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Equivalence of A-approximate continuity for self-adjoint expansive linear maps

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Abstract

Let $A\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $d \geqslant 1$, be an expansive linear map. The notion of A-approximate continuity was recently used to give a characterization of scaling functions in a multiresolution analysis (MRA). The definition of A-approximate continuity at a point $\mathbf{x}$ – or, equivalently, the definition of the family of sets having $\mathbf{x}$ as point of A-density – depend on the expansive linear map A. The aim of the present paper is to characterize those self-adjoint expansive linear maps $A_1, A_2\colon \mathbb{R}^d \to \mathbb{R}^d$ for which the respective concepts of A_μ-approximate continuity ($\mu = 1, 2$) coincide. These we apply to analyze the equivalence among dilation matrices for a construction of systems of MRA. In particular, we give a full description for the equivalence class of the dyadic dilation matrix among all self-adjoint expansive maps. If the so-called "four exponentials conjecture" of algebraic number theory holds true, then a similar full description follows even for general self-adjoint expansive linear maps, too.

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1. Introduction

A multiresolution analysis (MRA) is a general method introduced by Mallat [16] and Meyer [17] for constructing wavelets. On $\mathbb{R}^d$ ($d \geqslant 1$) equipped with the Euclidean norm $\|\cdot\|$, an MRA means a sequence of closed subspaces V_j, $j \in \mathbb{Z}$ of the Hilbert space $L^2(\mathbb{R}^d)$ that satisfies the following conditions:

(i) $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$;
(ii) $\forall j \in \mathbb{Z},\ f(\mathbf{x}) \in V_j \Leftrightarrow f(2\mathbf{x}) \in V_{j+1}$;
(iii) $W = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
(iv) there exists a *scaling function* $\phi \in V_0$, such that $\{\phi(\mathbf{x}-\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthonormal basis for V_0.

We could consider MRA in a general context, where instead of the dyadic dilation one considers a fixed linear map $A: \mathbb{R}^d \to \mathbb{R}^d$ such that A is an expansive map, i.e. all (complex) eigenvalues have absolute value greater than 1, and

$$A(\mathbb{Z}^d) \subset \mathbb{Z}^d, \tag{1}$$

i.e. the corresponding matrix of A with respect to the canonical basis has every entries belonging to $\mathbb{Z}$. Given such a linear map A one defines an A-MRA as a sequence of subspaces V_j, $j \in \mathbb{Z}$ of the Hilbert space $L^2(\mathbb{R}^d)$ (see [15,10,21,24]) that satisfies the conditions (i), (iii), (iv) and

(ii_1) $\forall j \in \mathbb{Z},\ f(\mathbf{x}) \in V_j \Leftrightarrow f(A\mathbf{x}) \in V_{j+1}$.

A characterization of scaling functions in a multiresolution analysis in a general context was given in [2], where the notion of A-approximate continuity is introduced as a generalization of the notion of approximate continuity.

In this work $|G|_d$ denotes the d-dimensional Lebesgue measure of the set $G \subset \mathbb{R}^d$, and $B_r := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| < r\}$ stands for the ball of radius r with the center in the origin. Also, we write $F + \mathbf{x}_0 = \{\mathbf{y} + \mathbf{x}_0 : \mathbf{y} \in F\}$ for any $F \subset \mathbb{R}^d$, $\mathbf{x}_0 \in \mathbb{R}^d$.

Definition 1. Let an expansive linear map $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be given. It is said that $\mathbf{x}_0 \in \mathbb{R}^d$ is a point of A-density for a measurable set $E \subset \mathbb{R}^d$, $|E|_d > 0$ if for all $r > 0$

$$\lim_{j\longrightarrow\infty} \frac{|E \cap (A^{-j}B_r + \mathbf{x}_0)|_d}{|A^{-j}B_r + \mathbf{x}_0|_d} = 1. \tag{2}$$

Given an expansive linear map $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$, and given $\mathbf{x}_0 \in \mathbb{R}^d$, we define the *family of A-dense sets at* $\mathbf{x}_0$ *as*

$$\mathscr{E}_A(\mathbf{x}_0) = \{E \subset \mathbb{R}^d \text{ measurable set}: \mathbf{x}_0 \text{ is a point of } A-\text{density for } E\}.$$

Furthermore, we will write $\mathscr{E}_A$ when $\mathbf{x}_0$ is the origin.

Definition 2. Let $A: \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map and let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be a measurable function. It is said that $\mathbf{x}_0 \in \mathbb{R}^n$ is a point of A-approximate continuity of the function f if there exists a measurable set $E \subset \mathbb{R}^n$, $|E|_n > 0$, such that $\mathbf{x}_0$ is a point of A-density for the set E and

$$\lim_{\mathbf{x}\to\mathbf{x}_0, \mathbf{x}\in E} f(\mathbf{x}) = f(\mathbf{x}_0). \tag{3}$$

The relation between the behavior of the Fourier transform $\hat{\phi}$ of the scaling function ϕ in the neighborhood of the origin and the condition (iii) is described in the following theorem of [2].

Theorem A. *Let V_j be a sequence of closed subspaces in $L^2(\mathbb{R}^d)$ satisfying the conditions* (i), (ii$_1$) *and* (iv). *Then the following conditions are equivalent*:

(A) $W = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
(B) *Setting* $|\hat{\phi}(\mathbf{0})| = 1$, *the origin is a point of A^*-approximate continuity of the function* $|\hat{\phi}|$.

As it was observed in [2, Remark 5, p. 1016], that the definition of points of A-approximate continuity depends of the expansive linear map A.

The aim of the present paper is to study the following problem:

Problem 1. Characterize those expansive linear maps $A_1, A_2 \colon \mathbb{R}^d \to \mathbb{R}^d$ for which the concept of A_1-approximate continuity coincides with the concept of A_2-approximate continuity.

Remark 1. From the definition of point of A-approximate continuity of a measurable function on $\mathbb{R}^d$, it is easy to see that given $\mathbf{x}_0 \in \mathbb{R}^d$ and given a measurable set $E \subset \mathbb{R}^d$, the point $\mathbf{x}_0$ is a point of A-approximate continuity for the function

$$f(\mathbf{x}) = \chi_{E\cup\{\mathbf{x}_0\}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in E \bigcup \{\mathbf{x}_0\}, \\ 0 & \text{if } \mathbf{x} \notin E, \end{cases}$$

if and only if $E \in \mathscr{E}(\mathbf{x}_0)$. Therefore, it suffices to study the notion of A-density, and once $\mathscr{E}_{A_1}(\mathbf{x}_0) = \mathscr{E}_{A_2}(\mathbf{x}_0)$, also the notions of A_1-approximate continuity and A_2-approximate continuity coincide.

Moreover, clearly, $E \in \mathscr{E}_A$ if and only if $E + \mathbf{x}_0 \in \mathscr{E}_A(\mathbf{x}_0)$.

Thus, we can simplify Problem 1 in the following way.

Problem 1*: Describe under what conditions on two expansive linear maps $A_1, A_2 \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$, we have that $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$.

In Corollary 20, we solve the problem for *expansive self-adjoint linear maps* on $\mathbb{R}^d$, without the extra condition (1). In the last section we discuss the additional, essentially number theoretical restrictions, brought into play by condition (1).

Characterization of expansive matrices satisfying (1) have been studied by several authors.

In [14], a complete classification for expanding 2×2-matrices satisfying (1) and $|\det M| = 2$ is given. Their result is the following.

Call two integer matrices A and M *integrally equivalent* if there exists an integer unimodular matrix C such that $C^{-1}AC = M$. Now, denote

$$A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}.$$

Lemma B. *Let M be an expanding 2×2-matrix satisfying (1). If* $\det M = -2$ *then M is integrally equivalent to A_1. If* $\det M = 2$ *then M is integrally equivalent to one of the following matrices*: $A_2, \pm A_3, \pm A_4$.

On the other side, a complete characterization for expanding 2×2-matrices satisfying (1) and

$$M^l = nI \quad \text{for some } l, n \in \mathbb{N} \tag{4}$$

is given in [6]. Their answer is given in the following theorem where they do not write the trivial case that M is a diagonal matrix.

