}, +\frac{1}{2N})$, that is, $\Delta(t) := (1 - 2N|t|)_+$. We finally consider the function

$$f(t) := \Delta(t-a)g(2Nt) + \Delta(t+a)g(2Nt) + 2\Delta(t)G(2Nt),$$

where a is the center of the interval I . Then an elementary computation of Fourier coefficients, using the fact that Δ has positive Fourier coefficients while the modulus of those of g and G are equal, allows to see that f is positive definite. Let us prove that one has (3). The left hand side is bounded by $\frac{2}{N}\|G\|_p^p$, while $\int_{\mathbb{T}} |f|^q$ is bounded below by $\frac{1}{2N}\|g\|_q^q - \frac{2}{N}\|G\|_q^q$. We conclude the proof choosing n, N sufficiently large.

Let us now consider $p > 2$ not an even integer. Mockenhaupt and Schlag in [10] have given counter-examples to the Hardy–Littlewood majorant conjecture, which are based on the following property: for $j > p/2$ an odd integer, the two trigonometric polynomials $g_0 := (1 + e_j)(1 - e_{j+1})$ and $G_0 := (1 + e_j)(1 + e_{j+1})$ satisfy the inequality $\|G_0\|_p < \|g_0\|_p$. By continuity, this inequality remains valid when p is replaced by q in the right hand side, with $q > q(p)$, for some $q(p) < p$. By a standard Riesz product argument, for K large enough, as well as $N_1, N_2, \dots, N_K$, depending on ε , the functions $g(t) := g_0(t)g_0(N_1t) \cdots g_0(N_Kt)$ and $G(t) := G_0(t)G_0(N_1t) \cdots G_0(N_Kt)$ satisfy the inequality $\|G\|_p \leq \varepsilon\|g\|_q$. From this point the proof is identical. $\square$

We do not know whether, for $p > 2$ not an even integer, that is $2k < p < 2k + 2$, we can take $q(p) = 2k$. We do not know either whether the statement is valid for functions with arbitrary large gaps.

3. Negative results in Wiener's problem

In the next theorems, $H^q(\mathbb{T})$ denotes the space of periodic distributions f whose negative coefficients are zero, and such that the function f_r are uniformly in $L^q(\mathbb{T})$ for $0 < r < 1$, where $f_r(t) := \sum_n \hat{f}(n)r^n e^{2i\pi nt}$.

Moreover, the norm (or quasi-norm) of f is given by $\|f\|_{H^q(\mathbb{T})}^q := \sup_{0 < r < 1} \int_0^1 |f_r|^q$. It is well known that, for $f \in H^q(\mathbb{T})$, the functions f_r have an a.e. limit f^* for r tending to 1. The function f^* , which we call the pointwise boundary value, belongs to $L^q(\mathbb{T})$. When $q \geq 1$, then f is the distribution defined by f^* , and $H^q(\mathbb{T})$ coincides with the subspace of functions in $L^q(\mathbb{T})$ whose negative coefficients are zero. In all cases the space $H^q(\mathbb{T})$ identifies with the classical Hardy space when identifying the distribution f with the holomorphic function $\sum_{n \geq 0} \hat{f}(n)z^n$ on the unit disc. This explains the use of the term of boundary value.

Theorem 8. Let $0 < p < \infty$, and $p \notin 2\mathbb{N}$. Then for any symmetric, measurable set $E \subset \mathbb{T}$ with $|E| > 0$ and any $q < p$, there exists a function f in the Hardy space $H^q(\mathbb{T})$, with positive Fourier coefficients, so that its pointwise boundary value f^* is in $L^p({}^c E)$ while $f^* \notin L^p(\mathbb{T})$. Moreover, f can be chosen with gaps tending to ∞ .

The key of the proof is Theorem 6. Observe that we can assume that $p > q > 1$. Indeed, f^ℓ is a positive definite function when f is, and counter-examples for some $p > 1$ will lead to counter-examples for p/ℓ . We do not give the details of the proof, which is analogous to the one of the next theorem.

