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Extremal k-edge-hamiltonian hypergraphs

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Abstract

An r-uniform hypergraph is k-edge-hamiltonian iff it still contains a hamiltonian-chain after deleting any k edges of the hypergraph. What is the minimum number of edges in such a hypergraph? We give lower and upper bounds for this question for several values of r and k.

Keywords: k-Edge-hamiltonian; Hamiltonian cycle; Hypergraph

1. Introduction

Let $\mathscr{H}$ be an r-uniform hypergraph on the vertex set $V(\mathscr{H})=\{v_1, v_2, \ldots, v_n\}$ where $n > r$. For simplicity of notation v_{n+x} with $x \geqslant 0$ denotes the same vertex as v_x (unless stated otherwise). The set of the edges, r-element subsets of V, is denoted by $\mathscr{E}(\mathscr{H}) = \{E_1, E_2, \ldots, E_m\}$. We will write simply V for $V(\mathscr{H})$ and $\mathscr{E}$ for $\mathscr{E}(\mathscr{H})$ if no confusion can arise.

In [1] the authors defined the notion of a hamiltonian-chain.

Definition 1. A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a *hamiltonian-chain* iff for each $1 \leqslant i \leqslant n$ $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} =: E_j$ is an edge of $\mathscr{H}$. An ordering $(v_1, v_2, \ldots, v_{l+1})$ of a subset of the vertex set is called an *open chain of length* l between v_1 and v_{l+1} iff for each $1 \leqslant i \leqslant l - r + 2$ there exists an edge E_j of $\mathscr{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. An open chain of length $n - 1$ is an *open hamiltonian-chain*. A cyclic ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a *chain of length* l iff for every $1 \leqslant i \leqslant l$ there exists an edge E_j of $\mathscr{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. (Now v_{l+x} denotes the same vertex as v_x).

Definition 2. A hypergraph is *hamiltonian* if it contains a hamiltonian-chain and it is *k-edge-hamiltonian* if by the removal of any k edges a hamiltonian hypergraph is obtained.

The notion of the degree is also extended, it is defined below in full generality; however, only some special cases will be used.

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Definition 3. The *degree of a fixed l-tuple* of distinct vertices, $\{v_1, v_2, \dots, v_l\}$, in an r-uniform hypergraph is the number of edges of the hypergraph containing the set $\{v_1, v_2, \dots, v_l\}$. It is denoted by $d_r(v_1, v_2, \dots, v_l)$. Furthermore $\delta_r^{(l)}(\mathscr{H})$ denotes the minimum of $d_r(v_1, v_2, \dots, v_l)$ over all l-tuples of vertices in $\mathscr{H}$. The *neighborhood* of a vertex v is defined by

$$N_{\mathscr{H}}(v) := \{E - \{v\} | v \in E, E \in \mathscr{E}(\mathscr{H})\}.$$

The main aim of the present article is to investigate minimum size k-edge-hamiltonian hypergraphs. In [2,4] the authors settle this question for graphs.

Theorem 4 (*Paoli et al. [2], Wong and Wong [4]*). *The number of edges in a minimum k-edge-hamiltonian graph on $n \geqslant k+3$ vertices is $\lceil n(k+2)/2 \rceil$.*

Since the degree of any vertex in an r-uniform hamiltonian-chain is r, the minimum degree in a k-edge-hamiltonian hypergraph is at least $r+k$, so the number of edges is at least $\lceil n(r+k)/r \rceil$. For $r=2$ this shows that the constructions in the above theorem are best possible. However, for $r>2$ this lower bound is not best possible.

2. 3-Uniform hypergraphs

If a hypergraph contains $k+1$ edge-disjoint hamiltonian-chains, then it is clearly k-edge-hamiltonian. This observation leads to the trivial upper bound on the minimum number of edges: $(k+1)n$. If $k=1$ then the following slightly better upper bound is obtained.