Theorem C. *Given $l, n \in \mathbb{N}$, an expanding 2×2-matrix which satisfy (1) and (4) exists if and only if there exist two numbers $\lambda_1 \neq \lambda_2$ whose sum and product are integral and satisfy $\lambda_1^l = \lambda_2^l = n$. Furthermore, then λ_1 and λ_2 are the eigenvalues of the matrix M in (4).*

The following corollary is a classification of the expanding matrix M satisfying (1), $\det M < 0$ and (4).

Corollary D. *Let M be an expanding 2×2-matrix satisfying (1) with $\det M < 0$. Then M satisfies condition (4) if and only if trace $M = 0$ and $\det M = -n$. Especially*

$$M^2 = nI.$$

Moreover, they give a classification of the expanding matrix M satisfying (1), $\det M > 0$ and (4). They write the following theorem with the restriction to the case that the eigenvalues are complex numbers because in other way, by (4), M is a diagonal matrix. Here, l will denote the minimal index for which (4) holds, i.e. they neglect the trivial cases generated by powers of (4).

Theorem E. *Let M be an expanding 2×2-matrix satisfying (1) with $\det M > 0$ and eigenvalues $\lambda_i \notin \mathbb{R}$, $i = 1, 2$. Then condition (4) can only hold for $l = 3, 4, 6, 8$ and 12. These cases can be classified as follows:*

$$\begin{aligned}
M^3 &= nI \quad \textit{if and only if trace } M = -n^{1/3} \textit{ and } \det M = n^{2/3},\\
M^4 &= nI \quad \textit{if and only if trace} M = 0 \textit{ and } \det M = n^{1/2},\\
M^6 &= nI \quad \textit{if and only if trace } M = n^{1/6} \textit{ and } \det M = n^{1/3},\\
M^8 &= nI \quad \textit{if and only if } (\textit{trace } M)^2 = 2n^{1/4} \textit{ and } \det M = n^{1/4},\\
M^{12} &= nI \quad \textit{if and only if } (\textit{trace } M)^2 = 3n^{1/6} \textit{ and } \det M = n^{1/6}.
\end{aligned}$$

Moreover, the following factorization of expanding integer matrices for some particular cases appears in [6].

Theorem F. *Every expanding 2×2-matrix M satisfying (1) with $\det M = -2^s$, $s \in \{1, 2, 3\}$, which satisfies (4) possesses a factorization*

$$M = ADPA^{-1},$$

where A is a unimodular matrix, P a permutation matrix – that is, $(P\mathbf{x})_i = x_{\pi_i}$ for some permutation π of $\{1, \ldots, d\}$ and for all $\mathbf{x} \in \mathbb{R}^d$, – and D is a diagonal matrix with entries $d_i \in \mathbb{Z}$ along the diagonal satisfying $|d_i d_{\pi(i)} \cdots d_{\pi^{d-1}(i)}| > 1$ for each $i = 1, \ldots, d$.

Furthermore, in [5], the following lemma is proved.

Lemma G. *Suppose that M is an expanding $d \times d$-matrix satisfying (1) with the property*

$$M^d = \pm 2I.$$

If there exists a representative $e \in \mathbb{Z}^d/M\mathbb{Z}^d$, so that the matrix $(e, Me, \ldots, M^{d-1}e)$ is unimodular, then M possesses the factorization

$$M = AS\Pi A^{-1},$$

where $A \in SL(d, \mathbb{Z})$, $S = \operatorname{diag}(\pm 2, \pm 1, \ldots, \pm 1)$ and Π is an irreducible permutation matrix.

If M is an expanding $d \times d$-matrix satisfying (1) with the property $M^d = 2I$, a family of compactly supported pairs of dual wavelet frames have been constructed from interpolating scaling functions in arbitrary dimensions with arbitrarily high smoothness satisfying many optimality conditions (see [9]). Some results for interpolating scaling functions, although not that closely related to our topic, can also be found in [7,8]; in particular, the examples in [8] cover MRA even for some non-self-adjoint matrices, like the quincunx matrix.

2. Basic notions

As a general reference regarding linear algebra, we refer to [11,13]. For further use, and to fix notation, let us briefly cover some basic facts.

Given $r > 0$, we denote $Q_r = \{\mathbf{x} \in \mathbb{R}^d : |x_i| < r \ \forall i = 1, \ldots, d\}$ the cube of side length $2r$ with the center in the origin.

Given a map A, we write $d_A = |\det A|$. If A is a matrix of an expansive linear map, then obviously $d_A > 1$. The volume of any measurable set S changes under A according to $|AS|_d = d_A|S|_d$.

A subspace $W \subset \mathbb{R}^d$ is called an *invariant subspace* under A if $AW \subset W$. As is usual, $W^\perp$ is called the *orthogonal complement* of W with respect to the canonical inner product on $\mathbb{R}^d$. The orthogonal projection of $\mathbf{w}$ onto W is $P_W(\mathbf{w}) := \mathbf{u}$.

Let W_1, W_2 be vector spaces then $W_1 \oplus W_2$ is the direct sum of W_1 and W_2. If $A_\mu : W_\mu \longrightarrow W_\mu$, $\mu = 1, 2$ are linear maps, then we denote by $A_1 \otimes A_2$ the map on $W_1 \oplus W_2$ defined for any $\mathbf{w}_\mu \in W_\mu$, $\mu = 1, 2$ as $(A_1 \otimes A_2)(\mathbf{w}_1 + \mathbf{w}_2) = A_1\mathbf{w}_1 + A_2\mathbf{w}_2$.

If W is an Euclidean space and $A: W \longrightarrow W$ is a linear map, then A^* will be the *adjoint* of A. A is a self-adjoint map if $A = A^*$. Let $A_1, A_2 : W \longrightarrow W$ be two linear maps. A_1 and A_2 are said to be *simultaneously diagonalizable* (see [13, p. 177]) if there exists a basis $\mathbf{u}_1, \ldots, \mathbf{u}_d$ of W such that $\mathbf{u}_l$, $l = 1, \ldots, d$, are eigenvectors of both A_1 and A_2.

The *Spectral Theorem* for self-adjoint maps (see [11, Theorem 1, p. 156]) tells us that for any self-adjoint linear map A on $\mathbb{R}^d$, if $\beta_1 < \cdots < \beta_k$ are all the *distinct* eigenvalues of A with respective multiplicities $m_1, \ldots, m_k$, then for each $i = 1, \ldots, k$, there exists an orthonormal basis

$$\mathbf{u}_{m_0+\cdots+m_{i-1}+1}, \ldots, \mathbf{u}_{m_0+\cdots+m_{i-1}+m_i},$$

where $m_0 = 0$, for the subspace U_i of all eigenvectors associated with the eigenvalue β_i, moreover, then $\mathbb{R}^d = U_1 \oplus \cdots \oplus U_k$ with

$$U_i = [\mathbf{u}_{m_0+\cdots+m_{i-1}+1}, \ldots, \mathbf{u}_{m_0+\cdots+m_{i-1}+m_i}]$$

being mutually orthogonal, invariant subspaces. Furthermore, we can then write $A = \otimes_{i=1}^k A_i$ where $A_i := A|_{U_i}$ are homothetic transformations $\mathbf{x} \to \beta_i\mathbf{x}$, $\forall \mathbf{x} \in U_i$, $i = 1, \ldots, k$.

For a general linear map M on $\mathbb{R}^d$, one can similarly find a decomposition $\mathbb{R}^d = U_1 \oplus \cdots \oplus U_k$ of invariant subspaces, which, however, is not necessarily be an orthogonal decomposition, see [11, Theorem 2, p. 113].

Recall that a linear map $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called *positive* map if it is self-adjoint and all its (necessarily real) eigenvalues are also positive.

Let $J: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a positive map having a diagonal matrix J and let $\lambda_1, \ldots, \lambda_d \in [0, \infty)$ be the elements in the diagonal. Then, if $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear map such that $A = CJC^{-1}$ where C is a $d \times d$ invertible matrix, then the powers A^t, $t \in \mathbb{R}$ are defined as $A^t = CJ^tC^{-1}$, where J^t is a diagonal matrix with elements $\lambda_1^t, \ldots, \lambda_d^t$ in the diagonal.

3. Properties of sets having 0 as a point of A-density

The next monotonicity property is clear.

Proposition 3. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an expansive linear map. Let $E, F \subset \mathbb{R}^d$ be measurable sets such that $E \subset F$ and $E \in \mathscr{E}_A$. Then $F \in \mathscr{E}_A$.*

In the following propositions we give different equivalent conditions for the origin to be a point of A-density for a measurable set $E \subset \mathbb{R}^d$. Put $E^c := \mathbb{R}^d \setminus E$.

