

## Decomposition as the sum of invariant functions with respect to commuting transformations

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**Summary.** As a natural generalization of various investigations in different function spaces, we study the following problem. Let  $A$  be an arbitrary non-empty set, and  $T_j$  ( $j = 1, \dots, n$ ) be arbitrary commuting mappings from  $A$  into  $A$ . Under what conditions can we state that a function  $f : A \rightarrow \mathbb{R}$  is the sum of “periodic”, that is,  $T_j$ -invariant functions  $f_j$ ? (A function  $g$  is periodic or invariant mod  $T_j$ , if  $g \circ T_j = g$ .) An obvious necessary condition is that the corresponding multiple difference operator annihilates  $f$ , i.e.,  $\Delta_{T_1} \dots \Delta_{T_n} f = 0$ , where  $\Delta_{T_j} f := f \circ T_j - f$ . However, in general this condition is not sufficient, and our goal is to complement this basic condition with others, so that the set of conditions will be both necessary and sufficient.

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### 1. Introduction

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function which is a sum of finitely many periodic functions

$$f = f_1 + f_2 + \dots + f_n, \quad f_i(x + \alpha_i) = f_i(x) \quad \text{for all } x \in \mathbb{R}, i = 1, \dots, n \quad (1)$$

with some fixed numbers  $\alpha_i \in \mathbb{R}$ . For  $\alpha \in \mathbb{R}$  let  $\Delta_\alpha$  denote the (forward) difference operator

$$\Delta_\alpha : \mathbb{R}^{\mathbb{R}} \rightarrow \mathbb{R}^{\mathbb{R}}, \quad \Delta_\alpha g(x) := g(x + \alpha) - g(x).$$

Then the  $\alpha_i$ -periodicity of  $f_i$  above means  $\Delta_{\alpha_i} f_i = 0$ , and because the difference operators are commuting, we also have that

$$\Delta_{\alpha_1} \Delta_{\alpha_2} \dots \Delta_{\alpha_n} f = 0. \quad (2)$$

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The starting point of this work is the problem, whether the converse of the above statement is also true, i.e., does (2) imply the existence of a periodic decomposition (1)?

Naturally, this question can be posed in any given function class  $X \subset \mathbb{R}^{\mathbb{R}}$ , when we have  $f \in X$  and (2) and want a decomposition (1) within the class, i.e., we require also  $f_i \in X$  ( $i = 1, \dots, n$ ). If the answer is affirmative, the class  $X$  is said to have the *decomposition property*. It is easy to see that  $\mathbb{R}^{\mathbb{R}}$  or  $C(\mathbb{R})$  (space of continuous functions) do *not* have this property. Indeed, let  $n = 2$  and  $\alpha_1 = \alpha_2 = \alpha$ . The identity function  $f(x) := x$  satisfies  $\Delta_\alpha \Delta_\alpha f = 0$ , but  $f(x) = x$  fails to be the sum of two  $\alpha$ -periodic functions (as then  $f$  would be periodic itself). This shows that the implication (2)  $\Rightarrow$  (1) fails even in the simplest possible case, and that further conditions either on the transformations or on the functions are needed in general.

The study of such decomposition problems originated from I. Z. Ruzsa. In particular, he showed that the identity function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x$  has a decomposition into the sum of  $\alpha$ - respectively  $\beta$ -periodic functions provided that  $\alpha, \beta$  are incommensurable. Later M. Wierdl observed that in the space of arbitrary real-valued functions the difference equation (2) implies (1) if the steps  $\alpha_i$  are linearly independent over  $\mathbb{Q}$  (see [9, Lemma, p. 109]). We extend this result in Corollary 14 to the case when the shifts  $\alpha_i$  are only assumed to be pairwise incommensurable.

Answering also a question of M. Laczkovich, Wierdl [9] showed that for  $n = 2$  identity (2) implies (1) in the space  $BC(\mathbb{R})$  (bounded continuous functions). The proof for general  $n$ , i.e., that  $BC(\mathbb{R})$  has the decomposition property was later done by M. Laczkovich and Sz. Gy. Révész in [7]. Moreover, Laczkovich and Révész generalized the problem to many other function classes in fact considering the derived problem in topological vector spaces in [8]. Later Z. Gajda ([1]) gave alternative proofs based on Banach limits that the spaces  $B(\mathbb{R})$  (bounded functions) and  $UCB(\mathbb{R})$  (uniformly continuous and bounded functions) have the decomposition property. Recently, interesting results and examples were found by V. M. Kadets and S. B. Shumyatskiy in [2, 3] about the decomposition problem in various Banach spaces.

In a different direction, T. Keleti (see [4, 5, 6]) studied related problems and was led to a negative answer regarding the existence of a continuous periodic decomposition of continuous functions on  $\mathbb{R}$  provided only that a measurable decomposition exists on  $\mathbb{R}$ , see [4, Theorem 4.8].

In the present paper we do not restrict to any particular function class, and we neither assume any particular structural properties like smoothness etc. of the transformations. The present work attacks the decomposition problem in the whole space of functions  $\mathbb{R}^A$  with respect to arbitrary commuting operators in  $A^A$ .

Throughout this note  $A$  will denote a fixed nonempty set. We will consider various self maps  $T : A \rightarrow A$ , called *transformations*, and to such a transformation

we associate the corresponding *shift operator*  $T$  (denoted by the same symbol) as  $T(f) := f \circ T$  and the *T-difference operator*  $\Delta_T : \mathbb{R}^A \rightarrow \mathbb{R}^A$  defined as

$$(\Delta_T f) := T(f) - f, \quad (\Delta_T f)(x) = f(Tx) - f(x).$$

A function  $f$  satisfying  $\Delta_T f = 0$  is called *T-invariant*.

