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Minimization of maxima of nonnegative and positive definite cosine polynomials with prescribed first coefficients

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Dedicated to Prof. K. Tandori on the occasion of his seventieth birthday and to Prof. L. Leindler on the occasion of his sixtieth birthday

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Abstract. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle group, $\mathbf{C}(\mathbb{T})$ be the set of continuous functions on $\mathbb{T}$, and $\mathcal{T}$ be the set of the trigonometrical polynomials. Let $f \in \mathcal{T}$ be nonnegative, even and positive (semi)definite. In number theory and analysis itself various extremal problems are related to the determination of the least or largest possible value of $f(0)$ under various conditions on the degree, spectrum set, or the value of some prescribed coefficients. We define

$$\mathcal{F}(a) = \{f \in \mathbf{C}(\mathbb{T}) : f(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx \geq 0 \ (\forall x), \quad a_k \geq 0 \ (k \in \mathbb{N})\}$$

and the extremal quantity

$$\alpha(a) = \inf\{f(0) : f \in \mathcal{F}(a)\}.$$

The aim of the paper is to collect as much information about $\alpha(a)$ as possible. Although the motivation of studying $\alpha(a)$ stems from the application of $\alpha(a)$ to extremal problems, e.g. one posed by Landau, we do not describe these connections here. Let us only mention that the new results of the paper, in

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particular concerning the behavior of α at $a \to 2$, provide essential help in those questions. However, the general approach here, and in particular the precise answer to the problem of estimation of Fourier coefficients of a nonnegative Fourier series from information regarding the first coefficient, as formulated in Theorem 2.1, seems to be of independent interest. Special emphasis is given to the description of $\alpha(a)$ at $a \to 2$. This analysis leads to considerable improvements upon earlier results of French and Steckin. However, our results settle only the order of magnitude of $\alpha(a)$, and a precise asymptotic description remained an open question.

0. Introduction

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle group, $\mathbf{C}(\mathbb{T})$ be the set of continuous functions on $\mathbb{T}$, and $\mathcal{T}$ be the set of the trigonometrical polynomials. Let $f \in \mathcal{T}$ be nonnegative, even and positive (semi)definite, i.e. for some $n \in \mathbb{N}$

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx \geq 0 \qquad (\forall x), \qquad a_k \geq 0 \quad (k = 0, 1, \dots, n). \tag{0.1}$$

In number theory and analysis itself various extremal problems are related to the determination of the least or largest possible value of $f(0)$ under various conditions on the degree, [F,T,L1,L2], spectrum set, [K–M,Ru1,Ru2], or the value of some prescribed coefficients [L1,L2,W,St]. We define

$$\mathcal{F}(a) = \{f \in \mathbf{C}(\mathbb{T}) : f(x) \sim 1 + a\cos x + \sum_{k=2}^{\infty} a_k \cos kx \geq 0 \quad (\forall x),\ a_k \geq 0\ (k \in \mathbb{N})\} \tag{0.2}$$

and the extremal quantity

$$\alpha(a) = \inf\{f(0) : f \in \mathcal{F}(a)\}. \tag{0.3}$$

It is easy to see that for a lying in the interior of the domain of definition for α, (0.3) is equivalent to $\alpha^*(a) = \inf\{f(0) : f \in \mathcal{T}(a)\}$, and that (0.3) is actually a minimum, cf. Proposition 4.2 below. Because of this latter minimum attaining property we choosed in (0.2)-(0.3) the use of $\mathbf{C}(\mathbb{T})$ in place of the equivalent formulation with $\mathcal{T}$.

The aim of the paper is to collect as much information about $\alpha(a)$ as possible. For the application of $\alpha(a)$ we refer to [R2], where extremal problems posed by Landau in [L1] are investigated. Although the motivation of studying $\alpha(a)$

stems from these problems, we do not describe these connections here. Let us only mention that the new results of the paper, in particular concerning the behavior of α at $a \to 2$, provide essential help in those questions. However, the general approach here, and in particular the precise answer to the problem of estimation of Fourier coefficients of a nonnegative Fourier series from information regarding the first coefficient, as formulated in Theorem 2.1, seems to be of independent interest.

As mentioned above, a special emphasis is given to the description of $\alpha(a)$ at $a \to 2$. This analysis is far from being trivial, as can be seen from the fact that even $\alpha(a) \to \infty$ as $a \to 2$ requires a proof, or that the estimates obtained lead to considerable improvements upon earlier results of French [F] and Steckin [St] regarding a problem of Landau, cf. [R2]. However, our results settle only the order of magnitude of $\alpha(a)$, and a precise asymptotic description remained an open question.

1. Notation

If we write the symbol of integration without specifying the domain of integration we always mean integration on $\mathbb{T}$ with respect to the Lebesgue measure, i.e.

$$\int = \int_{\mathbb{T}} dx. \tag{1.1}$$

The scalar product of $f \in \mathbf{C}(\mathbb{T})$ and $\mu \in \mathbf{BM}(\mathbb{T})$ is

$$\langle f, d\mu \rangle = \frac{1}{2\pi} \int f d\mu, \tag{1.2}$$

and Fourier coefficients of μ (or f) are always

$$a_k(\mu) = \langle 2\cos kx, d\mu(x) \rangle = \frac{1}{\pi} \int_{\mathbb{T}} \cos kx d\mu(x), \qquad b_k(\mu) = \langle 2\sin kx, d\mu(x) \rangle \tag{1.3}$$