Proposition 4. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an expansive linear map. Let $E \subset \mathbb{R}^d$ be a measurable set. Then for any $r > 0$ the following four conditions are equivalent:*

$$\text{(i)} \quad \lim_{j \longrightarrow \infty} \frac{|E \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} = 1; \tag{5}$$

$$\text{(ii)} \quad \lim_{j \longrightarrow \infty} |A^j E \cap B_r|_d = |B_r|_d; \tag{6}$$

$$\text{(iii)} \quad \lim_{j \longrightarrow \infty} \frac{|E^c \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} = 0; \tag{7}$$

$$\text{(iv)} \quad \lim_{j \longrightarrow \infty} |A^j E^c \cap B_r|_d = 0. \tag{8}$$

Proof. (i) $\Longleftrightarrow$ (ii) This is a direct consequence of the fact that for any $r > 0$, and for any $j \in \mathbb{N}$

$$\frac{|E \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} = \frac{|A^j E \cap B_r|_d}{|B_r|_d}.$$

(i) $\Longleftrightarrow$ (iii) Obviously, for any $r > 0$, and for any $j \in \mathbb{N}$

$$1 = \frac{|E \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} + \frac{|E^c \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d}.$$

(iii) $\Longleftrightarrow$ (iv) This follows since for any $r > 0$, and for any $j \in \mathbb{N}$

$$\frac{|E^c \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} = \frac{|A^j E^c \cap B_r|_d}{|B_r|_d}. \qquad \square$$

Corollary 5. *In order to $E \in \mathscr{E}_A$, the validity of any of the above conditions* (i)–(iv), *but required for all $r > 0$, are necessary and sufficient.*

Two sets are termed *essentially disjoint*, if their intersection is of measure zero.

Corollary 6. *For any expansive map A and two sets $E, F \subset \mathbb{R}^d$, which are essentially disjoint, at most one of the sets can belong to $\mathscr{E}_A$.*

Proof. Assume, e.g. $E \in \mathscr{E}_A$. Note that $F \in \mathscr{E}_A$ if and only if $\widetilde{F} := F \setminus (E \cap F) \in \mathscr{E}_A$, since deleting the measure zero intersection does not change the measures, hence neither the limits in the definition of $\mathscr{E}_A$. But $\widetilde{F} \subset E^c$, and $E \in \mathscr{E}_A$ entails that the limits (iii) and (iv) in Proposition 4 are zero, hence $E^c \notin \mathscr{E}_A$. Obviously (or by the monotonicity formulated in Proposition 3), then neither $\widetilde{F} \subset E^c$ can belong to $\mathscr{E}_A$. Whence $F \notin \mathscr{E}_A$. □

Proposition 7. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an expansive linear map. Let $E \subset \mathbb{R}^d$ be a measurable set and assume that for a certain $r_0 > 0$ some (and hence all) of conditions* (i)–(iv) *of Proposition 4 are satisfied. Then $E \in \mathscr{E}_A$. Conversely, if for any $r_0 > 0$ any of the conditions* (i)–(iv) *of Proposition 4 fails, then $E \notin \mathscr{E}_A$.*

Proof. Let $r \in \mathbb{R}$ and $0 < r < r_0$, and let $j \in \mathbb{N} \setminus \{0\}$, then $A^{-j}B_r \subset A^{-j}B_{r_0}$, hence by condition (iii)

$$\frac{|E^c \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} \leqslant \left(\frac{r_0}{r}\right)^d \frac{|E^c \cap A^{-j}B_{r_0}|_d}{|A^{-j}B_{r_0}|_d} \longrightarrow 0, \quad \text{when } j \longrightarrow +\infty.$$

Now let $r \in \mathbb{R}$ and $r > r_0$, and let $j \in \mathbb{N} \setminus \{0\}$. As the map A is an expansive map, $\exists m = m(r) \in \mathbb{N}$ such that $B_r \subset A^m B_{r_0}$. Then similarly to the above

$$\frac{|E^c \cap A^{-j}B_r|_d}{|A^{-j}B_r|_d} \leqslant d_A^m \frac{|E^c \cap A^{-j+m}B_{r_0}|_d}{|A^{-j+m}B_{r_0}|_d} \longrightarrow 0, \quad \text{when } j \longrightarrow +\infty. \qquad \square$$

Proposition 8. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an expansive linear map, and let $E \subset \mathbb{R}^d$ be a measurable set. Assume that $K \subset \mathbb{R}^d$ is another measurable set, and that there exist r_1, r_2 where $0 < r_1 < r_2 < \infty$ such that $B_{r_1} \subset K \subset B_{r_2}$. Then $E \in \mathscr{E}_A$ if and only if*

$$\lim_{j \longrightarrow \infty} \frac{|E \cap A^{-j}K|_d}{|A^{-j}K|_d} = 1 \tag{9}$$

or equivalently

$$\lim_{j \longrightarrow \infty} \frac{|E^c \cap A^{-j}K|_d}{|A^{-j}K|_d} = 0. \tag{10}$$

Proof. ($\Longrightarrow$) In view of the condition $B_{r_1} \subset K \subset B_{r_2}$ we have

$$\begin{aligned}\limsup_{j \longrightarrow \infty} \frac{|E^c \cap A^{-j}K|_d}{|A^{-j}K|_d} &\leqslant \limsup_{j \longrightarrow \infty} \frac{|E^c \cap A^{-j}B_{r_2}|_d}{|A^{-j}B_{r_1}|_d} \\ &\leqslant \left(\frac{r_2}{r_1}\right)^d \lim_{j \longrightarrow \infty} \frac{|E^c \cap A^{-j}B_{r_2}|_d}{|A^{-j}B_{r_2}|_d} = 0,\end{aligned}$$

because $E \in \mathscr{E}_A$.

($\Longleftarrow$) Again, by assumption we have

$$\limsup_{j\longrightarrow\infty}\frac{|E^c\cap A^{-j}B_{r_1}|_d}{|A^{-j}B_{r_1}|_d}\leqslant\left(\frac{r_2}{r_1}\right)^d\lim_{j\longrightarrow\infty}\frac{|E^c\cap A^{-j}K|_d}{|A^{-j}K|_d}=0,$$

using now (10). Finally, Proposition 7 tells us that the origin is a point of A-density for E. □

Lemma 9. *Let $A\colon\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be an expansive linear map. Assume that $Y\subset\mathbb{R}^d$, $Y\cong\mathbb{R}^p$, $1\leqslant p<d$, is an invariant subspace under A, and that also $Y^\perp$ is an invariant subspace under A. Let $E\subset\mathbb{R}^d$ be a measurable set of the form $E:=Y+F$, where $F\subset Y^\perp$. Then $E\in\mathscr{E}_A$ if and only if $F\in\mathscr{E}_{A|_{Y^\perp}}$.*

Proof. As $\mathbb{R}^d=Y\oplus Y^\perp$, and moreover, the subspaces Y and $Y^\perp$ are invariant subspaces under A, we can write $A=A|_Y\otimes A|_{Y^\perp}$.

We put $K:=K_1+K_2$, where $K_1:=\{\mathbf{y}\in Y:\|\mathbf{y}\|\leqslant 1\}$, and $K_2:=\{\mathbf{y}\in Y^\perp:\|\mathbf{y}\|\leqslant 1\}$. Observe that K satisfies the conditions of Proposition 8.