A  $(T_1, \dots, T_n)$ -invariant decomposition of some function  $f$  is a representation

$$f = f_1 + f_2 + \dots + f_n, \quad \text{where} \quad \Delta_{T_j} f_j = 0 \quad (j = 1, \dots, n). \quad (3)$$

As mentioned above, we do not assume any properties like smoothness, boundedness, injectivity or surjectivity etc., neither on the transformations nor on the functions, except that all the occurring transformations and functions are defined on the whole set  $A$  and that the occurring transformations must commute. For pairwise commuting transformations  $T_i$  the functional equation

$$\Delta_{T_1} \dots \Delta_{T_n} f = 0 \quad (4)$$

evidently implies for every  $k_1, \dots, k_n \in \mathbb{N}$  (where also  $0 \in \mathbb{N}$  in our present terminology) the equation

$$\Delta_{T_1^{k_1}} \dots \Delta_{T_n^{k_n}} f = 0. \quad (5)$$

Now in this general setting our basic question sounds: Does the functional equation (4) imply the existence of some  $(T_1, \dots, T_n)$ -invariant decomposition (3)? As mentioned above, in general this is not the case. Therefore, we look for further conditions, which, together with (4), are not only necessary, but also sufficient to ensure that such an invariant decomposition exists. More precisely, in the next section we focus on complementary conditions – functional equations – on the functions, which they must satisfy in case of existence of an invariant decomposition (3) and which equations will also imply existence of such a decomposition. In the third section we define a further, still quite general property of transformations, implying that the difference equation (4) also suffices for the existence of an invariant decomposition (3).

With this general framework the pure combinatorial nature of the problem is quite apparent. Since similar questions arise quite often in various settings, the present formulation may help understanding some related problems as well. To emphasize the combinatorial structure, one may reformulate the whole problem so as to consider  $A$  as the vertices of a directed and colored graph, with  $T_j$  being the set of directed edges, colored by the  $j^{\text{th}}$  color. Transformations are defined uniquely on  $A$ , which means that the out-degree of any color is exactly 1 at all points of  $A$ . The pairwise commutativity assumption then means that starting out from a given point and traveling along one blue and one red edge, we arrive at the same point independently of the order we chose of these colors. Looking for (color-) invariant functions is the search for  $f_j$  which assume equal values on points connected by a directed path of the  $j^{\text{th}}$  color. Mentioning this interpretation may reveal the combinatorial nature of the question, although we do not emphasize this language any longer.

## 2. Results for arbitrary transformations

**Theorem 1.** *Let  $A$  be a nonempty set and  $S, T : A \rightarrow A$  commuting transformations,  $f \in \mathbb{R}^A$ . The following are equivalent.*

- i) *There exists a decomposition  $f = g + h$ , with  $g$  and  $h$  being  $S$ - and  $T$ -invariant, respectively.*
- ii)  *$\Delta_S \Delta_T f = 0$ , and if for some  $x \in A$  and  $k, n, k', n' \in \mathbb{N}$  the equality*

$$T^k S^n x = T^{k'} S^{n'} x \quad (6)$$

*holds, then*

$$f(T^k x) = f(T^{k'} x) \quad (7)$$

*must also be satisfied.*

*Proof.* i)  $\Rightarrow$  ii): The first part is obvious. Indeed, as  $T$  and  $S$  commute,

$$\Delta_S \Delta_T f = \Delta_S \Delta_T g + \Delta_S \Delta_T h = \Delta_T \Delta_S g + \Delta_S \Delta_T h = 0 + 0 = 0.$$

Suppose now that (6) holds for some  $x \in A$  and  $k, n, k', n' \in \mathbb{N}$ . Then using commutativity of  $S$  and  $T$ , the  $S$ -invariance of  $g$  and the  $T$ -invariance of  $h$

$$\begin{aligned} f(T^k x) &= g(T^k x) + h(T^k x) = g(S^n T^k x) + h(x) \\ &= g(S^{n'} T^{k'} x) + h(T^{k'} x) = g(T^{k'} x) + h(T^{k'} x) = f(T^{k'} x) \end{aligned}$$

follows.

ii)  $\Rightarrow$  i): Before we can give the decomposition of  $f$  we partition the set  $A$ . We say that two elements  $x, y \in A$  are equivalent, if for some  $k, n, k', n' \in \mathbb{N}$  the equality

$$T^k S^n x = T^{k'} S^{n'} y \quad (8)$$

holds. Needless to say that we indeed defined an equivalence relation  $\sim$ , hence the set  $A$  splits into equivalence classes  $A/\sim$ , from which by the axiom of choice we choose a representation system. Obviously it is enough to define  $g$  and  $h$  on each of these equivalence classes. Indeed, for  $x \in A$  the elements  $x, Tx$  and  $Sx$  are all equivalent, so the invariance of the desired functions  $g, h$  is decided already in the common equivalence class. So our task is now reduced to defining the functions  $g$  and  $h$  on a fixed, but arbitrary equivalence class  $B$ . Let  $x \in B$  and  $x_0$  be the representative of  $B$ . By definition,  $x \in B$  means  $x \sim x_0$ , hence the existence of  $k, n, k', n' \in \mathbb{N}$  satisfying (8) with  $x_0$  in place of  $y$ . Set now

$$G(n, k, n', k', x) := f(T^{k'} x_0) - f(T^k x) + f(x). \quad (9)$$

Note that here appearance of  $n, k, n', k'$  in the argument refers to a particular combination of powers of  $S$  and  $T$  showing  $x \sim x_0$  rather than arbitrary free variables. First we show that whenever  $l, m, l', m' \in \mathbb{N}$  provide an alternative relation

$$T^l S^m x = T^{l'} S^{m'} x_0, \quad (10)$$

then

$$G(n, k, n', k', x) = G(m, l, m', l', x). \quad (11)$$

By assumption

$$T^{k+l'} S^{n+m'} x = T^{k'+l'} S^{n'+m'} x_0 = T^{k'+l} S^{n'+m} x \quad (12)$$

holds, so using 1) we obtain  $f(T^{k+l'} x) = f(T^{k'+l'} x)$ . This, together with the two sides of (12), substituted into the definition (9) of  $G$ , yield

$$\begin{aligned} & G(n + m', k + l', n' + m', k' + l', x) \\ &= f(T^{k'+l'} x_0) - f(T^{k+l'} x) + f(x) = f(T^{k'+l'} x_0) - f(T^{k'+l} x) + f(x) \\ &= G(n' + m, k' + l, n' + m', k' + l', x). \end{aligned} \quad (13)$$

Again using (9), (12) and the assumption  $\Delta_T \Delta_S f = 0$  (in the form that  $\Delta_{T^a} f(z) = \Delta_{T^a} f(S^b z)$  for any  $a, b \in \mathbb{N}$ ) we obtain

$$\begin{aligned} & G(n + m', k + l', n' + m', k' + l', x) - G(n, k, n', k', x) \\ &= \left( f(T^{k'+l'} x_0) - f(T^{k+l'} x) + f(x) \right) - \left( f(T^{k'} x_0) - f(T^k x) + f(x) \right) \\ &= \Delta_{T^{l'}} f(T^{k'} x_0) - \Delta_{T^{l'}} f(T^k x) = \Delta_{T^{l'}} f(T^{k'} S^{n'} x_0) - \Delta_{T^{l'}} f(T^k S^n x) = 0. \end{aligned}$$

This shows  $G(n + m', k + l', n' + m', k' + l', x) = G(n, k, n', k', x)$ , while the same way also  $G(n' + m, k' + l, n' + m', k' + l', x) = G(m, l, m', l', x)$  follows, so now (13) implies (11).