(or the similar integral with $f(x)dx$ in place of $d\mu(x)$). As we normalize measures and functions so that their constant term be 1, in our terminology the Dirac delta measure is the normalized measure with

$$\delta = \delta_0, \qquad \delta_z(H) = \begin{cases} 2\pi & \text{if } z \in H, \\ 0 & \text{if } z \notin H \end{cases} \tag{1.4}$$

for $H \subset \mathbb{T}$ and measurable. This way δ_z has the normalized Fourier expansion

$$d\delta_z(x) \sim 1 + 2\sum_{k=1}^{\infty} (\cos kz \cos kx + \sin kz \sin kx) \ , \tag{1.5}$$

and the even part of δ_z is

$$d\nu_z(x) \sim \frac{1}{2}\left(d\delta_z(x) + d\delta_{-z}(x)\right) \sim 1 + 2\sum_{k=1}^{\infty} \cos kz \cos kx \ , \tag{1.6}$$

the "even atomic measure at z". For the convolution operator we use the same normalized integration, hence

$$(f * \mu)(x) = \frac{1}{2\pi}\int f(x-t)d\mu(t) = \langle f(x-t), d\mu(t)\rangle_t \tag{1.7}$$

and for *even* measures

$$(f*\mu)(x) \sim 1+\sum_{k=1}^{\infty} \frac{a_k(\mu)}{2}\left(a_k(f)\cos kx + b_k(f)\sin kx\right) \quad (d\mu(x) = d\mu(-x)) \ . \tag{1.8}$$

In particular, convolution by $\delta = (\delta_0 = \nu_0)$ is the identity operator. The Dirichlet, Fejér and Poisson kernels are

$$D_N(x) = 1 + 2\sum_{n=1}^{N} \cos nx = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}, \tag{1.9}$$

$$F_N(x) = 1 + 2\sum_{n=1}^{N}\left(1 - \frac{n}{N+1}\right)\cos nx = \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)^2, \tag{1.10}$$

and

$$P_r(x) = 1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1-r^2}{1-2r\cos x + r^2} \ , \tag{1.11}$$

respectively.

2. Trinomials

We define

$$\mathcal{G}(k) = \{g \in \mathcal{T} : g(x) = 1 - a\cos x + b\cos kx, \quad a \geq 0, \quad b \in \mathbb{R}\} \ . \tag{2.1}$$

Lemma 2.1. i) *For any* $g \in \mathcal{G}(k)$ $\min_{\mathbb{T}} g = \min_{[0,\frac{\pi}{k}]} g$.

ii) *For any non-constant* $g \in \mathcal{G}(k)$ *and* $0 < z \leq \frac{\pi}{k}$ *with* $g'(z) = 0$ *either* z *is the unique minimum place of* g *in* $[0, \frac{\pi}{k}]$, *or* z *is a maximum place of* g *in* $[0, \frac{\pi}{k}]$ *and* $\min_{\mathbb{T}} g = g(0)$. *In particular,* $\min_{\mathbb{T}} g = \min\{g(0), g(z)\}$ *for all* $g \in \mathcal{G}(k)$.

Proof. i) For any $x \in \mathbb{R}$ let us choose the unique $m \in \mathbb{Z}$ for which $(2m-1)\pi < kx \leq (2m+1)\pi$. We have $g(x) - g(x - \frac{2m\pi}{k}) = a\left(\cos(x - \frac{2m\pi}{k}) - \cos x\right) \geq 0$ and $g(x - \frac{2m\pi}{k}) = g(|x - \frac{2m\pi}{k}|)$ since g is even.

ii) By condition, for a certain $0 < z \leq \frac{\pi}{k}$ we have

$$0 = g'(z) = a \sin z - bk \sin kz. \tag{2.2}$$

If $b = 0, \sin z \neq 0$ entails $a = 0$ and thus $g \equiv 1$. Hence $b \neq 0$, and for $\varphi(x) = \frac{\sin kx}{\sin x}$ we get from (2.2) that

$$\frac{a}{kb} = \varphi(z)\ . \tag{2.3}$$

For $0 < x \leq \frac{\pi}{k}$
(2.4)

$$\varphi'(x) = \frac{k \cos kx \sin x - \sin kx \cos x}{\sin^2 x} = \begin{cases} -\frac{\cos \frac{\pi}{2k}}{\sin^2 \frac{\pi}{2k}}, & \left(x = \frac{\pi}{2k}\right) \\ \frac{\cos kx \cos x}{\sin^2 x}\left(k \tan x - \tan kx\right) & \left(x \neq \frac{\pi}{2k}\right). \end{cases}$$

$\varphi' < 0$ and φ is strictly decreasing, hence (2.3) can hold for no other value of x in $(0, \frac{\pi}{k}]$ than z itself. It follows that the only zero place of g' in $(0, \frac{\pi}{k}]$ is z and g' changes sign there. If $g(z)$ is a minimum, then it is a strict minimum point on the whole $[0, \frac{\pi}{k}]$, and if it is a maximum, then the minimum of g in $[0, \frac{\pi}{k}]$ is either $g(0)$ or $g(\frac{\pi}{k})$. The second alternative can not be the case. Indeed, then $\min_{\mathbb{T}} g = g(\frac{\pi}{k})$ and hence $g'(\frac{\pi}{k}) = 0$, a contradiction if $z \neq \frac{\pi}{k}$, while in case $z = \frac{\pi}{k}$, $g(z) = g(\frac{\pi}{k})$ would be a maximum and also a minimum, hence g would be a constant, a case excluded already by $b \neq 0$. Hence ii) follows. ∎

Now we define

$$\mathcal{H} = \bigcup_{0 \leq z \leq \frac{\pi}{k}} \mathcal{H}(z), \quad \mathcal{H}(z) = \{h \in \mathcal{G}(k) : h(z) = h'(z) = 0\}. \tag{2.5}$$