With this notation, given $j\in\mathbb{N}$ we arrive at

$$\begin{aligned}E\cap A^{-j}K&=(Y+F)\cap(A|_Y\otimes A|_{Y^\perp})^{-j}(K_1+K_2)\\&=(Y+F)\cap((A|_Y)^{-j}K_1+(A|_{Y^\perp})^{-j}K_2)\\&=(A|_Y)^{-j}K_1+(F\cap(A|_{Y^\perp})^{-j}K_2).\end{aligned}$$

As the summands are subsets of Y and $Y^\perp$, respectively, this last sum is also a direct sum. Hence we are led to

$$\frac{|E\cap(A^{-j}K)|_d}{|A^{-j}K|_d}=\frac{|F\cap(A|_{Y^\perp})^{-j}K_2|_{d-p}}{|(A|_{Y^\perp})^{-j}K_2|_{d-p}}.$$

Taking limits and applying Proposition 8 we conclude the proof. □

Lemma 10. *Let $A_1,A_2\colon\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be expansive linear maps and assume that $W\subset\mathbb{R}^d$ is a subspace of $\mathbb{R}^d$ such that both W and $W^\perp$ are invariant subspaces under both A_1 and A_2. If $\mathscr{E}_{A_1}=\mathscr{E}_{A_2}$ then $\mathscr{E}_{A_1|W}=\mathscr{E}_{A_2|W}$.*

Proof. We consider the cylindrical sets $E=F+W^\perp$. According to Lemma 9 we know that $E\in\mathscr{E}_{A_\mu}\Longleftrightarrow F\in\mathscr{E}_{A_\mu|W}$, $\mu=1,2$. Therefore, the lemma follows. □

Lemma 11. *Let $A,A'\colon\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be expansive linear maps and suppose that there is a linear map $C\colon\mathbb{R}^d\longrightarrow\mathbb{R}^d$ with $d_C>0$, such that $A'=C^{-1}AC$. Moreover, let $E\subset\mathbb{R}^d$, $|E|_d>0$, be a measurable set. Then $E\in\mathscr{E}_A$ if and only if $C^{-1}E\in\mathscr{E}_{A'}$, i.e. $\mathscr{E}_A=C\mathscr{E}_{A'}$.*

Proof. A, A', C and C^{-1} are invertible linear maps, thus we have that for any $j\in\mathbb{N}\setminus\{0\}$

$$\frac{|(C^{-1}E)^c\cap A'^{-j}B_1|_d}{|A'^{-j}B_1|_d}=\frac{|E^c\cap A^{-j}CB_1|_d}{|A^{-j}CB_1|_d}.\tag{11}$$

Moreover, as C is an invertible linear map, there exists $0<r_1<r_2<\infty$ such that $B_{r_1}\subset CB_1\subset B_{r_2}$. Therefore, the statement follows from (11) and Proposition 8. □

A direct consequence of Lemma 11 is the following corollary.

Corollary 12. *Let $A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be simultaneously diagonalizable expansive linear maps. If $|\lambda_i^{(1)}| = |\lambda_i^{(2)}|, i = 1, \ldots, d$, where $\lambda_i^{(\mu)}, \mu = 1, 2, i = 1, \ldots, d$ are the eigenvalues of $A_\mu, \mu = 1, 2$, then*

$$\mathscr{E}_{A_1} = \mathscr{E}_{A_2}.$$

Proof. As A_1 and A_2 are simultaneously diagonalizable, there exists a linear map $C: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with $d_C > 0$, such that $A_\mu = C^{-1} J_\mu C, \mu = 1, 2$. From Lemma 11, we know that

$$\mathscr{E}_{A_1} = \mathscr{E}_{A_2} \Longleftrightarrow \mathscr{E}_{J_1} = \mathscr{E}_{J_2}.$$

Finally, $\mathscr{E}_{J_1} = \mathscr{E}_{J_2}$ is true because from $|\lambda_i^{(1)}| = |\lambda_i^{(2)}|, i = 1, \ldots, d$ it follows that for any $j \in \mathbb{Z}$ and for any $r > 0$ we have $J_1^j B_r = J_2^j B_r$. □

4. Some particular cases

Lemma 13. *Let $A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diagonal, positive, expansive linear map with the corresponding matrix*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}, \; 1 < \lambda_1, \lambda_2.$$

Let for any $\alpha > 0$ $E_\alpha \subset \mathbb{R}^2$ be the set

$$E_\alpha = \{(x_1, x_2) \in \mathbb{R}^2: |x_2| \geqslant |x_1|^\alpha\}.$$

Denote $\alpha_{1,2} := \alpha_{1,2}(\lambda_1, \lambda_2) := \log \lambda_2 / \log \lambda_1$. Then $E_\alpha \in \mathscr{E}_A$ if and only if $\alpha > \alpha_{1,2}$.

Proof. For any $j \in \mathbb{N} \setminus \{0\}$, and because of the symmetry of the sets E_α^c and $A^{-j} Q_1$

$$|E_\alpha^c \cap A^{-j} Q_1|_2 = 4 \int_0^{\lambda_1^{-j}} \int_0^{\lambda_2^{-j}} \mathbf{1}_{\{x_2 < x_1^\alpha\}} \, \mathrm{d}x_2 \, \mathrm{d}x_1 = 4 \int_0^{\lambda_1^{-j}} \min(x_1^\alpha, \lambda_2^{-j}) \mathrm{d}x_1 \tag{12}$$

for any value of $\alpha > 0$. Let us consider first the boundary case $\alpha = \alpha_{1,2}$. Then $x_1^{\alpha_{1,2}} \leqslant \lambda_1^{-j\alpha_{1,2}} = \lambda_2^{-j}$, hence the minimum is just $x_1^{\alpha_{1,2}}$, and we get

$$|E_{\alpha_{1,2}}^c \cap A^{-j} Q_1|_2 = 4 \int_0^{\lambda_1^{-j}} x_1^{\alpha_{1,2}} \, \mathrm{d}x_1 = 4 \frac{\lambda_1^{-j(\alpha_{1,2}+1)}}{\alpha_{1,2} + 1}.$$

Therefore

$$\frac{|E_{\alpha_{1,2}}^c \cap A^{-j} Q_1|_2}{|A^{-j} Q_1|_2} = \frac{\lambda_1^j \lambda_2^j}{\lambda_1^{j(\alpha_{1,2}+1)} (\alpha_{1,2} + 1)} = \frac{1}{\alpha_{1,2} + 1} \left(\frac{\lambda_2}{\lambda_1^{\alpha_{1,2}}} \right)^j = \frac{1}{\alpha_{1,2} + 1}$$

in view of $\lambda_2 = \lambda_1^{\alpha_{1,2}}$. The quotient of the measures on the left being constant, obviously the limit is positive but less than 1, hence by Propositions 8 and 4 (i), (iii) neither $E_{\alpha_{1,2}}$, nor its complement $E_{\alpha_{1,2}}^c$ can belong to $\mathscr{E}_A$.

Note that when $\alpha > \alpha_{1,2}$, then $x_1^\alpha \leqslant x_1^{\alpha_{1,2}}$ (as $x_1 < 1$), hence in (12) the minimum is again x_1^α. Therefore, a very similar calculation as above yields

$$\lim_{j \longrightarrow \infty} \frac{|E_\alpha^c \cap A^{-j} Q_1|_2}{|A^{-j} Q_1|_2} = \frac{1}{\alpha + 1} \lim_{j \longrightarrow \infty} \left(\frac{\lambda_2}{\lambda_1^\alpha} \right)^j = 0,$$

because now we have $\lambda_2/\lambda_1^{\alpha} = \lambda_1^{\alpha_{1,2}-\alpha} < 1$. Whence Propositions 8 and 4 (iii) now gives $E_\alpha \in \mathscr{E}_A$.

Finally, let $\alpha < \alpha_{1,2}$. Observe that the coordinate changing isometry of $\mathbb{R}^2$ provides a symmetry for our subject: changing the role of the coordinates we can consider now $\widetilde{E}_\beta := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \geqslant |x_2|^\beta\}$. Then obviously $E_\alpha^c = \operatorname{int} \widetilde{E}_{1/\alpha} \subset \widetilde{E}_{1/\alpha}$, and $\widetilde{\alpha}_{1,2} = \alpha_{2,1} = \log \lambda_1 / \log \lambda_2 = 1/\alpha_{1,2}$, hence from the previous case and Proposition 3 we obtain $E_\alpha^c \in \mathscr{E}_A$. But then $E_\alpha \notin \mathscr{E}_A$. That finishes the proof of the Lemma. $\square$

Lemma 14. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a positive expansive linear map. With the notation in Section 2, given $\delta > 0$, we define the measurable set*

$$G_\delta := \{\mathbf{x} = \mathbf{y} + \mathbf{z} : \mathbf{y} \in U_1, \mathbf{z} \in U_1^\perp, \|\mathbf{z}\| < \delta \|\mathbf{y}\|\}$$
$$= \left\{ \mathbf{x} = \sum_{i=1}^k \mathbf{y}_i : \mathbf{y}_i \in U_i, i = 1, \ldots, k, \|\mathbf{y_2} + \cdots + \mathbf{y_k}\| < \delta \|\mathbf{y_1}\| \right\}.$$

Then in case $\dim U_1 < d$, i.e. when not all the eigenvalues are equal to β_1, we have $G_\delta \in \mathscr{E}_A$.