All in all, the function

$$g(x) := G(n, k, n', k', x)$$

is well defined on  $B$  (whence on the whole of  $A$ ). Now  $h$  cannot be else than  $h := f - g$ . To complete the proof, we show that  $g$  and  $h$  have all the necessary properties. Let  $x \in A$  and  $x_0$  be the representative of the class  $B$  of  $x$ : for some  $n, n', k, k' \in \mathbb{N}$  we have  $T^k S^n x = T^{k'} S^{n'} x_0$ , and hence also

$$T^k S^n (Tx) = T^{k'+1} S^{n'} x_0 \quad \text{and equivalently} \quad T^{k+1} S^n x = T^{k'+1} S^{n'} x_0,$$

so we can write by the definition of  $g$

$$\begin{aligned} \Delta_T g(x) &= (f(T^{k'+1} x_0) - f(T^k(Tx)) + f(Tx)) \\ &\quad - (f(T^{k'+1} x_0) - f(T^{k+1} x) + f(x)) = \Delta_T f(x). \end{aligned}$$

As  $h := f - g$ ,  $\Delta_T h = 0$  is immediate. Finally, we prove that  $\Delta_S g = 0$ . For  $x \in B$  we have by (8) with  $x_0 = y$ , similarly to the above that

$$\begin{aligned} \Delta_S g(x) &= (f(T^{k'} x_0) - f(T^k Sx) + f(Sx)) \\ &\quad - (f(T^{k'} x_0) - f(T^k x) + f(x)) = -\Delta_{T^k} \Delta_S f(x) = 0 \end{aligned}$$

in view of ii). □

**Remark 2.** If  $TS \neq ST$  then 1) does not imply  $\Delta_S \Delta_T f = 0$  in general.

**Remark 3.** Condition 1) is symmetric with respect to the pairs  $g, S$  and  $h, T$ . This gives the further equivalent assertion

iii)  $\Delta_S \Delta_T f = 0$ , and if for some  $x \in A$  and  $k, n, k', n' \in \mathbb{N}$  (6) holds, then

$$f(S^n x) = f(S^{n'} x)$$

must be satisfied.

**Theorem 4.** Let  $T_1, \dots, T_n$  be commuting transformations of  $A$  and let  $f$  be a real function on  $A$ . In order to have a  $(T_1, \dots, T_n)$ -invariant decomposition (3) of  $f$ , the following Condition (\*) is necessary.

For every  $N \leq n$ , disjoint  $N$ -term partition  $B_1 \cup B_2 \cup \dots \cup B_N = \{1, 2, \dots, n\}$ , distinguished elements  $h_j \in B_j$  ( $j = 1, \dots, N$ ), indices  $0 < k_j, l_j, l'_j \in \mathbb{N}$ , ( $j = 1, \dots, N$ ) and  $z \in A$  once the conditions

$$(*) \quad T_{h_j}^{k_j} T_i^{l_i} z = T_i^{l'_i} z \quad \text{for all } i \in B_j \setminus \{h_j\}, \text{ for all } j = 1, \dots, N \quad (14)$$

are satisfied, then

$$\Delta_{T_{h_1}^{k_1}} \dots \Delta_{T_{h_N}^{k_N}} f(z) = 0 \quad (15)$$

must also hold.

**Remark 5.** In case all the blocks  $B_j$  are singletons the condition (14) is empty, so (15) expresses exactly (5). In particular, Condition (\*) contains (4).

*Proof of Theorem 4.* We argue by induction on  $n$ . For  $n = 1$  the assertion is trivial and for  $n = 2$  we refer to Theorem 1 and Remark 3. Let  $n > 2$  and assume that the statement of the theorem is true for all  $n' < n$ . Suppose that for the partition  $B_1 \cup B_2 \cup \dots \cup B_N = \{1, 2, \dots, n\}$ , for  $h_j \in B_j$ ,  $k_j, l_i, l'_i \in \mathbb{N}$ , ( $j = 1, \dots, N$ ,  $i \in B_j \setminus \{h_j\}$ ) and for  $z \in A$  the conditions in (14) hold. We need to prove (15).

First, suppose that there are at least two non-empty blocks in the partition, and, say,  $h_1 \in B_1$ . We will apply the induction hypothesis in the following situation. We define  $A'$  as the orbit of  $z$  under  $B_2 \cup \dots \cup B_N$  and  $f' := (\Delta_{T_{h_1}^{k_1}} f)|_{A'}$ . Since  $f$  has a  $(T_1, \dots, T_n)$ -decomposition, the function  $f'$  has an invariant decomposition with respect to the transformations belonging to  $B_2 \cup \dots \cup B_N$ . Indeed, by assumption  $f = f_1 + \dots + f_n$  with  $f_i$  being  $T_i$ -invariant. We show that  $(\Delta_{T_{h_1}^{k_1}} f_i)|_{A'} = 0$  for  $i \in B_1$ . This is obvious for  $i = h_1 \in B_1$ , so let us assume  $i \in B_1 \setminus \{h_1\}$ . Then by (14) for  $x \in A'$  we have  $x = Sz$  with a suitable product  $S$  of mappings  $T_j$ ,  $j \in B_2 \cup \dots \cup B_N$  that

$$\begin{aligned} \Delta_{T_{h_1}^{k_1}} f_i(x) &= f_i(T_{h_1}^{k_1} x) - f_i(x) = f_i(T_{h_1}^{k_1} Sz) - f_i(Sz) \\ &= f_i(T_{h_1}^{k_1} T_i^{l_i} Sz) - f_i(T_i^{l'_i} Sz) \\ &= f_i(T_i^{l'_i} Sz) - f_i(T_i^{l'_i} Sz) = 0. \end{aligned}$$

