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A new short proof of the EKR theorem ☆

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ABSTRACT

A family $\mathcal{F}$ is *intersecting* if $F \cap F' \neq \emptyset$ whenever $F, F' \in \mathcal{F}$. Erdős, Ko, and Rado (1961) [6] showed that

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} \tag{1}$$

holds for an intersecting family of k-subsets of $[n] := \{1, 2, 3, \dots, n\}$, $n \geqslant 2k$. For $n > 2k$ the only extremal family consists of all k-subsets containing a fixed element. Here a new proof is presented by using the Katona's shadow theorem for t-intersecting families.

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1. Definitions: shadows, *b*-intersecting families

$\binom{X}{k}$ denotes the family of k-element subsets of X. For a family of sets $\mathcal{A}$ its s-*shadow* $\partial_s \mathcal{A}$ denotes the family of s-subsets of its members $\partial_s \mathcal{A} := \{S\colon\ |S| = s,\ \exists A \in \mathcal{A},\ S \subseteq A\}$. E.g., $\partial_1 \mathcal{A} = \bigcup \mathcal{A}$. Suppose that $\mathcal{A}$ is a family of a-sets such that $|A \cap A'| \geqslant b \geqslant 0$ for all $A, A' \in \mathcal{A}$. Katona [10] showed that then

$$|\mathcal{A}| \leqslant |\partial_{a-b} \mathcal{A}|. \tag{2}$$

We show that this inequality quickly implies the EKR theorem. This way it is even shorter than the classical proof of Katona [11] using cyclic permutations, or the one found by Daykin [2] applying the Kruskal–Katona theorem.

2. The proof

Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting. Define a partition $\mathcal{F}_0 := \{F \in \mathcal{F}\colon\ 1 \notin F\}$, $\mathcal{F}_1 := \{F \in \mathcal{F}\colon\ 1 \in F\}$ and define $\mathcal{G}_1 := \{F \setminus \{1\}\colon\ 1 \in F \in \mathcal{F}\}$. Consider $\mathcal{F}_0$ as a family on $[2, n]$. Its complementary family $\mathcal{G}_0 :=$

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$\{[2,n] \setminus F\colon\ F \in \mathcal{F}_0\}$ is $(n-1-k)$-uniform. The intersection property of $\mathcal{F}$ implies that any member of $\mathcal{G}_1$ is not contained in any member of $\mathcal{G}_0$. We obtain

$$\mathcal{G}_1 \cap \partial_{k-1}\mathcal{G}_0 = \emptyset.$$

Since both $\mathcal{G}_1$ and $\partial_{k-1}\mathcal{G}_0$ are subfamilies of $\binom{[2,n]}{k-1}$ we obtain that $|\mathcal{G}_1| + |\partial_{k-1}\mathcal{G}_0| \leqslant \binom{n-1}{k-1}$. The intersection size $|G \cap G'|$ of $G, G' \in \mathcal{G}_0$ is at least $n-2k$, since

$$\left|G \cap G'\right| = \left|\bigl([2,n] \setminus F\bigr) \cap \bigl([2,n] \setminus F'\bigr)\right| = (n-1) - 2k + \left|F \cap F'\right|.$$

Then (2) gives (with $a = n-k-1$, $b = n-2k \geqslant 0$) that $|\mathcal{G}_0| \leqslant |\partial_{k-1}\mathcal{G}_0|$. Summarizing

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_0| = |\mathcal{G}_1| + |\mathcal{G}_0| \leqslant |\mathcal{G}_1| + |\partial_{k-1}\mathcal{G}_0| \leqslant \binom{n-1}{k-1}. \qquad \square \tag{3}$$

Extremal families. Equality holds in (2) if and only if $a = b$, or $\mathcal{A} = \emptyset$, or $\mathcal{A} \equiv \binom{[2a-b]}{a}$. Thus, for $n > 2k$, equality in (3) implies either $\mathcal{G}_0 = \emptyset$ and $1 \in \bigcap \mathcal{F}$, or $\mathcal{G}_0 \equiv \binom{[2,n-1]}{n-1-k}$ and $n \in \bigcap \mathcal{F}$.

3. Two algebraic reformulations

Given two families of sets $\mathcal{A}$ and $\mathcal{B}$, the *inclusion matrix* $I(\mathcal{A},\mathcal{B})$ is a 0-1 matrix of dimension $|\mathcal{A}| \times |\mathcal{B}|$, its rows and columns are labeled by the members of $\mathcal{A}$ and $\mathcal{B}$, respectively, the element $I_{A,B} = 1$ if and only if $A \supseteq B$. In the case $\mathcal{F} \subseteq 2^{[n]}$ the matrix $I(\mathcal{F}, \binom{[n]}{1})$ is the usual *incidence matrix* of $\mathcal{F}$, and $I(\mathcal{F}, \binom{[n]}{s})$ is the *generalized* incidence matrix of order s.