Theorem 5. *There exists a* 1-*edge-hamiltonian* 3-*uniform hypergraph $\mathscr{H}$ on n vertices with*

$$|\mathscr{E}(\mathscr{H})| = \tfrac{11}{6}n + \mathrm{o}(n).$$

Proof. Let $\mathscr{V}(\mathscr{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$ where $p = \lceil n/6 \rceil$ and $q = n - p$. There are two types of edges in $\mathscr{H}$. The first kind of edges form a chain on $\{v_1, \dots, v_q\}$,

$$\mathscr{E}_1(\mathscr{H}) := \{\{v_i, v_{i+1}, v_{i+2}\} | 1 \leqslant i \leqslant q\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathscr{E}_2(\mathscr{H}) := \{\{w_i, v_{5(i-1)+j}, v_{5(i-1)+j+1}\} | 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant 6\}.$$

This means that the neighborhood of w_i is an ordinary graph, a path of length 6 formed by vertices $v_{5(i-1)+1}, \dots, v_{5(i-1)+7}$. The neighborhood of w_{i+1} is also a path of length 6, which begins at $v_{5(i-1)+6}$, so $v_{5(i-1)+6}, v_{5(i-1)+7} \in N(w_i) \cap N(w_{i+1})$ (except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $6 \nmid n$). Let $\mathscr{E}(\mathscr{H}) := \mathscr{E}_1(\mathscr{H}) \cup \mathscr{E}_2(\mathscr{H})$, then it is clear that $|\mathscr{E}(\mathscr{H})| = q + 6p = n + 5\lceil n/6 \rceil = 11n/6 + \mathrm{o}(n)$ (see Fig. 1).

This hypergraph contains many hamiltonian-chains which can be obtained in the following way. Start with the chain formed by $\{v_1, \dots, v_q\}$ and extend this cycle by inserting the rest of the vertices one by one. It is obvious that we can insert w_i between any two consecutive vertices of $v_{5(i-1)+2}, v_{5(i-1)+3}, v_{5(i-1)+4}, v_{5(i-1)+5}, v_{5(i-1)+6}$ (but we cannot insert it between $v_{5(i-1)+1}$ and $v_{5(i-1)+2}$ or $v_{5(i-1)+6}$ and $v_{5(i-1)+7}$). Note that the new chain contains three "consecutive edges" of $N(w_i)$ but it does not contain two "consecutive edges" from the original chain (those which contain both neighbors of w_i in the new chain (see Fig. 2).

Now we prove that $\mathscr{H}$ is 1-edge-hamiltonian, that is, $\mathscr{H} - E$ contains a hamiltonian-chain for any $E \in \mathscr{E}(\mathscr{H})$.

Suppose that $E = \{v_t, v_{t+1}, v_{t+2}\} \in \mathscr{E}_1(\mathscr{H})$. Then it is easy to check that there is a w_i which we can insert either between v_t and v_{t+1} or v_{t+1} and v_{t+2}, so the new chain does not contain E any more. Further, we can insert all other w vertices into suitable places, hence we obtain the desired hamiltonian-chain (see Fig. 3), for example the following one

$$v_t, v_{t+1}, w_i, v_{t+2}, v_{t+3}, \dots, v_{t+5}, v_{t+6}, w_{i+1}, v_{t+7}, v_{t+8}, \dots, v_{t+5j}, v_{t+5j+1}, w_{i+j}, v_{t+5j+2}, \dots.$$

On the other hand, if $w_i \in E$ for some i then it is clear that $N(w_i) - E$ always contains three "consecutive edges", therefore w_i can be inserted into the chain formed by $\{v_1, \dots, v_q\}$. Inserting the rest of the vertices in the same way as in the other case, we obtain a hamiltonian-chain of $\mathscr{H} - E$. $\square$