Since we have the relations

$$T_{h_j}^{k_j} T_i^{l_i} z = T_i^{l'_i} z \quad \text{for all } i \in B_j, i \neq h_j, \text{ for all } j = 2, \dots, N,$$

the induction hypothesis gives

$$\Delta_{T_{h_2}^{k_2}} \dots \Delta_{T_{h_N}^{k_N}} \Delta_{T_{h_1}^{k_1}} f(z) = \Delta_{T_{h_2}^{k_2}} \dots \Delta_{T_{h_N}^{k_N}} f'(z) = 0,$$

hence the assertion.

Second, we suppose that there is only one block in the partition, i.e.,  $B_1 = \{1, \dots, n\}$ . We have to show  $\Delta_{T_{h_1}^{k_1}} f(z) = 0$ . We can suppose without loss of generality that  $h_1 = 1$  (we also write  $k = k_1$ ) so (14) becomes

$$T_1^k T_j^{l_j} z = T_j^{l'_j} z, \quad j = 2, \dots, n.$$

By the condition of decomposability, i.e., the validity of (3), it suffices to show only that

$$f_j(T_1^k z) = f_j(z) \quad \text{if } \Delta_{T_j} f_j = 0.$$

But, indeed, this holds by

$$f_j(T_1^k z) = f_j(T_1^k T_j^{l_j} z) = f_j(T_j^{l'_j} z) = f_j(z). \quad \square$$

We thank Tamás Keleti for suggesting the above formulation of Condition (\*). This allows the following nice reformulation in abelian groups, which was also suggested by him.

**Remark 6.** It is particularly interesting to formulate Condition (\*) in the special case, when  $A$  is an abelian group and the transformations  $T_i$  are translations by  $a_i \in A$ , i.e.,  $T_i x = x + a_i$  for  $x \in A$ ,  $i = 1, \dots, n$ . In this case, Condition (\*) takes the following form.

Suppose that whenever for a partition  $B_1 \cup B_2 \cup \dots \cup B_N = \{1, 2, \dots, n\}$  with distinguished elements  $h_j \in B_j$  and for the natural numbers  $k_j \in \mathbb{N}$ ,  $j = 1, \dots, N$

$$(**) \quad a_i \text{ divides}^1 k_j \cdot a_{h_j} \quad \text{for all } i \in B_j \setminus \{h_j\} \text{ and for all } j = 1, \dots, N, \quad (16)$$

then

$$\Delta_{T_{h_1}^{k_1}} \dots \Delta_{T_{h_N}^{k_N}} f = 0 \quad (17)$$

must also be satisfied.

We saw in Theorem 1 that Condition (\*) – or even a subset of the conditions listed in it – provides also a sufficient condition if  $n = 2$ . Next we push this further.

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<sup>1</sup> For  $a, b \in A$ , we say that  $a$  divides  $b$  if there is  $n \in \mathbb{N}$  with  $n \cdot a = b$ .

**Theorem 7.** *Suppose that  $T_1 = T$ ,  $T_2 = S$  and  $T_3 = U$  commute and the function  $f$  satisfies Condition (\*). Then  $f$  has a  $(T, S, U)$ -invariant decomposition.*

The proof will be based on the following series of lemmas.

**Lemma 8.** *Let  $g \in \mathbb{R}^A$  be a function and  $T$  be a transformation of the set  $A$ . The following statements are equivalent.*

- (i) *There exists a function  $h$  for which  $\Delta_T h = g$ .*
- (ii)  *$\sum_{i=0}^{k-1} g(T^i x) = 0$  whenever  $T^k x = x$ ,  $x \in A$ ,  $k \in \mathbb{N}$ .*

*Proof.* If i) holds and  $T^k x = x$  is satisfied for some  $x \in A$  and  $k \in \mathbb{N}$ , then

$$\sum_{i=0}^{k-1} g(T^i x) = \sum_{i=0}^{k-1} (h(T^{i+1} x) - h(T^i x)) = h(T^k x) - h(x) = 0.$$

Suppose now that ii) holds. We define the equivalence relation:  $x \sim y$  if  $T^k x = T^l y$  with some  $k, l \in \mathbb{N}$ . By the axiom of choice, we select a representative of each equivalence class. Then it suffices to give a proper construction of  $h$  on an arbitrarily given equivalence class  $B$  with representative  $x_0$ , say.

If both

$$T^m x = T^n x_0 \quad \text{and} \quad T^{m'} x = T^{n'} x_0, \quad (18)$$

then also

$$T^{m+n'} x = T^{n+n'} x_0 = T^{n+m'} x. \quad (19)$$

First suppose that  $m' \geq m$  and  $n' \geq n$ , and compute with  $z := T^m x = T^n x_0$  and  $M := \min(n' - n, m' - m)$  and  $N := \max(n' - n, m' - m)$  the relations

$$\begin{aligned} & \sum_{i=0}^{n'-1} g(T^i x_0) - \sum_{j=0}^{m'-1} g(T^j x) - \sum_{i=0}^{n-1} g(T^i x_0) + \sum_{j=0}^{m-1} g(T^j x) \\ &= \sum_{i=n}^{n'-1} g(T^i x_0) - \sum_{j=m}^{m'-1} g(T^j x) \\ &= \sum_{i=0}^{n'-n-1} g(T^i z) - \sum_{j=0}^{m'-m-1} g(T^j z) = \pm \sum_{l=M}^{N-1} g(T^l z). \end{aligned} \quad (20)$$

Now suppose, e.g., that  $N = n' - n \geq M = m' - m$  (the opposite case being similar). Because of (18) we have

$$T^M z = T^{m'-m} z = T^{m'} x = T^{n'} x_0 = T^{n'-n} z = T^N z,$$

which, in view of ii), immediately gives the vanishing of (20).