Lemma 2.2. i) *If $g \in \mathcal{G}(k)$ is nonnegative but not strictly positive, then $g \in \mathcal{H}$.*

ii) *For $0 < z \le \frac{\pi}{k}$, $\mathcal{H}(z)$ consists of the single element*

$$h_z(x) = 1 - a(z)\cos x + b(z)\cos kx \tag{2.6}$$

with

$$a(z) = \frac{k \sin kz}{d(z)}, \quad b(z) = \frac{\sin z}{d(z)}, \quad d(z) = k\cos z \sin kz - \cos kz \sin z. \tag{2.7}$$

iii) *$h \in \mathcal{H}(0)$ if and only if*

$$h = \lambda h_0 + (1-\lambda)H, \quad \lambda = \left(1 - \frac{1}{k^2}\right) a\ , \tag{2.8}$$

where

$$h_0(x) = 1 - \frac{k^2}{k^2-1}\cos x + \frac{1}{k^2-1}\cos kx \tag{2.9}$$

and

$$H(x) = 1 - \cos kx. \tag{2.10}$$

iv) *For $x \in \mathbb{T}$, $x \ne \pm z$ $h_z(x)$ is strictly positive for all $0 \le z \le \frac{\pi}{k}$.*

v) *$h \in \mathcal{H}(0)$ is nonnegative if and only if (2.8) is a convex combination, i.e. $0 \le \lambda \le 1$. For $h \in \mathcal{H}(0)$ and $h \ge 0$ $h(0) = 0$ is a strict global minimum of h.*

Proof. i) Follows from $\min g = 0$ and Lemma 2.1 i).

ii) For $h \in \mathcal{H}(z)$, we have by definition

$$\begin{aligned} 0 &= h(z) = 1 - a\cos z + b\cos kz, \\ 0 &= h'(z) = a\sin z - bk\sin kz, \end{aligned} \tag{2.11}$$

which is a two by two linear equation for a and b with determinant $d(z)$. For $0 < z \le \frac{\pi}{k}$

$$d(z) = \begin{cases} k\cos\frac{\pi}{2k} & \left(z = \frac{\pi}{2k}\right) \\ \cos kz \cos z\,(k\tan kz - \tan z) & \left(z \ne \frac{\pi}{2k}\right), \end{cases} \tag{2.12}$$

hence $d(z) > 0$ and (2.6)-(2.7) describes the unique solution of (2.11).

iii) If $h \in \mathcal{H}(0)$ we have from (2.11) only

$$0 = h(0) = 1 - a + b \tag{2.13}$$

since h is always even and $h'(0) = 0$ for all a and b. Now plainly (2.9) and (2.10) are elements of $\mathcal{H}(0)$, and so is (2.8). Conversely, for any h satisfying (2.13), equating coefficients yields the representation (2.8) with the given λ.

iv) First let $0 < z \le \frac{\pi}{k}$. According to Lemma 2.1 ii) it suffices to prove $h_z(0) > 0 = h_z(z)$. Now from ii) (2.6)

$$h_z(0) = \frac{d(z) - k\sin kz + \sin z}{d(z)} = \frac{\sin z(1-\cos kz) - k\sin kz(1-\cos z)}{d(z)}.$$

The denominator is positive and $\sin z > 0$, hence using also the identity $1-\cos 2w = \tan w \sin 2w$ we are entitled to check

$$0 < \frac{d(z)h_z(0)}{\sin z} = \begin{cases} 2 & \left(z = \frac{\pi}{k}\right) \\ \sin kz\left(\tan\frac{kz}{2} - k\tan\frac{z}{2}\right) & \left(0 < z < \frac{\pi}{k}\right). \end{cases}$$

Now we pass to the case $z = 0$. Observe that for $z \to 0$, $a(z) \to a(0) = \frac{k^2}{k^2-1}$ and $b(z) \to b(0) = \frac{1}{k^2-1}$, hence for any $x \in \mathbb{T}$, $h_0(x) = \lim_{z\to 0} h_z(x) \ge 0$ and $h_0(0)$ is a minimum-place of h_0. According to Lemma 2.1 ii) no other minimum-place can occure.

v) Plainly, if (2.8) is a convex combination, then h is nonnegative. Now suppose $h \ge 0$ and $h \in \mathcal{H}(0)$. We have from $h \ge 0$ that

$$0 \le \sum_{j=1}^{k} h\left(\frac{2j\pi}{k}\right) = k - a\sum_{j=1}^{k}\cos\left(\frac{2j\pi}{k}\right) + kb = k(1+b),$$

and from (2.13) and this we get $a \ge 0$ and so also $\lambda \ge 0$. Moreover, since $h(0) = 0$, $h'(0) = 0$ and $h \ge 0$, we must have $h''(0) \ge 0$, i.e.

$$0 \le \frac{1}{k^2}h''(0) = \frac{1}{k^2}(a - k^2 b) = \frac{1}{k^2}a + (1-a) = 1 - \lambda,$$

thus $0 \le \lambda \le 1$ and (2.8) is a convex combination. The second assertion holds for h_0 and H, and so a fortiori for any convex combination of them. This concludes the proof. ∎

In the following we consider nonnegative trinomials. Put

$$\mathcal{F}_k = \{f \in \mathcal{T} : f \ge 0,\ f(x) = 1 + a\cos x + b\cos kx,\ a, b \in \mathbb{R}\}. \tag{2.14}$$

Plainly, $\mathcal{F}_k$ is a compact subset of $\mathbf{C}(\mathbb{T})$, since for any nonnegative trigonometric polynomial scalar multiplication by $1 \pm \cos mx$ shows immediately that all coefficients are at most 2 in absolute value.