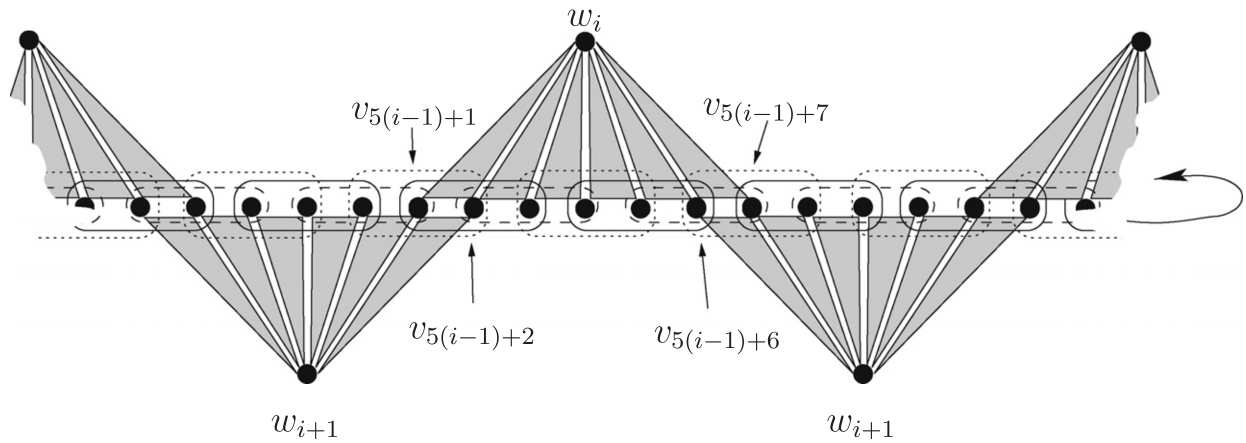

Fig. 1. 3-Uniform 1-edge-hamiltonian hypergraph.

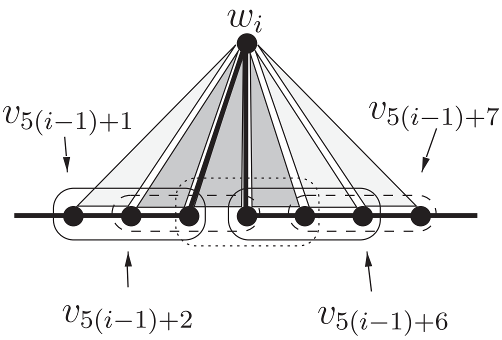

Fig. 2. How to insert w_i?

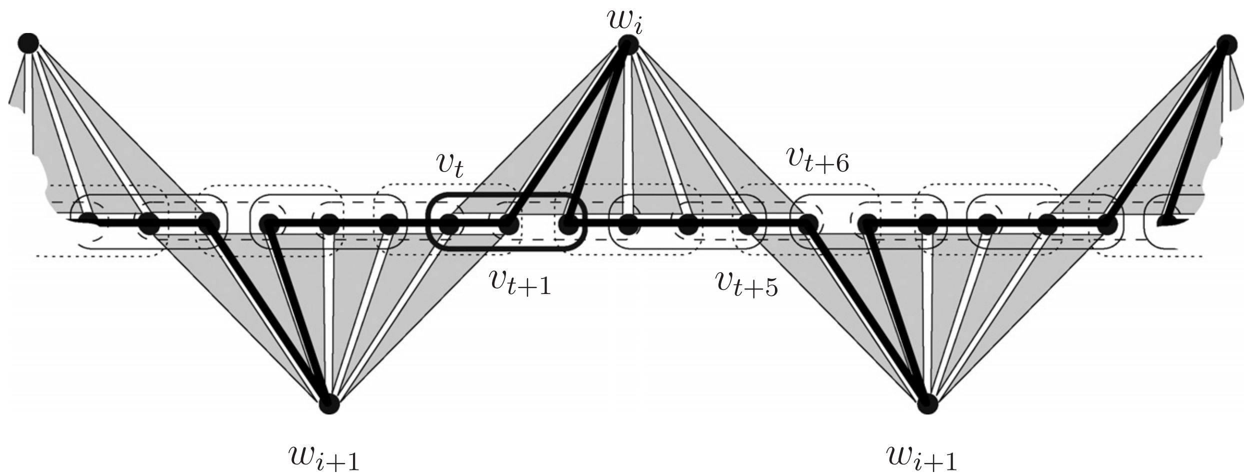

Fig. 3. Hamiltonian-chain in $\mathscr{H} - E$.

Theorem 6. *For any* 1*-edge-hamiltonian* 3*-uniform hypergraph* $\mathscr{H}$ *on* $n \geqslant 5$ *vertices*

$$|\mathscr{E}(\mathscr{H})| \geqslant \frac{14}{9} n$$

holds.