Proof. Clearly, $|B_1 \cap G_\delta|_d = \int_{B_1} \mathbf{1}_{G_\delta}\, d\mathbf{x}$, and $\mathbf{1}_{G_\delta} \longrightarrow \mathbf{1}_{\mathbb{R}^d}$ a.e. when $\delta \to \infty$, so by the Lebesgue dominated convergence theorem we conclude

$$\lim_{\delta \longrightarrow \infty} |B_1 \cap G_\delta|_d = |B_1|_d. \tag{13}$$

Next we prove that for any given $\delta > 0$, $G_{\frac{\beta_2}{\beta_1}\delta} \subset AG_\delta$. We can write AG_δ as

$$AG_\delta = \left\{ \sum_{i=1}^k \beta_i \mathbf{y}_i : \mathbf{y}_i \in U_i, i = 1, \ldots, k, \sum_{i=2}^k \|\mathbf{y}_i\|^2 < \delta^2 \|\mathbf{y}_1\|^2 \right\}$$
$$= \left\{ \sum_{i=1}^k \mathbf{z}_i : \mathbf{z}_i \in U_i, i = 1, \ldots, k, \sum_{i=2}^k \frac{1}{\beta_i^2} \|\mathbf{z}_i\|^2 < \delta^2 \frac{1}{\beta_1^2} \|\mathbf{z}_1\|^2 \right\}.$$

Let $\mathbf{x} \in G_{\frac{\beta_2}{\beta_1}\delta}$. Then

$$\mathbf{x} = \mathbf{z}_1 + \cdots + \mathbf{z}_k \text{ such that } \frac{1}{\beta_2^2}\|\mathbf{z}_2\|^2 + \cdots + \frac{1}{\beta_2^2}\|\mathbf{z}_k\|^2 < \frac{1}{\beta_1^2}\delta^2\|\mathbf{z}_1\|^2$$

and as $\beta_i > \beta_2, i = 3, \ldots, k$, then

$$\frac{1}{\beta_2^2}\|\mathbf{z}_2\|^2 + \cdots + \frac{1}{\beta_k^2}\|\mathbf{z}_k\|^2 \leqslant \frac{1}{\beta_2^2}\|\mathbf{z}_2\|^2 + \cdots + \frac{1}{\beta_2^2}\|\mathbf{z}_k\|^2 < \frac{1}{\beta_1^2}\delta^2\|\mathbf{z}_1\|^2.$$

Hence we arrive at $\mathbf{x} \in AG_\delta$ proving $G_{\frac{\beta_2}{\beta_1}\delta} \subset AG_\delta$ indeed. If now we iterate this and use (13), we infer

$$\lim_{j \longrightarrow \infty} |B_1 \cap A^j G_\delta|_d \geqslant \lim_{j \longrightarrow \infty} |B_1 \cap G_{(\frac{\beta_2}{\beta_1})^j \delta}|_d = |B_1|_d,$$

so by Propositions 4 (ii) and 7 we get $G_\delta \in \mathscr{E}_A$. $\square$

Lemma 15. *Let $A: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a positive expansive linear map, and similarly to Section 2 let the different eigenvalues be listed as $1 < \beta_1 < \cdots < \beta_k$, $U_1 \subset \mathbb{R}^d$ being the eigenspace* *belonging to* β_1. *Moreover, let* $V \subset U_1^{\perp}$ *be any subspace of* $\mathbb{R}^d$ *orthogonal to* U_1, *and write* $W := (U_1 \oplus V)^{\perp}$. *We finally set for any* $\delta > 0$

$$F_\delta = \{\mathbf{x} = \mathbf{u} + \mathbf{v} + \mathbf{w} \colon \mathbf{u} \in U_1, \mathbf{v} \in V, \mathbf{w} \in (U_1 \oplus V)^{\perp}, \|\mathbf{v}\| < \delta\|\mathbf{u}\|\}.$$

Then $F_\delta \in \mathscr{E}_A$.

Proof. We can combine Proposition 3 and Lemma 14, because G_δ is contained in F_δ. □

5. The main result

Theorem 16. *Let* $A_1, A_2 \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be positive expansive linear maps. Then* $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ *if and only if* $\exists t > 0$ *such that*

$$(A_1)^t = A_2.$$

For the proof of Theorem 16, we first settle the case of diagonal matrices in the following lemma. After that, we will apply the spectral theorem to prove even the general case.

Lemma 17. *Let* $A_1, A_2 \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be positive diagonal expansive linear maps with the corresponding matrices*

$$A_\mu = \begin{pmatrix} \lambda_1^{(\mu)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(\mu)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_d^{(\mu)} \end{pmatrix},$$

where $\lambda_i^{(\mu)} \in \mathbb{R}$, $1 < \lambda_1^{(\mu)} \leqslant \lambda_2^{(\mu)} \leqslant \cdots \leqslant \lambda_d^{(\mu)}$, *for* $\mu = 1, 2$. *Then* $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ *if and only if* $\exists t > 0$ *such that*

$$(A_1)^t = A_2.$$

Proof. ($\Rightarrow$) For an indirect proof, we assume that it is false that $\exists t > 0$ such that $(A_1)^t = A_2$. Then $\exists i, l \in \{1, \ldots, d\}, i < l$, such that $(\lambda_i^{(1)})^{t_1} = \lambda_i^{(2)}$ and $(\lambda_l^{(1)})^{t_2} = \lambda_l^{(2)}$ with $0 < t_1, t_2$ but $t_1 \neq t_2$, i.e.

$$t_1 = \frac{\ln \lambda_i^{(2)}}{\ln \lambda_i^{(1)}} \neq \frac{\ln \lambda_l^{(2)}}{\ln \lambda_l^{(1)}} = t_2$$

or equivalently

$$\alpha_1 := \frac{\ln \lambda_l^{(1)}}{\ln \lambda_i^{(1)}} \neq \alpha_2 := \frac{\ln \lambda_l^{(2)}}{\ln \lambda_i^{(2)}}.$$

Without loss of generality, we can assume that $(1 \leqslant)\ \alpha_1 < \alpha_2$. Let $\alpha > 0$ and let us define

$$F := \{(x_i, x_l) \in \mathbb{R}^2 \colon |x_l| \geqslant |x_i|^{\alpha}\}$$

and

$$E := \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \colon |x_l| \geqslant |x_i|^{\alpha}, x_j \in \mathbb{R} (j \neq i, l)\} \cong F \oplus \mathbb{R}^{d-2}.$$

Then Lemma 9 tells us that $E\in\mathscr{E}_{A_\mu}$, $\mu=1,2\Longleftrightarrow F\in\mathscr{E}_{M_\mu}$, $\mu=1,2$, where $M_1, M_2:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ are expansive linear maps with matrices

$$M_\mu=\begin{pmatrix}\lambda_i^{(\mu)} & 0\\ 0 & \lambda_l^{(\mu)}\end{pmatrix},\quad \mu=1,2.$$

However, making use of $\alpha_1<\alpha_2$, we can choose a value $\alpha_1<\alpha<\alpha_2$, and then Lemma 13 gives $F\in\mathscr{E}_{M_1}$ but $F\notin\mathscr{E}_{M_2}$, contradicting to the assumption $\mathscr{E}_{A_1}=\mathscr{E}_{A_2}$.

($\Leftarrow$) As $A_2=(A_1)^t$ if and only if $A_1=(A_2)^{1/t}$, it suffices to see that $\mathscr{E}_{A_1}\subset\mathscr{E}_{A_2}$. So let $E\in\mathscr{E}_{A_1}$.