Now let us introduce also

$$\mathcal{Z}_0(k) = \{H\} \cup \{h_z : 0 \le z \le \frac{\pi}{k}\}, \qquad \mathcal{Z}_\pi(k) = \{h(x+\pi) : h \in \mathcal{Z}_0(k)\} \quad \text{and} \quad \mathcal{Z}(k) = \mathcal{Z}_0(k) \cup \mathcal{Z}_\pi(k). \tag{2.15}$$

Lemma 2.3. *Let F be any linear functional on $\mathbf{C}(\mathbb{T})$. The maximum of F on $\mathcal{F}_k$ is attained at some function of $\mathcal{Z}(k)$.*

Proof. Let

$$F(\mathbf{1}) = X, \ F(\cos x) = Y, \ F(\cos kx) = Z \tag{2.16}$$

and suppose that F attains its maximum on f^*, i.e.

$$M = \max_{\mathcal{F}_k} F = F(f^*) = X + a^*Y + b^*Z, \quad \text{where} \quad f^*(x) = 1 + a^* \cos x + b^* \cos kx. \tag{2.17}$$

Let $m = \min f^*$, and suppose first that $m > 0$. Since also $g = \frac{f^*-m}{1-m}$ is in $\mathcal{F}_k$, we are led to

$$M = F(f^*) = F\left((1-m)g + m\right) = (1-m)F(g) + mX \le (1-m)M + mX, \tag{2.18}$$

hence $M \le X$. Further, $F(\mathbf{1} \pm \cos x) = X \pm Y$, $F(\mathbf{1} \pm \cos kx) = X \pm Z$ and as $\mathbf{1}\pm\cos x$ and $\mathbf{1}\pm\cos kx$ are in $\mathcal{F}_k$, we get $M \ge X \pm Y$, $M \ge X \pm Z$, hence $Y = Z = 0$ and F is constant $M = X$ on the whole of $\mathcal{F}_k$. Consequently, if $m > 0$ then any $h \in \mathcal{Z}(k)$ is also a maximum place of F, too. Now consider the case $m = 0$. Then $f^* \in \mathcal{F}_k$ but f^* is not strictly positive. If $a^* \le 0$, then f^* also belongs to $\mathcal{G}(k)$, hence in view of Lemma 2.2 i) $f^* \in \mathcal{H}$, and it suffices to show that

$$\max_{\mathcal{H}} F = \max_{\mathcal{Z}_0(k)} F. \tag{2.19}$$

This follows from (2.5), (2.15) and Lemma 2.2 iii) and v) since

$$\max_{\mathcal{H}(0)} F = \max_{0 \le \lambda \le 1} F\left(\lambda h_0 + (1-\lambda)H\right) = \max\{F(h_0), F(H)\}.$$

Finally if $a^* > 0$, we consider the modified functional

$$\tilde{F}\left(f(x)\right) = F\left(f(x+\pi)\right) = X - aY + b(-1)^k Z \quad \left(f(x) = 1 + a\cos x + b\cos kx\right). \tag{2.20}$$

$\mathcal{F}_k$ is translation-invariant, hence $\max F = \max \tilde{F}$ on $\mathcal{F}_k$. Moreover, $\tilde{F}$ attains its maximum on

(2.21) $\tilde{f}(x) = f^*(x-\pi) = 1 - a^* \cos x + b^*(-1)^k \cos kx = 1 + \tilde{a} \cos x + \tilde{b} \cos kx$,

hence the argument for the case $a^* \le 0$ works for $\tilde{F}$, $\tilde{f}$ and $\tilde{a} = -a^* \le 0$. We gain that

$$\max_{\mathcal{F}_k} \tilde{F} = \max_{\mathcal{Z}_0(k)} \tilde{F} = \max_{\mathcal{Z}_\pi(k)} F \ . \tag{2.22}$$

This concludes the proof of Lemma 2.3. ∎

Note that the same statement holds true if we seek the minimum of F. Let us record here two corollaries for later reference.

Proposition 2.1. *Among the nonnegative cosine trinomials of the form* $1+a\cos x + b\cos kx$ *the biggest possible value of* a *belongs to*

$$h_{\frac{\pi}{2k}}(x+\pi) = 1 + \frac{1}{\cos \frac{\pi}{2k}} + \frac{(-1)^k}{k} \tan \frac{\pi}{2k} \cos kx \ . \tag{2.23}$$

Note that the similar extremum for $1 - a\cos x + b\cos kx$ can be obtained by translation by π. Hence $|a| \le \frac{1}{\cos\frac{\pi}{2k}}$ for all $f \in \mathcal{F}_k$.

Proof. By Lemma 2.3 we have to consider the functional $F(1+a\cos x + b\cos kx) = a$, and have to find

(2.24)
$$\max_{\mathcal{F}_k} F = \max_{\mathcal{Z}(k)} F = \max\{F(H), F(H(x+\pi)), \max_{0\le z\le \frac{\pi}{k}} F(h_z), \max_{0\le z\le \frac{\pi}{k}} F(h_z(x+\pi))\}$$
$$= \max\{0, \max_{0\le z\le \frac{\pi}{k}} a(z), \max_{0\le z\le \frac{\pi}{k}} (-a(z))\} = \max_{0\le z\le \frac{\pi}{k}} a(z).$$

Since $a(\frac{\pi}{k}) = 0$, $a(0) = \frac{k^2}{k^2-1} < \frac{1}{\cos\frac{\pi}{2k}} = a(\frac{\pi}{2k})$, we certainly have a maximum point of a in $(0, \frac{\pi}{k})$, which corresponds to a minimum place of $A(z) = \frac{1}{a(z)}$. Hence for that point z the derivative must vanish,

$$\begin{aligned} 0 = A'(z) &= (\cos z - \sin z \cot kz)' = -\sin z - \frac{1}{k}\cos z \cot kz + \frac{\sin z}{\sin^2 kz} \\ &= \frac{\sin z \cos^2 kz - \frac{1}{k}\cos z \cos kz \sin kz}{\sin^2 kz} \\ &= \begin{cases} \frac{\cos z \cos^2 kz}{\sin^2 kz}\{\tan z - \frac{\tan kz}{k}\} & \left(z \ne \frac{\pi}{2k}\right), \\ 0 & \left(z = \frac{\pi}{2k}\right). \end{cases} \end{aligned} \tag{2.25}$$

Since the only root of A' in $(0, \frac{\pi}{k})$ is $z = \frac{\pi}{2k}$, this is the unique minimumpoint of A. Proposition 2.1 is proved. ∎

Theorem 2.1. *Let $\mu \in \mathbf{BM}(\mathbb{T})$ be an even nonnegative measure with Fourier series*

$$d\mu(x) \sim 1 + \sum_{k=1}^{\infty} a_k \cos kx. \tag{2.26}$$

If $0 \le a_1 \le 2$ *and* $k\arccos(\frac{a_1}{2}) \le \pi$*, then*

$$a_k \ge 2\cos(k\arccos(\frac{a_1}{2}))\ . \tag{2.27}$$

Moreover, if equality occures for any particular k in the above range, then equality holds true for all $k \in \mathbb{N}$ and $\mu = \nu_{\arccos(\frac{a_1}{2})}$ of (1.5).