In case we do not have both conditions  $m' > m$  and  $n' > n$ , let us take  $m'' := m + m' + n'$ ,  $n'' := n + n' + m'$ , and, taking also (19) into account, apply

the known case to  $n, m$  and  $n'', m''$  as well as to  $n', m'$  and  $n'', m''$  separately. These considerations then tell us that (20) is always 0 and so

$$h(x) := \sum_{i=0}^{n-1} g(T^i x_0) - \sum_{j=0}^{m-1} g(T^j x), \quad \text{whenever } T^m x = T^n x_0 \quad (21)$$

is a correct definition of a function  $h$ . In  $T^m x = T^n x_0$  we may suppose that  $m > 0$ , so by  $T^{m-1}(Tx) = T^n x_0$  and using (21) we have

$$h(Tx) - h(x) = \sum_{i=0}^{n-1} g(T^i x_0) - \sum_{j=0}^{m-2} g(T^j Tx) - \sum_{i=0}^{n-1} g(T^i x_0) + \sum_{j=0}^{m-1} g(T^j x) = g(x),$$

hence the assertion follows.  $\square$

**Lemma 9.** *If  $G : A \rightarrow \mathbb{R}$  is an arbitrary function and  $T : A \rightarrow A$  is an arbitrary transformation, then there is a function  $g$  and a  $T$ -invariant function  $\gamma$  such that*

$$\Delta_T g = G + \gamma.$$

*Proof.* First of all we define an equivalence relation  $\sim$ . We say that  $x$  and  $y$  are equivalent,  $x \sim y$ , if there exists  $n, m \in \mathbb{N}$  such that  $T^n y = T^m x$ . Of course this is indeed an equivalence relation, and the equivalence class of an element  $x \in A$  is denoted by  $B_x$ . In view of Lemma 8 we are looking for some  $T$ -invariant  $\gamma$  satisfying

$$\sum_{i=0}^{k-1} \gamma(T^i x) = - \sum_{i=0}^{k-1} G(T^i x), \quad \text{whenever } T^k x = x, k \in \mathbb{N}.$$

By  $\Delta_T \gamma = 0$  this is equivalent to the assertion that for every equivalence class  $B_z$  there is a constant  $\gamma = \gamma(B_z)$  such that

$$-\gamma = \frac{1}{k} \sum_{i=0}^{k-1} G(T^i x), \quad \text{if } T^k x = x, k \in \mathbb{N}^+, x \in B_z.$$

Suppose that  $y \sim x$ ,  $T^k x = x$  and  $T^l y = y$ ,  $k, l \in \mathbb{N}^+$ . By  $x, y \in B_z$  we have  $T^a x = T^b y$  for some  $a, b \in \mathbb{N}$ , and for  $K = kl$  the equations  $T^K x = x$ ,  $T^K y = y$  hold. Now

$$\begin{aligned} \frac{1}{l} \sum_{i=0}^{l-1} G(T^i y) &= \frac{1}{K} \sum_{i=0}^{K-1} G(T^i y) = \frac{1}{K} \sum_{i=b}^{K-1+b} G(T^i y) \\ &= \frac{1}{K} \sum_{i=0}^{K-1} G(T^i T^a x) = \frac{1}{k} \sum_{i=0}^{k-1} G(T^i x), \end{aligned} \quad (22)$$

which means that this quantity is constant for  $x, y$  with the above properties. Define

$$\gamma(x) := \frac{1}{k-l} \sum_{i=l}^{k-1} G(T^i x), \quad \text{if } T^k x = T^l x, k, l \in \mathbb{N}, k > l,$$

and  $\gamma(x)$  arbitrary if there are no such  $k, l$ . By the above argument this definition is independent of the particular choice of  $k, l$ . If  $x \sim y$  and for  $x$  there are no  $k, l$  satisfying  $T^k x = T^l x$ , neither can exist such  $k, l$  for  $y$ . Thus we see that  $\gamma$  can be chosen to be constant on  $B_z$ . The proof is hence complete.  $\square$

**Lemma 10.** *Let  $T$  and  $S$  be commuting transformations of the set  $A$ , and let  $G : A \rightarrow \mathbb{R}$  be an  $S$ -invariant function. Then there exist functions  $\gamma$  and  $g$  such that  $\Delta_S \gamma = \Delta_S g = 0$ ,  $\Delta_T \gamma = 0$  and*

$$\Delta_T g = G + \gamma.$$

*Proof.* Again we define an equivalence relation,  $x \sim y$  if  $S^n x = S^m y$  for some  $n, m \in \mathbb{N}$ . Because of commutativity we have  $Tx \sim Ty$  whenever  $x \sim y$ . Let us consider  $\tilde{A} := A/\sim$  and define  $\tilde{T}(B_x) := B_{Tx}$ , where, in general,  $B_z$  stands for the equivalence class of  $z$ . By the preceding remarks the transformation  $\tilde{T}$  is well-defined. Since  $G$  is  $S$ -invariant, it is constant on each equivalence class  $B_x$ , so  $\tilde{G}(B_x) := G(x)$  is a correct definition of a real-valued function on  $\tilde{A}$ . Applying Lemma 9 to  $\tilde{A}$ ,  $\tilde{T}$ ,  $\tilde{G}$  we obtain the functions  $\tilde{\gamma}$  and  $\tilde{g}$  with  $\Delta_{\tilde{T}} \tilde{g} = \tilde{G} + \tilde{\gamma}$  and  $\Delta_{\tilde{T}} \tilde{\gamma} = 0$ . Defining  $g$  and  $\gamma$  to be constant on each equivalence class of  $\sim$ :

$$g(x) := \tilde{g}(B_x), \quad \gamma(x) = \tilde{\gamma}(B_x)$$

we see immediately that  $\Delta_S \gamma = \Delta_S g = 0$ . Obviously  $\Delta_{\tilde{T}} \tilde{g} = \tilde{G} + \tilde{\gamma}$  implies  $\Delta_T g = G + \gamma$  and  $\Delta_{\tilde{T}} \tilde{\gamma} = 0$  implies  $\Delta_T \gamma = 0$ . This completes the proof.  $\square$

**Lemma 11.** *Let  $T, S$  be commuting transformations of  $A$  and let  $G : A \rightarrow \mathbb{R}$  be a function satisfying  $\Delta_S G = 0$ . Then there exists a function  $g : A \rightarrow \mathbb{R}$  satisfying both  $\Delta_S g = 0$  and  $\Delta_T g = G$  if and only if*

$$\sum_{i=0}^{k-1} G(T^i x) = 0 \quad \text{whenever} \quad T^k S^l x = S^{l'} x, \quad x \in A, \quad k, l, l' \in \mathbb{N}. \quad (23)$$

*Proof.* It is easy to check that existence of a function  $g$  with the above requirements implies (23) (cf Lemma 8), hence we have to prove sufficiency of this condition only.