Since A_1 is a positive, expansive diagonal mapping, obviously for any $0\leqslant s<1$ we have $B_1\subset(A_1)^sB_1\subset A_1B_1$. Now write, for any $j\in\mathbb{N}\setminus\{0\}$, the exponent tj as $tj=l_j-s_j$ with $l_j:=\lceil tj\rceil$, the least integer $\geqslant tj$, and $s_j:=\lceil tj\rceil-tj\in[0,1)$. So we have

$$\frac{|E^c\cap A_2^{-j}B_1|_d}{|A_2^{-j}B_1|_d}=\frac{|E^c\cap A_1^{-l_j+s_j}B_1|_d}{|A_1^{-l_j+s_j}B_1|_d}\leqslant d_{A_1}\frac{|E^c\cap A_1^{-l_j+1}B_1|_d}{|A_1^{-l_j+1}B_1|_d}.$$

Since $\{-l_j+1\}_{j\in\mathbb{N}}$ is an integer sequence and $-l_j+1\to-\infty$ when $j\to\infty$, by condition $E\in\mathscr{E}_{A_1}$, Proposition 4(iii) entails that the right-hand side converges to 0 with $j\to\infty$, whence

$$\lim_{j\longrightarrow\infty}\frac{|E^c\cap A_2^{-j}B_1|_d}{|A_2^{-j}B_1|_d}=0.$$

According to Proposition 7 this means $E\in\mathscr{E}_{A_2}$. $\square$

Lemma 18. *Let $A_1, A_2:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ be positive expansive linear maps such that $\mathscr{E}_{A_1}=\mathscr{E}_{A_2}$. Then* $\dim(U_1^{(1)}\cap U_1^{(2)})\geqslant 1$.

Proof. Assume the contrary, i.e. $U_1^{(1)}\cap U_1^{(2)}=\{\mathbf{0}\}$, hence $V:=U_1^{(1)}+U_1^{(2)}=U_1^{(1)}\oplus U_1^{(2)}$. Recall that by definition both $U_1^{(1)}$ and $U_1^{(2)}$ are of dimension at least one, and now $\dim U_1^{(1)}+\dim U_1^{(2)}=\dim V:=p\leqslant d$, hence now neither of them can have full dimension. Without loss of generality we can assume $V=\mathbb{R}^p$. First we work in V. Denote $V_\mu:=(U_1^{(\mu)})^\perp$ (the orthogonal complement understood within V) for $\mu=1,2$.

If S is the unit sphere of V, $S:=\{\mathbf{x}\in V:\|\mathbf{x}\|=1\}$, then by he indirect assumption also the traces $T_\mu:=S\cap U_1^{(\mu)}$ are disjoint for $\mu=1,2$. As these sets are compact, too, there is a positive distance $0<\rho:=\operatorname{dist}(T_1,T_2)\leqslant\sqrt{2}$ between them.

Let us fix some parameter $0<\kappa<1$, to be chosen later. Next we define the sets

$$K_\mu:=\{\mathbf{u}+\mathbf{v}:\mathbf{u}\in U_1^{(\mu)},\mathbf{v}\in V_\mu,\|\mathbf{v}\|\leqslant\kappa\|\mathbf{u}\|\}\quad(\mu=1,2).$$

We claim that these sets are essentially disjoint, more precisely $K_1\cap K_2=\{\mathbf{0}\}$, if κ is chosen appropriately. So let now $\mu=1$ or $\mu=2$ be fixed, and consider any $\mathbf{x}\in K_\mu$ with $\gamma:=\|\mathbf{x}\|\neq0$, i.e. $\mathbf{x}\in K_\mu\setminus\{\mathbf{0}\}$. From the representation of $\mathbf{x}$ as the sum of the orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$, we get $\|\mathbf{u}\|\leqslant\|\mathbf{x}\|=\sqrt{\|\mathbf{u}\|^2+\|\mathbf{v}\|^2}\leqslant\sqrt{\|\mathbf{u}\|^2+\kappa^2\|\mathbf{u}\|^2}=\sqrt{1+\kappa^2}\|\mathbf{u}\|$. We put $\beta:=\|\mathbf{u}\|$. Let now $\mathbf{y}:=(1/\gamma)\mathbf{x}\in S$ be the homothetic projection of $\mathbf{x}$ on S. Then

$$\begin{aligned}\operatorname{dist}(\mathbf{y},T_\mu)&\leqslant\left\|\mathbf{y}-\frac{1}{\beta}\mathbf{u}\right\|\leqslant\left\|\mathbf{y}-\frac{1}{\beta}\mathbf{x}\right\|+\left\|\frac{1}{\beta}\mathbf{x}-\frac{1}{\beta}\mathbf{u}\right\|\\&\leqslant\left|1-\frac{\gamma}{\beta}\right|+\frac{1}{\beta}\|\mathbf{v}\|\leqslant(\sqrt{1+\kappa^2}-1)+\kappa<2\kappa.\end{aligned}$$

Therefore, if we choose $\kappa < \rho/4$, then $\mathbf{y}$ falls in the $\rho/2$ neighborhood of T_μ, whence the homothetic projections $\mathbf{y}_\mu$ of elements $\mathbf{x}_\mu \in K_\mu$, $\mu = 1, 2$, can never coincide. But K_μ are cones, invariant under homothetic dilations, therefore, this also implies that $K_1 \cap K_2 \subset \{\mathbf{0}\}$, as we needed.

Let us write $W := (U_1^{(1)} \oplus U_1^{(2)})^\perp$. Now we consider the sets

$$\begin{aligned} H_\mu &:= K_\mu \oplus W = \{\mathbf{x} + \mathbf{w} : \mathbf{x} \in K_\mu, \mathbf{w} \in W\} \\ &= \{\mathbf{u} + \mathbf{v} + \mathbf{w} : \mathbf{u} \in U_1^{(\mu)}, \mathbf{v} \in V_\mu, \mathbf{w} \in W, \|\mathbf{v}\| \leqslant \kappa \|\mathbf{u}\|\} \quad (\mu = 1, 2), \end{aligned}$$

which are also essentially disjoint, as $H_1 \cap H_2 = W$ and $|W|_d = 0$ because $\dim W < d$. These sets are exactly of the form F_δ in Lemma 15, thus $H_\mu \in \mathscr{E}_{A_\mu}$ for $\mu = 1, 2$. It remains to recall Corollary 6, saying that essentially disjoint sets cannot simultaneously be elements of the same $\mathscr{E}_{A_\mu}$, that is, $H_1 \in \mathscr{E}_{A_1}$ but then $H_2 \notin \mathscr{E}_{A_1}$, and $H_2 \in \mathscr{E}_{A_2}$, but $H_2 \notin \mathscr{E}_{A_1}$. Here we arrived at a contradiction with $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$, which concludes our proof. □

Lemma 19. *Let $A_1, A_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be positive expansive linear maps such that $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$. Then A_1 and A_2 are simultaneously diagonalizable maps.*

Proof. We prove the lemma by induction with respect to the dimension. Obviously, the lowest dimensional case of $d = 1$ is true. Now let $d \geqslant 1$ and assume that for any two positive expansive linear maps $M_1, M_2 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\mathscr{E}_{M_1} = \mathscr{E}_{M_2}$, M_1 and M_2 are simultaneously diagonalizable. We will prove that the statement is true for dimension $d + 1$. Let $A_1, A_2 : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1}$ be positive expansive linear maps such that $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$. From Lemma 18 we know that there exists a one-dimensional subspace, say $[u]$, so that $[u] \subset U_1^{(1)} \cap U_1^{(2)}$.

As u is an eigenvector of the positive self-adjoint linear maps A_1 and A_2, $[u]$ is an invariant subspace of both A_1 and A_2, and we have that also $[u]^\perp$ is an invariant subspace under both A_1 and A_2. Hence from Lemma 10, we obtain that $\mathscr{E}_{M_1} = \mathscr{E}_{M_2}$ where $M_\mu := A_\mu|_{[u]^\perp}$, $\mu = 1, 2$. Then by hypothesis of induction we know that the positive expansive linear maps $M_1, M_2 : [u]^\perp \longrightarrow [u]^\perp$ are simultaneously diagonalizable maps. Furthermore, as we can write $A_\mu = A_\mu|_{[u]} \otimes M_\mu$, $\mu = 1, 2$, and $u \in [u]$ is an eigenvector of A_1 and A_2, we can conclude that A_1 and A_2 are simultaneously diagonalizable maps. □

Proof of Theorem 16 ($\Longleftarrow$) From the *spectral theorem* we know that there exists a linear map $C : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with $d_C > 0$, such that $A_1 = C J_1 C^{-1}$ where $J_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is an expansive linear map with corresponding matrix

$$J_1 = \begin{pmatrix} \lambda_1^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_d^{(1)} \end{pmatrix}, \quad \lambda_i^{(1)} \in \mathbb{R}, \quad 1 < \lambda_1^{(1)} \leqslant \lambda_2^{(1)} \leqslant \cdots \leqslant \lambda_d^{(1)}.$$

According to the condition $A_2 = A_1^t$ with $t > 0$, we can write the corresponding matrix of the map A_2 as $A_2 = C(J_1)^t C^{-1}$.