Proof. Since μ is nonnegative, for $n \in \mathbb{N}$ we have

$$0 \le 2\langle 1 \pm \cos nx, d\mu(x)\rangle = 2 \pm a_n, \tag{2.28}$$

hence $|a_n| \le 2$ and $a_1 = 2$ only if $\operatorname{supp}\mu \subseteq \{x : 1 - \cos x = 0\}$, i.e. $\operatorname{supp}\mu = \{0\}$ and $\mu = \delta_0$. Now let $0 < a_1 < 2$, μ and k be fixed, and consider for $0 \le z \le \frac{\pi}{k}$ the scalar product

$$0 \le \langle h_z,\ d\mu\rangle = 1 - \frac{a_1 a(z)}{2} + \frac{a_k b(z)}{2}\ , \tag{2.29}$$

where h_z is the function definied in (2.6) – (2.9). The coefficients a, b are continuous on $[0, \frac{\pi}{k}]$, and $b(z) > 0$, hence

$$a_k \ge \frac{a_1 a(z) - 2}{b(z)} = S(z)\ , \tag{2.30}$$

S is a continuous function on $[0, \frac{\pi}{k}]$, and from the explicit form (2.7) of $a(z)$ and $b(z)$ we get the differentiable expression

$$S(z) = \frac{a_1 k \sin kz - 2d(z)}{\sin z} \quad (0 < z \le \frac{\pi}{k})\ . \tag{2.31}$$

We seek for the maximum of $S(z)$. Differentiating and using (2.7) we see

$$\begin{aligned} S'(z)\sin^2 z &= \left(a_1 k^2 \cos kz - 2d'(z)\right)\sin z - (a_1 k \sin kz - 2d(z))\cos z \\ &= a_1\left(k^2\cos kz \sin z - k \sin kz \cos z\right) - \\ &\qquad - 2\cos z\left(k^2 \cos kz \sin z - k \sin kz \cos z\right) \\ &= 2k\left(\frac{a_1}{2} - \cos z\right)(k\cos kz \sin z - \sin kz \cos z)\,. \end{aligned} \tag{2.32}$$

Since the last factor is negative, S has a maximum for $z_0 = \arccos(\frac{a_1}{2})$, what is in the interval $(0, \frac{\pi}{k}]$ for $0 < a_1 < 2$ and $k \le \frac{\pi}{z_0}$. Putting the obtained value of z_0 into (2.30) – (2.31) we are led to

$$a_k \ge \frac{2\cos z_0 k \sin kz_0 - 2d(z_0)}{\sin z_0} = 2\cos\left(k \arccos\left(\frac{a_1}{2}\right)\right), \tag{2.33}$$

proving (2.27). In case of equality also (2.29) holds with equality, hence $\operatorname{supp}\mu \subseteq \{h_{z_0}(x) = 0\} = \{z_0,\ -z_0\}$, and as μ must be even, we get $\mu = \nu_{z_0}$ as stated. Now we proved that if (2.27) holds with equality for any particular k, then $\mu = \nu_{z_0}$ and (2.27) holds with equality for all permissible k. Similarly, the statement, though giving almost no real information, is valid for $a_1 = 0$. ■

Remarks. For $a_1 < 0$ we can apply the same Theorem for $d\mu(x+\pi)$ to obtain

$$(-1)^k a_k \ge (-1)^k 2\cos\left(k \arccos\left(\frac{-a_1}{2}\right)\right) \tag{2.34}$$

for all k satisfying $k\arccos\left(\frac{-a_1}{2}\right) \le \pi$.

If $k\arccos\left(\frac{|a_1|}{2}\right) > \pi$, i.e. $|a_1| < 2\cos\frac{\pi}{k}$, then only the trivial estimate $|a_k| \le 2$, proved in (2.28), holds true. Indeed, for any choice of a_k, the four measures δ_0, δ_π, $\nu_{\frac{\pi}{k}}$ and $\nu_{\pi-\frac{\pi}{k}}$ have first and k^{th} coefficients $2, -2, 2\cos\frac{\pi}{k}, -2\cos\frac{\pi}{k}$ and $2, 2(-1)^k, -2, 2(-1)^{k+1}$, respectively, hence their convex hull always contains a normalized even nonnegative measure with the given a_1 and the prescribed a_k.

3. Some spline functions

Let us consider for arbitrary $h > 0$ the function

$$\chi_h(x) = \begin{cases} 1 & |x| < \frac{h}{2} \\ 0 & |x| \ge \frac{h}{2} \end{cases}, \qquad \chi(x) = \chi_1(x), \tag{3.1}$$

and for $0 < h \le \pi$ the 2π-periodic normalized variant

$$\varphi_h(x) = \frac{2\pi}{h}\chi_h(x) \qquad (x \in \mathbb{T}) \ . \tag{3.2}$$

The basic splines $M_n(x)$, $n = 2, 3, \ldots$ on $\mathbb{R}$ can be expressed, cf. [Sc] pp. 68–76, as

$$\begin{aligned} M_n(x) &= (\underbrace{\chi * \chi * \ldots * \chi}_{n})(x) = \frac{1}{(n-1)!}\Delta^n\left(\left(x^{n-1}\right)_+\right) \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left(\frac{2\sin\frac{u}{2}}{u}\right)^n e^{iux}du, \end{aligned} \tag{3.3}$$

where $*$ stands for convolution on $\mathbb{R}$, Δ is the symmetric (or central) difference operator of step unity and $y_+ = \max\{y, 0\}$.