We can argue similarly as in the proof of Lemma 10, but using Lemma 8 in place of Lemma 9.

Namely, as  $G$  is  $S$ -invariant, we can consider the equivalence relation  $x \sim y$  if  $S^n x \sim S^m y$  for some  $n, m \in \mathbb{N}$ , and define  $\tilde{G}$  on the set of equivalence classes  $\tilde{A} := A/\sim$  as the common value  $G(x)$  of the function  $G$  on the whole class  $B_x$ . Also, by commutativity,  $T$  generates a well-defined transformation  $\tilde{T}$  of  $\tilde{A}$ . With this definition, Lemma 8 applies to  $\tilde{G}$  and  $\tilde{T}$ ; note that in  $\tilde{A}$  two classes are related with respect to  $\tilde{T}$  as  $\tilde{T}^{k'}(B_x) = B_x$  if the condition in (23) holds. Therefore, (23) is equivalent to condition ii) of Lemma 8 when applied to the function  $\tilde{G}$  on the

set  $\tilde{A}$  and the transformation  $\tilde{T}$ . The “lift up”  $g$  of  $\tilde{g}$  as in the proof of Lemma 10 will be appropriate.  $\square$

**Remark 12.** Combining the last two lemmas, one can see that  $\Delta_T g = G$  is equivalent to the requirement that  $\gamma(x) = 0$  whenever  $T^k S^l x = S^{l'} x$ ,  $x \in A$  and  $k, l, l' \in \mathbb{N}$ ; moreover, any proper  $\gamma$  in Lemma 10 must satisfy

$$\gamma(x) = -\frac{1}{k} \sum_{i=0}^{k-1} G(T^i x) \quad \text{whenever } T^k S^l x = S^{l'} x, x \in A, k \in \mathbb{N}^+, l, l' \in \mathbb{N}. \quad (24)$$

In particular,

$$\gamma(x) = -\frac{1}{k-k'} \sum_{i=k'}^{k-1} G(T^i x) \quad \text{if } T^k S^l x = T^{k'} S^{l'} x, k, k', l, l' \in \mathbb{N}, k-k' > 0. \quad (25)$$

Furthermore, looking at the proof of Lemma 9, we see that if no such conditions as in (25) are satisfied for  $x$ , then  $\gamma(x)$  can be chosen to be an arbitrary constant on each equivalence class of  $\sim_{TS}$ , where  $x \sim_{TS} y$  if  $S^a T^b x = S^{a'} T^{b'} y$ , for some  $a, b, a', b' \in \mathbb{N}$ .

*Proof of Theorem 7.* Take  $\Delta_T f = F$ . By taking  $B_1 = \{1\}$ ,  $B_2 = \{2\}$  and  $B_3 = \{3\}$  in Condition (\*) we have

$$\Delta_T \Delta_S \Delta_U f = 0, \quad \text{that is } \Delta_S \Delta_U F = 0. \quad (26)$$

Further, also by Condition (\*) for the partition  $B_1 = \{1\}$ ,  $B_2 = \{2, 3\}$ ,  $h_2 = 3$ , if  $U^{k+k'} S^n x = U^{k'} S^{n'} x$ , then

$$\Delta_T \Delta_{U^k} f(U^{k'} x) = 0, \quad \text{that is } F(U^{k+k'} x) = F(U^{k'} x). \quad (27)$$

The equations (26) and (27) show that 1) of Theorem 1 is fulfilled. This implies the existence of  $S$ - and  $U$ -invariant functions  $H$  and  $L$  respectively with

$$F = H + L.$$

We apply Lemma 10 to obtain the real-valued functions  $h, l, \chi, \lambda$  with

$$\begin{aligned} \Delta_T h &= H + \chi, & \Delta_S h &= \Delta_S \chi = \Delta_T \chi = 0, \\ \Delta_T l &= L + \lambda, & \Delta_U l &= \Delta_U \lambda = \Delta_T \lambda = 0. \end{aligned} \quad (28)$$

Define  $g := f - h - l$ , then  $f = g + h + l$  and  $\Delta_S h = \Delta_U l = 0$ , while  $\Delta_T g = \Delta_T(f - h - l) = F - \Delta_T h - \Delta_T l$ . So using now the decomposition  $F = H + L$  and (28), we arrive at  $\Delta_T g = (H + L) - (H + \chi) - (L + \lambda) = -\chi - \lambda$ .

To illustrate the merit of the next argument, let us assume first that  $T^k x = x$  for some  $x \in A$  and  $k \in \mathbb{N}^+$ . Then we can refer to Lemma 8. We have seen that the function  $\gamma := -(\chi + \lambda) = \Delta_T g$ , hence condition i) of the lemma is satisfied and  $\gamma$  must satisfy the equivalent condition ii). On the other hand,  $\gamma$  is also  $T$ -invariant by construction (see (28)), hence ii) of Lemma 8 can be satisfied if only

$\gamma(T^i x) = 0$  for all  $i = 0, \dots, k$ . Therefore, in case  $T^k x = x$  for some  $x \in A$  and  $k \in \mathbb{N}^+$ , we already have  $\Delta_T g(x) = \gamma(x) = 0$ . Note also that, because of the  $T$ -invariance of  $\gamma$ , for any  $y \in A$  with  $T^n x = T^m y$  for some  $n, m \in \mathbb{N}$  one must have  $\Delta_T g(y) = \Delta_T g(x)$ , in particular  $\Delta_T g(y) = 0$  if  $x$  is as before.