Lemma 17 tells us that $\mathscr{E}_{J_1} = \mathscr{E}_{(J_1)^t}$. Also we have the equivalence

$$\mathscr{E}_{J_1} = \mathscr{E}_{(J_1)^t} \Longleftrightarrow C\mathscr{E}_{J_1} = C\mathscr{E}_{(J_1)^t}.$$

Finally, Lemma 11 implies $C\mathscr{E}_{J_1} = \mathscr{E}_{A_1}$ and $C\mathscr{E}_{(J_1)^t} = \mathscr{E}_{A_2}$, hence

$$C\mathscr{E}_{J_1} = C\mathscr{E}_{(J_1)^t} \Longleftrightarrow \mathscr{E}_{A_1} = \mathscr{E}_{A_2}.$$

This concludes the proof of the ($\Longleftarrow$) direction.

($\Longrightarrow$) According to Lemma 19, there exists an orthonormal basis for $\mathbb{R}^d$, $\mathbf{u}_1, \dots, \mathbf{u}_d$, such that A_1 and A_2 have a common diagonal representation matrix C in this basis. More precisely, the linear map $C: \mathbb{R}^d \longrightarrow \mathbb{R}^d$, has matrix $C := (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d)$ (where $\mathbf{u}_l$, $l = 1, \dots, d$ are column vectors, formed from the common eigenvectors of A_1 and A_2), and we can write $A_\mu = C J_\mu C^{-1}$, $\mu = 1, 2$, where $J_\mu: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are expansive diagonal linear maps with the corresponding matrices being

$$J_\mu = \begin{pmatrix} \lambda_1^{(\mu)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(\mu)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_d^{(\mu)} \end{pmatrix}, \quad \lambda_i^{(\mu)} \in \mathbb{R}, \quad 1 < \lambda_1^{(\mu)} \leqslant \lambda_2^{(\mu)} \leqslant \cdots \leqslant \lambda_d^{(\mu)}.$$

Note that $d_C = 1 > 0$ because the orthogonality of the column vectors $\mathbf{u}_l$ $(l = 1, \dots, d)$. From Lemma 11, we get

$$\mathscr{E}_{A_1} = \mathscr{E}_{A_2} \Longleftrightarrow C\mathscr{E}_{J_1} = C\mathscr{E}_{J_2} \Longleftrightarrow \mathscr{E}_{J_1} = \mathscr{E}_{J_2}.$$

And finally, Lemma 17 tells us that

$$\mathscr{E}_{J_1} = \mathscr{E}_{J_2} \Longleftrightarrow \exists t > 0 \text{ such that } (J_1)^t = J_2.$$

Therefore, we can write $A_2 = C(J_1)^t C^{-1} = (A_1)^t$, which concludes the proof of Theorem 16. □

For a slightly more general result, let now $A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be self-adjoint expansive linear maps, without assuming that they are positive. We now consider the diagonal matrices:

$$J_\mu = \begin{pmatrix} \lambda_1^{(\mu)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(\mu)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_d^{(\mu)} \end{pmatrix}, \quad \lambda_i^{(\mu)} \in \mathbb{R}, \tag{14}$$

where $A_\mu = C_\mu J_\mu C_\mu^{-1}$, $\mu = 1, 2$, with some invertible mappings $C_1, C_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Let us denote

$$J'_\mu = \begin{pmatrix} |\lambda_1^{(\mu)}| & 0 & 0 & \cdots & 0 \\ 0 & |\lambda_2^{(\mu)}| & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |\lambda_d^{(\mu)}| \end{pmatrix}. \tag{15}$$

Then as a consequence of Theorem 16 and Corollary 12, we can say the following.

Corollary 20. *Let $A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be self-adjoint expansive linear maps. Let $C_1, C_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be such that $A_\mu = C_\mu J_\mu C_\mu^{-1}$, $\mu = 1, 2$, where J_μ are the respective diagonal maps as in (14). Then $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ if and only if $\exists t > 0$ such that*

$$(A'_1)^t = A'_2,$$

where $A'_\mu = C_\mu J'_\mu C_\mu^{-1}$, $\mu = 1, 2$, with J'_μ in (15).

6. Application to multiresolution analysis

In this section we study equivalence among expansive matrices satisfying (1). In general, the problem is still open. We look for some description of self-adjoint expansive linear maps

$A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfying (1), such that $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$. Hence we can get equivalent self-adjoint expansive linear maps for MRA.

Above we obtained that if $A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are expansive positive linear maps, then $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ if and only if there exists $t > 0$ such that $A_2 = (A_1)^t$. The general case of self-adjoint maps reduces to this case according to Corollary 20, so in the following discussion we restrict to this case of positive equivalent mappings. To meaningfully interpret the general requirement, one assumes $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ – so according to Theorem 16 we have $A_2 = (A_1)^t$, $t > 0$ – and now we look for further properties to ensure (1), too. So in the following let us assume that (1) is satisfied by A_1 and by A_2.

To fix notations we have already settled with choosing $\mathbb{Z}^d$ to be the fundamental lattice for our MRA. Therefore, we can assume that A_1 is written in diagonal form in the canonical basis of $\mathbb{Z}^d$ (otherwise considerations should change to the fundamental lattice spanned by the orthogonal basis of eigenvectors for A_1). As a consequence of $A_2 = (A_1)^t$, also A_2 is in diagonal form with respect to the canonical basis. Therefore, (1) means that we require these diagonal entries – eigenvalues of A_μ – belong to $\mathbb{Z}$, or, actually, to $\mathbb{N}$ as they are positive matrices.

In case all eigenvalues of A_1 are equal, i.e. $\beta_1^{(1)}$, by $A_2 = (A_1)^t$ we have the same property also for A_2, and the equation we must solve is that $\beta_1^{(2)} = (\beta_1^{(1)})^t \in \mathbb{N}$ and $\beta_1^{(1)} \in \mathbb{N}$ simultaneously. Clearly, with $t := \log \beta_1^{(2)} / \log \beta_1^{(1)}$ this can always be solved, so any two integer dilation matrices define equivalent MRA. Let us remark that in the thesis [20] there is a complete analysis of equivalence (with respect to the notion of points of A-density) to the dyadic dilation matrix, among all expansive linear mappings, self-adjoint or not. However, our focus here is different, as here we consider, under assumptions of self-adjointness, equivalence of arbitrary, not necessarily dilation mappings.

In the general case when A_1 (and hence also A_2) are not dilations, there must be two different entries (eigenvalues) in the diagonal of A_1 (and of A_2). As equivalence is hereditary in the sense that the restricted mappings on eigensubspaces of A_μ must also be equivalent, we first restrict to the case of dimension 2.

In dimension 2, we thus assume that A_1 has diagonal elements $a \neq b$ belonging to $\mathbb{N} \setminus \{0, 1\}$ and zeroes off the diagonal, and we would like to know when do we have with some $t > 0$ that $a, b, a^t, b^t \in \mathbb{Z}$ (or $\in \mathbb{N}$). Obviously, if $t \in \mathbb{N} \setminus \{0\}$ then this condition holds for any $a, b \in \mathbb{N}$. Also, in case a and b are full qth powers, we can as well take $t = p/q \in \mathbb{Q}$ with arbitrary $p \in \mathbb{N}$. That system of solutions – $a = \alpha^q$, $b = \beta^q$, $t = p/q$ with $\alpha, \beta, p \in \mathbb{N}$ – form one trivial set of solutions for our equivalence.

Another trivial set of solutions arises when $b = a^k$ with some $k \in \mathbb{N}$. Then it suffices to have $a^t \in \mathbb{N}$, which automatically implies $b^t \in \mathbb{N}$. More generally, if $b = a^{k/m}$ is a rational relation between a and b, then by the unique prime factorization we conclude that a is a full mth power and that b is full kth power, and again we find a system of solutions for all $t \in \mathbb{Q}$ of the form $t = \ell/k$.