Using the 2π-periodic convolution (1.7) we define the convolution powers

$$\Psi_{h,n}(x) = \varphi_h^{*n}(x), \text{ i.e. } \quad \Psi_{h,1} = \varphi_h,\ \Psi_{h,n+1} = \Psi_{h,n} * \varphi_h \ . \tag{3.4}$$

It is well-known, that

$$\Psi_{h,n}(x) = 1 + 2\sum_{k=1}^{\infty} \left(\frac{2\sin\left(\frac{kh}{2}\right)}{kh} \right)^n \cos kx \ , \tag{3.5}$$

and for even $n = 2l$ $\Psi_{h,2l}$ is nonnegative and positive definite. Now if $lh < \pi$, we get from (3.1), (3.2) and (3.4) that the support of $\Psi_{h,2l}$ is $[-lh, lh] \subset (-\pi, \pi)$ and we are led to the following Lemma by simple change of variable and taking into account also (2.3) and (2.6).

Lemma 3.1. *For $l \in \mathbb{N}$, $h > 0$ with $hl \le \pi$ we have*

$$\Psi_{h,2l}(x) = \frac{2\pi}{h} M_{2l}\left(\frac{x}{h}\right) \quad (|x| \le \pi) \ . \tag{3.6}$$

Lemma 3.2.

i) *$\pi M_{2l}(0)$ is asymptotically equal to $\sqrt{3\pi}$ as $l \to \infty$.*

ii) $\min_{l\in\mathbb{N}} \pi M_{2l}(0)\sqrt{l} = \pi M_4(0)\sqrt{2}$.

Proof. i) Standard asymptotical evaluation of the ending integral formula of (3.3), cf. eg. [B], pp. 68–69.

ii) To obtain a lower estimate of $\pi M_{2l}(0)\sqrt{l}$ we use the last formula of (3.3) with the simple change of variable $t = 2u$ and gain

$$I(l) = \pi M_{2l}(0) = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^{2l} dt > \int_{-1.25}^{1.25} e^{-2lg(t)}\, dt \ , \tag{3.7}$$

where for $|t| \le 1.25$

$$g(t) = -\log\left(\frac{\sin t}{t}\right) = \sum_{k=1}^{\infty} \frac{1}{k}\left(1 - \frac{\sin t}{t}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k} w(t)^k \tag{3.8}$$

The alternating series expansion

$$w(t) = 1 - \frac{\sin t}{t} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} t^{2j}}{(2j+1)!} \tag{3.9}$$

provides the upper estimate (for $|t| < 1.25$)

$$w(t) < \frac{t^2}{6} - \frac{t^4}{120} + \frac{t^6}{7!} = \frac{t^2}{6}\left(1 - \frac{t^2}{20} + \frac{t^4}{840}\right) < \frac{t^2}{6}\left(1 - t^2 0.048\right) . \tag{3.10}$$

Besides, we get from (3.8) and $0 \le w \le 1 - \sin(1.25)/1.25 = 0.2408\ldots$ the estimate

$$g(t) < w + \frac{1}{2}w^2 + \frac{1}{3}\frac{w^3}{1-w} < w(1 + 0.606w), \tag{3.11}$$

and combining (3.10) and (3.11) leads to

$$g(t) < \frac{t^2}{6}\left(1 - 0.048t^2\right)\left(1 + 0.606\frac{t^2}{6}\right) < \frac{t^2}{6}1.063 \quad (|t| \le 1.25). \tag{3.12}$$

We apply this in (3.7) to get
(3.13)

$$I(l) > \int_{-1.25}^{1.25} e^{-1.063lt^2/3} dt = 2 \int_{0}^{1.25\sqrt{\frac{1.063l}{3}}} e^{-v^2} dv \sqrt{\frac{3}{1.063l}}$$

$$= \sqrt{\frac{3\pi}{1.063l}}\left(1 - \frac{2}{\sqrt{\pi}} \int_{1.25\sqrt{\frac{1.063l}{3}}}^{\infty} e^{-v^2} dv\right) > \frac{2.9776}{\sqrt{l}}\left(1 - 2Q\left(0.744\sqrt{l}\right)\right),$$

where Q is the error to the Gauss distribution, i.e.

$$Q(x) = \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} e^{-v^2} dv. \tag{3.14}$$

Since Q is decreasing, we obtain from (3.13) that for $l \ge 15$

$$I(l)\sqrt{l} > 2.9776\left(1 - 2Q\left(0.744\sqrt{15}\right)\right) > 2.965\,. \tag{3.15}$$

For the first fourteen values of l we can calculate $M_{2l}(0)$ from the last but one formula of (3.3). We get

$$\begin{aligned} &M_2(0) = 1, \quad \pi M_2(0) = \pi \\ (3.16)\quad &M_4(0) = \frac{1}{6}\left(\Delta^4(x^3)_+\right) = \frac{2}{3}, \quad \pi M_4(0)\sqrt{2} = \frac{2\sqrt{2}\pi}{3} = 2.9619\ldots \\ &M_6(0) = \frac{1}{120}\left(\Delta^6(x^5)_+\right)(0) = \frac{11}{20}, \quad \pi M_6(0)\sqrt{3} = \frac{11\sqrt{3}\pi}{20} = 2.9927\ldots, \end{aligned}$$

and $\pi M_{2l}(0)\sqrt{l} > 3$ for $l = 4, 5, \ldots, 14$.