Our aim is to obtain the same thing in general, for all over  $A$ . In the definition of  $\chi$  and  $\lambda$  we may have certain flexibility. Exploiting this and choosing both functions carefully we will have  $\gamma = -(\chi + \lambda) = 0$ . For this purpose, we define an equivalence relation

$$x \sim y \quad \text{if} \quad T^a S^b U^c x = T^{a'} S^{b'} U^{c'} y \quad \text{for some } a, b, c, a', b', c' \in \mathbb{N}. \quad (29)$$

It suffices to restrict considerations to one equivalence class  $B_z = \{y \in A : z \sim y\}$ , so without loss of generality we can work only on  $B_z$  assuming tacitly  $A = B_z$ .

By Remark 12, we *can not* choose  $\chi(x)$  for some  $x$  arbitrarily if some relation of the type

$$T^k S^l x = T^{k'} S^{l'} x, \quad k, k', l, l' \in \mathbb{N}, k > k' \quad (30)$$

holds. Let us call such points  $x$   $(S, T)$ -prescribed.

Suppose first that there are neither  $(S, T)$ -prescribed nor  $(U, T)$ -prescribed points. In this case recalling Lemma 11 we can choose both  $\chi$  and  $\lambda$  to be, e.g., constant 0 on  $A$ .

Next, by symmetry, we can assume that there are e.g.  $(S, T)$ -prescribed points. We will show that in this case  $\chi$  can be also chosen to be a constant. So let now  $x \in A$  be fixed and satisfying (30). Then

$$\chi(x) = -\frac{1}{k-k'} \sum_{i=k'}^{k-1} H(T^i x) \quad (31)$$

must hold by Remark 12. Moreover, relations as in (30) hold for all  $y \in A$ ,  $y \sim_{TS} x$  (where  $x \sim_{TS} y$  iff  $S^a T^b x = S^{a'} T^{b'} y$  for some  $a, b, a', b' \in \mathbb{N}$ , as in Remark 12). Conversely, if  $y$  is not  $(S, T)$ -prescribed, then  $y$  can not be in  $\sim_{TS}$ -relation with the above  $x$ , and by Remark 12, the value of  $\chi$  can be chosen to be arbitrary constant on the whole  $\sim_{TS}$ -class of  $y$ . So let this constant be  $\chi(x)$  ( $x$  is the above fixed element). Now, as there might exist elements  $y \in A$  having (30), we show that  $\chi(y) = \chi(x)$  for all such  $y$ , too, so for all  $y \in A$  regardless whether  $y \sim_{TS} x$  holds or not. To this end, notice first that the relation  $T^k S^l U x = T^{k'} S^{l'} U x$  is also valid. So using  $\Delta_U L = 0$  and thus  $\Delta_U H = \Delta_U F = \Delta_T \Delta_U f$ , we compute

$$\begin{aligned} \Delta_U \chi(x) &= \chi(Ux) - \chi(x) = -\frac{1}{k-k'} \sum_{i=k'}^{k-1} H(T^i Ux) + \frac{1}{k-k'} \sum_{i=k'}^{k-1} H(T^i x) \\ &= -\frac{1}{k-k'} \sum_{i=k'}^{k-1} \Delta_U H(T^i x) = -\frac{1}{k-k'} \sum_{i=k'}^{k-1} \Delta_U F(T^i x) \\ &= \Delta_U \Delta_{T^{k-k'}} f(T^{k'} x) = 0, \end{aligned} \quad (32)$$

the last step being an application of Condition (\*) for the partition  $B_1 = \{3\}$ ,  $B_2 = \{1, 2\}$ ,  $h_2 = 1$  and the equation  $T^k S^l x = T^{k'} S^{l'} x$  (remember the same argument works for  $y$ , too). Because by assumption  $T^a S^b U^c x = T^{a'} S^{b'} U^{c'} y$  holds for some  $a, b, c, a', b', c' \in \mathbb{N}$ , we obtain (using also  $\chi(Ux) = \chi(x)$ ,  $\chi(Uy) = \chi(y)$ , as proved above, and the  $S$ - and  $T$ -invariance of  $\chi$ ) that

$$\chi(x) = \chi(T^a S^b U^c x) = \chi(T^{a'} S^{b'} U^{c'} y) = \chi(y),$$

which shows that  $\chi$  is indeed constant on the whole of  $A$ .

Now, if there is no  $(U, T)$ -prescribed point, then there is absolute freedom in choosing  $\lambda$  (as long as it is constant on the equivalence classes), so we can define it to be the negative of the constant value of  $\chi$ . We obtain  $\gamma = -(\chi + \lambda) = 0$  as required.

Finally, suppose that there are both  $(S, T)$ -prescribed and  $(U, T)$ -prescribed points, say  $x$  and  $x'$ , which then satisfy (30) and

$$T^m U^n x' = T^{m'} U^{n'} x' \quad \text{for some } m, m', n, n' \in \mathbb{N}^+, m > m', \quad (33)$$

respectively. Since  $x \sim x'$ , by definition  $T^a S^b U^c x = T^{a'} S^{b'} U^{c'} x' (= : y)$  holds for some  $a, b, c, a', b', c' \in \mathbb{N}$ . But then  $y$  is both  $(S, T)$ - and  $(U, T)$ -prescribed, that is both (30) and (33) holds with  $y$  replacing  $x$  and  $x'$ , respectively. From this we conclude that

$$\begin{aligned} T^{(m-m')(k-k')} U^{n(k-k')} T^{m'+k'} y &= U^{n'(k-k')} T^{m'+k'} y, \quad \text{and} \\ T^{(m-m')(k-k')} S^{l(m-m')} T^{m'+k'} y &= S^{l'(m-m')} T^{m'+k'} y. \end{aligned} \quad (34)$$

To see the first identity here, we can write, by (33) (with  $x'$  replaced by  $y$ ) and applying  $T^{k'}$  to both sides, that

$$T^{m-m'} U^n T^{m'+k'} y = U^{n'} T^{m'+k'} y.$$

This is just the required equation provided  $k - k' = 1$ , and the general case follows from this inductively. The second line in (34) is proved analogously.

Recall that in case of presence of  $(S, T)$ -prescribed points the function  $\chi$  can be chosen having a constant value. The same applies for  $\lambda$  in case there exists some  $(U, T)$ -prescribed point. So let us now assume that both functions are defined as constants all over  $A$ . Then it remains to show that these constant functions sum to 0 at some, hence on all points of  $A$ .