All these trivial solutions can be summarized as cases of rational relations between a, b and t: once there is such a relation, one easily checks, if the respective matrix entries really become integers. So we find that systems of trivial solutions do exist if either t is rational, or if $\log a$ and $\log b$ are rationally dependent (are of rational multiples of each other). We can thus call these cases the *trivial equivalence* of self-adjoint expansive linear maps with respect to MRA construction. These explain the next definition.

Definition 21. Let $A_\mu (\mu = 1, 2)$ be two self-adjoint expansive linear maps, with $A_\mu = C_\mu J_\mu C_\mu^{-1}$, where J_μ are the respective diagonal maps as in (14), and $A'_\mu = C_\mu J'_\mu C_\mu^{-1}$, $\mu = 1, 2$, with J'_μ

in (15) for $\mu = 1, 2$. We say that A_1 and A_2 are *trivially equivalent*, if either $(A_1)^t = A_2$ with a rational $t = p/q \in \mathbb{Q}$, with all diagonal entries $|\lambda_j^{(1)}| \in \mathbb{N}$ $(j = 1, \ldots, d)$ being full qth powers (of some, perhaps different natural entries), or if with some natural numbers $a, b \in \mathbb{N}$ we have $|\lambda_j^{(1)}| = a^{n_j}$ with $n_j \in \mathbb{N}$ $(j = 1, \ldots, d)$ satisfying $(n_1, \ldots, n_d) = q$, and with $t = m/q \cdot \log b / \log a$, where $m \in \mathbb{N}$.

With this notion we can summarize our findings in the next statement.

Proposition 22. *Let* $A_1, A_2: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be self-adjoint expansive linear maps, with* $A_\mu = C_\mu J_\mu C_\mu^{-1}$, $\mu = 1, 2$, *where* J_μ *are the respective diagonal maps as in* (14). *Then according to Corollary* 20, $\mathscr{E}_{A_1} = \mathscr{E}_{A_2}$ *if and only if there exists* $t > 0$ *such that* $A_1'^t = A_2'$, *where* $A_\mu' = C_\mu J_\mu' C_\mu^{-1}$, $\mu = 1, 2$, *with* J_μ' *in* (15). *Moreover, if the respective matrices are trivially equivalent in the above sense, then they both satisfy* (1), *and thus form two equivalent expansive linear maps for MRA.*

The next question is to describe solutions of (1) for a and b in the diagonal of a 2 by 2 matrix A_1 with linearly independent logarithms over $\mathbb{Q}$, and $t \notin \mathbb{Q}$. We can conjecture that such equivalences do not occur, i.e. if A_1 and A_2 are equivalent positive expansive matrices in $\mathbb{R}^{2\times 2}$, satisfying (1), then they are from the above described trivial classes (including, of course, both the cases when A_μ are dilations, as then $a = b$, and when $A_1 = A_2$, as then $t = 1$ is rational).

We cannot prove this conjecture, but we can say that a well-known conjecture of number theory would imply this, too. Namely, we can now recall the so-called "Four Exponentials Conjecture", see e.g. [22, p. 14].

Conjecture 23 (Four Exponentials Conjecture). *Let* x_1, x_2 *be two* $\mathbb{Q}$*-linearly independent complex numbers and* y_1, y_2 *also two* $\mathbb{Q}$*-linearly independent complex numbers. Then at least one of the four numbers*

$$\exp(x_i y_j) \quad (i = 1, 2,\ j = 1, 2)$$

is transcendental.

Indeed, if the conjecture is right, we can choose $x_1 := \log a$, $x_2 := \log b$, $y_1 := 1$ and $y_2 := t$. If x_1 and x_2 are linearly independent over $\mathbb{Q}$ and y_1 and y_2 are also linearly independent over $\mathbb{Q}$, then either of the four numbers $a = e^{\log a}$, $b = e^{\log b}$, $a^t = e^{t \log a}$ and $b^t = e^{t \log b}$ must be transcendental, therefore, one cannot have $a, b, a^t, b^t \in \mathbb{Z}$. So in case the Four Exponentials Conjecture holds true, we must necessarily have either $a = b^q$ where $q \in \mathbb{Q}$, or $t \in \mathbb{Q}$.

The same argument can be implemented even in dimension d.

Proposition 24. *Assume that the above Four Exponentials Conjecture holds true. Then the two self-adjoint expansive linear maps* A_1 *and* A_2 *generate equivalent MRA if and only if the conditions of Proposition* 22 *above hold true*: $(A_1')^t = A_2'$ *and, moreover,* A_1 *and* A_2 *are trivially equivalent matrices.*

Proof. Assume first that $t \notin \mathbb{Q}$. Then t and 1 are linearly independent (over $\mathbb{Q}$), hence for any pair of indices $1 \leqslant j, k \leqslant d$ applying the Four Exponential Conjecture we conclude linear dependence of $x_1 := \log|\lambda_j^{(1)}|$ and $x_2 := \log|\lambda_k^{(1)}|$, i.e. $|\lambda_k^{(1)}| = |\lambda_j^{(1)}|^{r_{j,k}/s_{j,k}}$, with $r_{j,k}, s_{j,k} \in \mathbb{N}$. So in view of the unique prime factorization, $|\lambda_j^{(1)}| = \alpha_{j,k}^{s_{j,k}}$ (and also $|\lambda_k^{(1)}| = \alpha_{j,k}^{r_{j,k}}$) with some $\alpha_{j,k} \in \mathbb{N}$. Now, for each $k \in \{1, \ldots, d\}$ we compare the different expressions $|\lambda_k^{(1)}| = \alpha_{j,k}^{r_{j,k}}$, $j \in \{1, \ldots, d\}$, thus, taking the l.c.m. of the numbers $r_{j,k}$, $j = 1, \ldots, d$ we can define $r_k := [\cdots r_{j,k} \cdots]$ and, again from the unique prime factorization, we find that $|\lambda_k^{(1)}|$ must be a full r_kth power, i.e. $|\lambda_k^{(1)}| = \alpha_k^{r_k}$ with some $\alpha_k \in \mathbb{N}$. Consequently, for any fixed pair of $k, j \in \{1, \ldots, d\}$, $\alpha_k^{r_k} = |\lambda_k^{(1)}| = |\lambda_j^{(1)}|^{r_{j,k}/s_{j,k}} = (\alpha_j^{r_j})^{r_{j,k}/s_{j,k}}$, so again by the unique prime factorization all the α_ks have the same prime divisors, and we can write $|\lambda_k^{(1)}| = \alpha^{u_k}$ (with some $\alpha \in \mathbb{N}$ and $u_k \in \mathbb{N}$) for $k = 1, \ldots, d$. We can even consider $\nu := \max\{\mu : \alpha = a^\mu, a, \mu \in \mathbb{N}\}$, and with this ν write $|\lambda_k^{(1)}| = a^{n_k}$, where $n_k := \nu u_k$ $(k = 1, \ldots, d)$.

Observe that the same reasoning applies to the diagonal entries of the second matrix A_2', hence we also find $|\lambda_k^{(2)}| = b^{m_k}$, where $b \in \mathbb{N}$ and $m_k \in \mathbb{N}$ $(k = 1, \ldots, d)$.

Now write $(n_1, \ldots, n_d) = q$, and apply the equivalence condition $(A_1)^t = A_2$ to get $tn_k \log a = m_k \log b$ for all $k = 1, \ldots, d$. By the linear representation of the g.c.d, we thus obtain $tq \log a = m \log b$, where $m \in \mathbb{N}$ is a linear combination of the exponents m_k (and hence is divisible by $p := (m_1, \ldots, m_d)$). In any case, we have obtained the case $t = m/q \log a / \log b$ of the trivial equivalence above.

Second, let $t = p/q \in \mathbb{Q}$. Then, for each $j = 1, \ldots, d$, we have the equation $|\lambda_j^{(2)}| = |\lambda_j^{(1)}|^t = |\lambda_j^{(1)}|^{p/q}$, so $|\lambda_j^{(2)}| = a_j^p$ and $|\lambda_j^{(1)}| = a_j^q$, with $a_j \in \mathbb{N}$ otherwise arbitrary: and this is the other case of trivial equivalence, as defined above. □

In all, we found that under the assumption of the truth of the Four Exponentials Conjecture, equivalence with respect to the notion of A_μ-approximate continuity (or, equivalently, A_μ-density at 0) and fulfilling condition (1) implies trivial equivalence of the expansive self-adjoint linear matrices A_1 and A_2.

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