■

4. Properties of α

Proposition 4.1. *$\mathcal{D}(\alpha)$, the domain of definition of α, is an interval $\mathcal{D}(\alpha) = [A, 2)$ or $(A, 2)$ where $-\sqrt{3} \le A \le -\sqrt{2}$.*

Proof. Convexity of $\mathcal{D}(\alpha)$ is clear from definitions (0.2) and (0.3) noting that $\mathcal{D}(\alpha) = \{a \in \mathbb{R} \colon \mathcal{F}(a) \ne \emptyset\}$. A glance at the argument around (2.28) yields $\mathcal{D}(\alpha) \subseteq (-2, 2)$ and $\sup \mathcal{D}(\alpha) = 2$ is obvious from the existence of the positive and positive definite kernels (1.10) or (1.11). It remains to estimate

$$A = \inf \mathcal{D}(\alpha) = \inf\{a : \mathcal{F}(a) \ne \emptyset\}\ . \tag{4.1}$$

First let us consider $h_{\pi/4}$ of $\mathcal{F}_2$, i.e.

$$h(x) = 1 - \sqrt{2}\cos x + \frac{1}{2}\cos 2x\ , \tag{4.2}$$

which shows $A \le -\sqrt{2}$. Now take $g(x) = h_{\pi/6}(x + \pi)$ of $\mathcal{F}_3$, and suppose that $f \in \mathcal{F}(a)$ with a Fourier series of the type given in (0.2). Using the explicit form

$$g(x) = 1 + \frac{2}{\sqrt{3}}\cos x - \frac{1}{3\sqrt{3}}\cos 3x, \tag{4.3}$$

we have

$$0 \le \langle f, g\rangle = 1 + \frac{a}{\sqrt{3}} - \frac{a_3}{6\sqrt{3}}\ . \tag{4.4}$$

Whence

$$a_3 \le 6(\sqrt{3} + a)\ , \tag{4.5}$$

and in view of $a_3 \ge 0$ we obtain $a \ge -\sqrt{3}$, which yields

$$A \ge -\sqrt{3}\ . \tag{4.6}$$

■

Proposition 4.2. i) *In the definition* (1.3) *of* $\alpha(a)$ *the infinum is actually a minimum, i.e.*

$$\alpha(a) = \min\{f(0) : f \in \mathcal{F}(a)\} \tag{4.7}$$

for all $a \in \mathcal{D}(\alpha)$.

ii) *If* $A \in \mathcal{D}(\alpha)$, *then* $\lim_{a\to A+} \alpha(a) = \alpha(A)$, *and if* $A \notin \mathcal{D}(\alpha)$, *then* $\lim_{a\to A+} \alpha(a) = \infty$.

Proof. The proof uses a compactness argument in the space of $\mathbf{BM}(\mathbb{T})$ utilizing that any $f \in \mathcal{F}(a)$ has an absolutely convergent Fourier series and $||f||_\infty = f(0)$. As the proof works in a somewhat more general setting, we refer to 2.5 and 2.6 Propositions of [Re1], where the condition of positive definiteness is used in a weakened form.

■

Proposition 4.3. $\alpha(a)$ *is convex on* $\mathcal{D}(\alpha)$.

Proof. The assertion is clear from the definitions, and also holds true in a more general setting, cf. 2.3 Proposition of [Re1].

■

Proposition 4.4.

i) $\alpha(a) = 1 + a$ *for* $-1 \le a \le 1$.

ii) $\alpha(a) = 2a$ *for* $1 \le a \le \frac{4}{3}$.

iii) $\alpha(a) > 2a$ *for* $\frac{4}{3} < a < 2$.

iv) $\alpha(a) = 0$ *for* $\frac{-4}{3} \le a \le -1$.

v) $\alpha(a) > 0$ *for* $a < \frac{-4}{3}$, $a \in \mathcal{D}(\alpha)$.

Proof. i) For $|a| \le 1$, $1 + a\cos x \in \mathcal{F}(a)$ and its value at 0 is clearly minimal and equals to $1 + a$.

ii) Let $f \in \mathcal{F}(a)$ be the extremal function with $f(0) = \alpha(a)$. (Such f exists in view of Propositon 4.2.) We have

$$0 \le f(\pi) = 1 - a + \sum_{k=2}^{\infty} (-1)^k a_k \le 1 - a + \sum_{k=2}^{\infty} a_k \tag{4.8}$$

and hence

$$\alpha(a) = f(0) = 1 + a + \sum_{k=2}^{\infty} a_k \ge 2a + \left(1 - a + \sum_{k=2}^{\infty} a_k\right) \ge 2a \ . \tag{4.9}$$

The above holds for all $a \in \mathcal{D}(\alpha)$. Now if $1 \le a \le 4/3$, let us consider the function

$$f(x) = 1 + a\cos x + (a-1)\cos 2x \tag{4.10}$$

and its translate

$$g(x) = f(x+\pi) = 1 - a\cos x + (a-1)\cos 2x\ . \tag{4.11}$$

Clearly $f(0) = 2a$, hence it suffices to show $f \in \mathcal{F}(a)$, for what we need to prove only $g \ge 0$. Now take $k = 2$ in Lemma 2.2 iii) and v) and observe that with $\lambda = \frac{3}{4}a$ (2.8) is a convex combination if and only if $0 \le a \le 4/3$. We are led to

$$\begin{aligned} 0 \le h = \lambda h_0 + (1-\lambda)H &= \frac{3}{4}a\left(1 - \frac{4}{3}\cos x + \frac{1}{3}\cos 2x\right) + \left(1 - \frac{3a}{4}\right)(1-\cos 2x) \\ &= g(x), \end{aligned} \tag{4.12}$$

which entails ii).

iii) We have already shown $\alpha(a) \ge 2a$ in (4.9), and thus the only thing to do is to exclude equality for $a > 4/3$. Now in case of equality (4.9) and so also (4.8) must hold with equality, and thus $a_k = 0$ for all odd indices k, $f(\pi) = 0$ and

$$\sum_{k=2}^{\infty} a_k = a - 1\ . \tag{4.13}$$

Since f is even, $f'(\pi) = 0$, and so $f \ge 0$ and $f(\pi) = 0$ entails $f''(\pi) \ge 0$, i.e. using also $a_k = 0$ for odd k's, we get

$$\begin{aligned} 0 \le f''(\pi) &= \left(-a\cos x - \sum_{k=2}^{\infty} k^2 a_k \cos kx\right)(\pi) \\ &= a - \sum_{k=2}^{\infty}(-1)^k k^2 a_k = a - \sum_{k=2}^{\infty} k^2 a_k \le a - 4\sum_{k=2}^{\infty} a_k. \end{aligned} \tag{4.14}$$