Suppose that L is a set of non-negative integers, $|L| = s$, and for any two distinct members A, A' of the family $\mathcal{A}$ one has $|A \cap A'| \in L$. The Frankl, Ray-Chaudhuri, and Wilson [8,13] theorem states that in the case of $\mathcal{A} \subseteq \binom{[n]}{k}$, $s \leqslant k$ the row vectors of the generalized incidence matrix $I(\mathcal{A}, \binom{[n]}{s})$ are linearly independent. Here the rows are taken as real vectors (in [13]) or as vectors over certain finite fields (in [8]). Note that this statement generalizes (2) with $L = \{b, b+1, \ldots, a-1\}$, $s = a-b$.

Matrices and the EKR theorem. Instead of using (2) one can prove directly that the row vectors of the inclusion matrix $I(G_0 \cup G_1, \binom{[2,n]}{k-1})$ are linearly independent. For more details see [8,13].

Linearly independent polynomials. One can define homogeneous, multilinear polynomials $p(F, \mathbf{x})$ of rank $k-1$ with variables $x_2, \ldots, x_n$

$$p(F,\mathbf{x}) = \begin{cases} \sum\{x_S\colon\ S \subset [2,n] \setminus F,\ |S| = k-1\} & \text{for } 1 \notin F \in \mathcal{F}, \\ x_{F \setminus \{1\}} & \text{for } 1 \in F \in \mathcal{F}, \end{cases}$$

where $x_S := \prod_{i \in S} x_i$. To prove (1) one can show that these polynomials are linearly independent. For more details see [9].

4. Remarks

The idea of considering the shadows of the complements (one of the main steps of Daykin's proof [2]) first appeared in Katona [10] (p. 334) in 1964. He applied a more advanced version of his intersecting shadow theorem (2), namely an estimate using $\partial_{a-b+1}\mathcal{A}$.

Linear algebraic proofs are common in combinatorics, see the book [1]. For recent successes of the method concerning intersecting families see Dinur and Friedgut [4,5]. There is a relatively short proof of the EKR theorem in [9] using linearly independent polynomials. In fact, our proof here can be considered as a greatly simplified version of that one.

Since the algebraic methods are frequently insensitive to the structure of the hypergraphs in question it is much easier to give an upper bound $\binom{n}{k-1}$ which holds for *all* n and k (see [3]). To decrease this formula to $\binom{n-1}{k-1}$ requires further insight. Our methods resemble to those of Parekh [12] and Snevily [14] who succeeded to handle this for various related intersection problems.

Generalized incidence matrices proved to be extremely useful, see, e.g., the ingenious proof of Wilson [15] for another Frankl–Wilson theorem, namely the exact form of the classical Erdős–Ko–Rado theorem concerning the maximum size of a k-uniform, t-intersecting family on n vertices. They proved [7,15] that the maximum size is exactly $\binom{n-t}{k-t}$ if and only if $n \geqslant (t+1)(k-t+1)$.

References

[1] L. Babai, P. Frankl, Linear Algebra Methods in Combinatorics, Dept. of Comp. Sci., The University of Chicago, Chicago, 1992.
[2] D. Daykin, Erdős–Ko–Rado from Kruskal–Katona, J. Combin. Theory Ser. A 17 (1974) 254–255.
[3] M. Deza, P. Frankl, Erdős–Ko–Rado theorem – 22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419–431.
[4] Irit Dinur, Ehud Friedgut, Intersecting families are essentially contained in juntas, Combin. Probab. Comput. 18 (2009) 107–122.
[5] Irit Dinur, Ehud Friedgut, On the measure of intersecting families, uniqueness and stability, Combinatorica 28 (2008) 503–528.
[6] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961) 313–320.
[7] P. Frankl, The Erdős–Ko–Rado theorem is true for $n = ckt$, in: Combinatorics, vol. I, Proc. Fifth Hungarian Colloq., Keszthely, 1976, in: Colloq. Math. Soc. J. Bolyai, vol. 18, North-Holland, Amsterdam–New York, 1978, pp. 365–375.
[8] P. Frankl, R.M. Wilson, Intersection theorems with geometric consequences, Combinatorica 1 (1981) 357–368.
[9] Z. Füredi, Kyung-Won Hwang, P.M. Weichsel, A proof and generalizations of the Erdős–Ko–Rado theorem using the method of linearly independent polynomials, in: Topics in Discrete Mathematics, in: Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 215–224.
[10] G. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hungar. 15 (1964) 329–337.
[11] G.O.H. Katona, A simple proof of the Erdős–Ko–Rado theorem, J. Combin. Theory Ser. B 13 (1972) 183–184.
[12] Ojash Parekh, Forestation in hypergraphs: linear k-trees, Electron. J. Combin. 10 (2003), N12, 6 pp.
[13] Ray-Chaudhuri, R.M. Wilson, On t-designs, Osaka J. Math. 12 (1975) 737–744.
[14] H. Snevily, A sharp bound for the number of sets that pairwise intersect at k positive values, Combinatorica 23 (2003) 527–533.
[15] R.M. Wilson, The exact bound in the Erdős–Ko–Rado theorem, Combinatorica 4 (1984) 247–257.