By (34) and writing  $z := T^{m'+k'} y$ , we arrive at  $T^K U^N z = U^{N'} z$  and  $T^K S^L z = S^{L'} z$ , with appropriate  $K, L, N, L', N' \in \mathbb{N}$ ,  $K > 0$ . Thus, by Remark 12, we must have

$$\begin{aligned} \chi(z) &= -\frac{1}{K} \sum_{i=0}^{K-1} H(T^i z), \quad \text{since } T^K S^L z = S^{L'} z, \quad \text{and} \\ \lambda(z) &= -\frac{1}{K} \sum_{i=0}^{K-1} L(T^i z), \quad \text{since } T^K U^N z = U^{N'} z. \end{aligned} \quad (35)$$

Summing these and using the decomposition of  $F$  we obtain

$$\gamma(z) = -(\chi(z) + \lambda(z)) = \frac{1}{K} \sum_{i=0}^{K-1} F(T^i z) = \frac{1}{K} \Delta_{T^K} f(z) = 0, \quad (36)$$

by Condition (\*) for the partition  $B_1 = \{1, 2, 3\}$  with  $h_1 = 1$  and in view of the equations on the right of (35). That is,  $\chi + \lambda$  is zero, and the proof is complete.  $\square$

**Question 13.** We close this section by the natural question whether Condition (\*) is equivalent to (3) for all  $n \in \mathbb{N}$  ( $n > 3$ ).

### 3. Further results for unrelated transformations

We call two commuting transformations  $S, T$  on  $A$  *unrelated*, if  $T^n S^k x = T^m S^l x$  can occur only if  $n = m$  and  $k = l$ . In particular, then neither of the two transformations can have any cycles in their orbits, nor do their joint orbits have any recurrence.

If all pairs among the transformations  $T_j$  ( $j = 1, \dots, n$ ) are unrelated, then Condition (\*) degenerates, as in (14) we necessarily have that all blocks  $B_j$  are singletons. We saw in Remark 5 that it is exactly the difference equation (4).

As an application in a special situation, consider now the case when the set  $A := \mathbb{R}$  and the transformations are just shifts by real numbers. It is easy to see that  $T_\alpha$  and  $T_\beta$ , the shift operators by  $\alpha \in \mathbb{R}$  and  $\beta \in \mathbb{R}$ , are unrelated if  $\alpha/\beta$  is irrational. Therefore, for  $n = 3$  we obtain the following special case from Theorem 7.

**Corollary 14.** *Let  $\alpha_i$  ( $i = 1, \dots, n$ ) be nonzero real numbers so that  $\alpha_i/\alpha_j$  are irrational whenever  $1 \leq i \neq j \leq n$ . Then the conditions (1) and (2) are equivalent.*

We stated the above corollary for general  $n$  since for unrelated transformations it can be proved for any  $n \in \mathbb{N}$ . In fact, the following more general form holds.

**Theorem 15.** *If the transformations  $T_j$  ( $j = 1, \dots, n$ ) are pairwise (commuting and) unrelated, then the difference equation (4) is equivalent to the existence of some invariant decomposition (3).*

*Proof.* We argue by induction. The cases of small  $n$  are obvious. Existence of an invariant decomposition (3) clearly implies the difference equation (4) for any set of pairwise commuting transformations, unrelated or not, hence it suffices to deal with the converse direction.

Let  $F := \Delta_{T_{n+1}} f$ . As the  $n+1$ -level difference equation of  $f$  is inherited by  $F$  as an  $n$ -level one, by the inductive hypothesis we can find an invariant decomposition

of  $F$  in the form

$$F = F_1 + \cdots + F_n, \quad \text{where} \quad \Delta_{T_j} F_j = 0 \quad (j = 1, \dots, n). \quad (37)$$

Since  $T_{n+1}$  and  $T_j$  are unrelated for  $j = 1, \dots, n$ , the condition (23) in Lemma 11 is void, and therefore the "lift ups"  $f_j$  with  $\Delta_{T_j} f_j = 0$ ,  $\Delta_{T_{n+1}} f_j = F_j$  exist for all  $j = 1, \dots, n$ . Therefore,  $f_{n+1} := f - f_1 - \cdots - f_n$  provides a function satisfying  $\Delta_{T_{n+1}} f_{n+1} = F - F_1 - \cdots - F_n = 0$ , whence a decomposition of  $f$  is established.  $\square$

#### 4. On invariant decompositions of bounded functions

Finally, let us mention a complementary result, which concerns bounded functions, thus is not fully in scope here, but is similar in nature regarding the absolutely unrestricted structural framework of transformations and functions.

**Proposition 16.** *Let  $A$  be any set,  $T, S : A \rightarrow A$  arbitrary commuting transformations, and let  $G : A \rightarrow \mathbb{R}$  be any function satisfying  $\Delta_S G = 0$ . Then the following two assertions are equivalent.*

- i) *There is a bounded function  $H : A \rightarrow \mathbb{R}$  such that  $\Delta_T H = G$  and  $\Delta_S H = 0$ .*
- ii) *There is a constant  $C < +\infty$  such that  $\left| \sum_{i=1}^{m-1} G(T^i x) \right| \leq C$  whenever  $x \in A$  and  $m \in \mathbb{N}$ .*

Moreover, one has the relations  $C \leq \|H\|_\infty \leq 2C$ .

*Proof.* The implication i)  $\Rightarrow$  ii) is immediate with  $C := 2\|H\|_\infty$ , since

$$\sum_{i=1}^{m-1} G(T^i x) = \sum_{i=1}^{m-1} \Delta_T H(T^i x) = H(T^m x) - H(x).$$

The proof of the converse direction ii)  $\Rightarrow$  i) goes along similar lines to the above, hence we skip the details.  $\square$

We mention this as an example of the case when the class of functions on  $A$  we deal with is  $B(A)$ , the set of all bounded functions. It is known that  $B(A)$  has the decomposition property, see [1] and [8], but the exact norm inequalities are not known and very likely depend on the transformations, in particular properties like unrelated and alike. On the other hand it is remarkable, that if  $f$  is bounded, then no further conditions, neither on the transformations nor on  $f$  are involved: (4) itself implies (3). It would be interesting, but perhaps difficult, to determine the best general bound for the norms of individual terms in (3) once  $\|f\|$  is given.

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