Now (4.13) and (4.14) together yield $a \le 4/3$ proving that equality can not hold for $a > 4/3$.

iv) By definition, $\alpha(a)$ is nonnegative. On the other hand for arbitrary $a \in [1, \frac{4}{3}]$ we presented a polynomial $g \in \mathcal{F}(-a)$ satisfying $g(0) = 0$.

v) Similarly to iii), take $f \in \mathcal{F}(a)$ with $f(0) = \alpha(a)$ and suppose that $\alpha(a) = 0$. Since f is even, $f'(0) = 0$, hence $f(0) = 0$ and $f \ge 0$ entails $f''(0) \ge 0$. That is, we have

$$0 = f(0) = 1 + a + \sum_{k=2}^{\infty} a_k, \tag{4.15}$$

and

$$0 \le f''(0) = -a - \sum_{k=2}^{\infty} k^2 a_k \le -a - 4 \sum_{k=2}^{\infty} a_k . \tag{4.16}$$

Expressing the sum from (4.15) and inserting it into (4.16) yields

$$0 \le -a + 4(a+1) = 3a + 4, \tag{4.17}$$

proving that equality can not hold for $a < -4/3$.

■

5. Order of magnitude of α at 2

Theorem 5.1. *For every* $0 < a < 2$ *we have*

$$\alpha(a) > \frac{\sqrt{2+a}}{\sqrt{2-a}}. \tag{5.1}$$

Proof. Let n be any integer not exceeding $\pi/\arccos(\frac{a}{2})$ and let $f \in \mathcal{F}(a)$ be chosen with $f(0) = \alpha(a)$. (Such f exists in view of Proposition 4.2.) We want to apply Theorem 2.1 for $d\mu(x) = f(x)dx$ and for $k = 2, 3, \dots, n$. Since $f \in \mathcal{F}(a)$ is positive definite,

$$\alpha(a) = f(0) = 1 + a + \sum_{k=2}^{\infty} a_k \ge 1 + a + \sum_{k=2}^{n} a_k \ , \tag{5.2}$$

hence by Theorem 2.1 and (5.2) we are led to

$$\alpha(a) \ge 1 + a + 2 \sum_{k=2}^{n} \cos kz \quad (z = \arccos(a/2)) . \tag{5.3}$$

Observe, that the right hand side of (5.3) is exacly $D_n(z)$, hence by (1.9)

$$\alpha(a) \ge D_n(z) = \frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\sin(z/2)} \quad (z = \arccos(a/2)) . \tag{5.4}$$

We specify n as $n = [\frac{\pi}{2z}] < \frac{\pi}{z}$. Then $|\frac{\pi}{2} - (n+\frac{1}{2})z| < \frac{z}{2}$, and

$$\sin\left(\left(n+\frac{1}{2}\right)z\right) > \sin\left(\frac{\pi}{2} - \frac{z}{2}\right) = \cos\frac{z}{2} \ . \tag{5.5}$$

From (5.4) and (5.5) we get

$$\alpha(a) > \frac{\cos z/2}{\sin z/2} = \frac{2\cos^2 z/2}{\sin z} = \frac{1+\cos z}{\sqrt{1-\cos^2 z}}. \tag{5.6}$$

and inserting $\cos z = a/2$ gives (5.1).

■

Theorem 5.2. *For $a > 1.315$ we have*

$$\alpha(a) < \frac{4\pi}{3\sqrt{3}}\frac{1}{\sqrt{2-a}}\ . \tag{5.7}$$

Proof. Let us consider the functions (3.6) definied in **3**. From the Fourier series expansion (3.5) we get that

$$a = 2\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^{2l} \tag{5.8}$$

and $\Psi_{h,2l} \in \mathcal{F}(a)$ with this a. By definition of $\alpha(a)$

$$\alpha(a) \le \Psi_{h,2l}(0) \tag{5.9}$$

follows, and using the alternating series expansion (3.9) we get

$$2 - a = 2\left(1 - \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^{2l}\right) < 4l\left(1 - \frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) < \frac{lh^2}{6}. \tag{5.10}$$

From (5.9) and (5.10) we obtain the estimate

$$\alpha(a)\sqrt{2-a} < \sqrt{\frac{l}{6}}h\Psi_{h,2l}(0) = \frac{2\pi}{\sqrt{6}}M_{2l}(0)\sqrt{l}, \tag{5.11}$$

where we used Lemma 3.1 at the end. Apart from the constant factor $2/\sqrt{6}$, this last expression was studied in Lemma 3.2, and we know that the optimal choice for l is $l = 2$ when minimizing this quantity. If $l = 2$, the condition $hl \le \pi$ means $h \le \pi/2$, and as $\sin(t)/t$ is decreasing for $0 < t < \pi$, we get from (5.8) that $0 < h \le \pi/2$ represents all values of a with

$$2 > a \ge 2\left(\frac{\sin\pi/4}{\pi/4}\right)^4 = \frac{2^7}{\pi^4} = 1.314045\ldots\ .$$

Thus for all $a > 1.315$ we have from (5.11) with $l = 2$ that

$$\alpha(a) < \frac{2\pi}{\sqrt{3}}M_4(0)\frac{1}{\sqrt{2-a}} = \frac{4\pi}{3\sqrt{3}}\frac{1}{\sqrt{2-a}},$$

also taking into account $M_4(0) = 2/3$ from (3.3) or (3.16). ∎

Corollary 5.3. *We have*

$$2 \le \liminf_{a\to 2-} \alpha(a)\sqrt{2-a} \le \limsup_{a\to 2-} \alpha(a)\sqrt{2-a} \le \frac{4\pi}{3\sqrt{3}} < 2.4184\,.$$

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