Using Theorem 7 instead of Theorem 6, we have the following:

Theorem 9.

- (i) Let $p > 2$, with $p \notin 2\mathbb{N}$, and let $\ell \in \mathbb{N}$ such that $2\ell < p < 2(\ell + 1)$. Then, for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and $q > q(p)$, there exists a positive definite function $f \in L^{2\ell}(\mathbb{T})$, whose negative coefficients are zero, such that $f \notin L^q(\mathbb{T})$ while f is in $L^p({}^cU)$.
- (ii) Let $0 < p < 2$. Then for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and any $s < q < p$, there exists a function f in the Hardy space $H^s(\mathbb{T})$ with non negative Fourier coefficients, so that $f \notin H^q(\mathbb{T})$ while f^* is in $L^p({}^cU)$.

Proof. Let us first prove (i). We can assume that cU contains a neighborhood of 0. So, by Wiener's property, if f is integrable and belongs to $L^p({}^cU)$, then f is in $L^{2\ell}(\mathbb{T})$. Let us prove that there exists such a function, whose Fourier coefficients satisfy the required properties, and which does not belong to $L^q(\mathbb{T})$. By using Theorem 7, we can find positive definite polynomials f_k such that $\|f_k\|_q = 2^{k/2} \rightarrow \infty$, while $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k/2}$ with $U_k \subset U$, so that $\sum \|f_k\|_{L^p({}^cU)} < \infty$. Moreover, we may choose the U_k disjoint and such that $|U_k| < 2^{-\alpha_k}$, with $\alpha(1 - 1/q) = 1$. Then, using Hölder's Inequality, we obtain

$$\int_{\mathbb{T}} |f_k| \leq 2^{-\alpha(1-1/q)k} \left(\int_{U_k} |f_k|^q \right)^{1/q} + \left(\int_{\mathbb{T} \setminus U_k} |f_k|^p \right)^{1/p} \leq 2 \times 2^{-k} \left(\int_{\mathbb{T}} |f_k|^q \right)^{1/q},$$

so that $\sum \|f_k\|_1 < \infty$. The function $f := \sum_{k \geq 1} e_{m_k} f_k$ has the required properties. Indeed,

$$\|f\|_{H^q(\mathbb{T})} \geq \|f^*\|_{L^q(U_k)} \geq \|f_k\|_q - \sum_j \|f_j\|_{L^q({}^cU_j)} \geq 2^{\frac{k}{2}} - \sum_{j>0} 2^{-\frac{j}{2}},$$

from which we conclude the fact that f is not in $H^q(\mathbb{T})$.

Let us now consider $1 < p < 2$ and $q < 1$, from which we conclude for (ii). We proceed as before, with f_k 's given by Theorem 7, such that $\|f_k\|_q = 2^{k/2}$ and $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k/2}$. The U_k 's are assumed to be disjoint and of small measure, so that $\sum_k \|f_k\|_{H^s}^s < \infty$. It follows that $f \in H^s(\mathbb{T})$. The proof follows the same lines, even if f is not a function, in general, but a distribution. $\square$

Remark 10. As Wainger in [13], we can prove a little more: the function f may be chosen such that $\sup_{r<1} |f_r|$ is in $L^p({}^cU)$. Let us give the proof in the case (i). We can assume that U may be written as $I \cup (-I)$ for some interval I . Let J be the interval of same center and length half, and take f constructed as wished, but for the open set $J \cup (-J)$. Finally, write $f = \phi + \psi$, with $\phi := f \chi_{c(J \cup (-J))}$. Then using the maximal theorem we know that $\sup_{r<1} |\phi_r| \in L^p(\mathbb{T})$, while the Poisson kernel $P_t(x - y)$ is uniformly bounded for $x \notin U$ and $y \in J \cup (-J)$, so that $\sup_{r<1} |\psi_r|$ is uniformly bounded outside U .

In the case (ii), we use the fact that derivatives of the Poisson kernel $P_t(x - y)$ are also uniformly bounded.

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