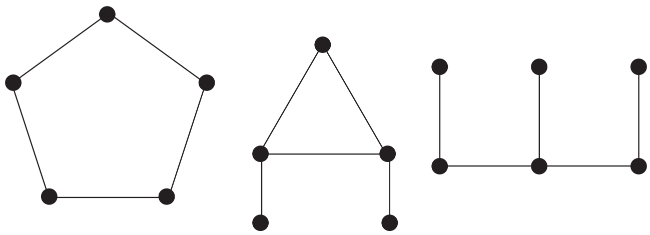

Fig. 4. Stable graphs with five edges.

Proof. Observe that the neighborhood of a vertex in a hamiltonian-chain is a path on four distinct vertices, a P_4. Let us call a graph *stable* if it contains a P_4 after deleting any edge of the graph. Thus, the neighborhood of every vertex of a 1-edge-hamiltonian graph is stable. We also call a vertex of the hypergraph *stable* iff its neighborhood is stable.

It is easy to check that the only stable graph with four edges is the C_4, the cycle with four edges. All other stable graphs contain at least five edges. Clearly any graph which contains C_4 as a subgraph is also 1-stable. There are such graphs. On the other hand, there are three other 1-stable graphs with five edges without a C_4 (see Fig. 4).

Let $\mathscr{H}$ be a 1-edge-hamiltonian 3-uniform hypergraph and let $v_1, \dots, v_n$ be a hamiltonian-chain.

Claim 1. $d_3(v_{i-2}) + d_3(v_i) + d_3(v_{i+2}) \geqslant 14$ *holds for any* i.

Proof. Note that, the only way to make $|N(v_i)| = 4$ is to add the edge $\{v_i, v_{i-2}, v_{i+2}\}$ to $\mathscr{H}$, because $N(v_i)$ already contains the edges $\{v_{i-2}, v_{i-1}\}$, $\{v_{i-1}, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\}$.

Suppose that $d_3(v_{i-2}) + d_3(v_i) + d_3(v_{i+2}) \leqslant 13$. Since $d_3(v_j) \geqslant 4$ for any j, there are only two cases.

If $d_3(v_{i-2}) = d_3(v_i) = 4 \leqslant d_3(v_{i+2})$ (or $d_3(v_{i-2}) \geqslant 4 = d_3(v_i) = d_3(v_{i+2})$) then $\{v_{i-2}, v_i, v_{i+2}\} \in \mathscr{E}(\mathscr{H})$ must hold, but this implies $d_3(v_{i-2}) \geqslant 5$, a contradiction.

The other case is when $d_3(v_{i-2}) = d_3(v_{i+2}) = 4 \leqslant d_3(v_i) \leqslant 5$. Since v_{i-2} and v_{i+2} is stable, $\{v_{i-4}, v_{i-2}, v_i\}$, $\{v_i, v_{i+2}, v_{i+4}\} \in \mathscr{E}(\mathscr{H})$ holds. However, this means that $N(v_i)$ is a path of length 5 with six distinct vertices. This is a contradiction, because this graph is not stable, therefore $d_3(v_i) \geqslant 6$. □

Using the above claim, we obtain that

$$9|\mathscr{E}(\mathscr{H})| = 3\sum_{i=1}^{n} d_3(v_i) = \sum_{i=3}^{n+2} d_3(v_{i-2}) + d_3(v_i) + d_3(v_{i+2}) \geqslant 14n,$$

proving the theorem. □

Theorem 7. *There exists a* 2*-edge-hamiltonian* 3*-uniform hypergraph* $\mathscr{H}$ *on n vertices with*

$$|\mathscr{E}(\mathscr{H})| = \tfrac{13}{4}n + \mathrm{o}(n).$$

Proof. The structure of the construction is very similar to that of Theorem 5. Let $\mathscr{V}(\mathscr{H}) := \{w_1, \dots, w_p, v_1, \dots, v_q\}$ where $p = \lceil n/4 \rceil$ and $q = n - p$. There are two types of edges in $\mathscr{H}$. The first kind of edges form a chain on $\{v_1, \dots, v_q\}$:

$$\mathscr{E}_1(\mathscr{H}) := \{\{v_i, v_{i+1}, v_{i+2}\} | 1 \leqslant i \leqslant q\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathscr{E}_2(\mathscr{H}) := \{\{w_i, v_{4(i-1)+j}, v_{4(i-1)+j+1}\} | 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant 9\}.$$

This means that the neighborhood of w_i is an ordinary graph, a path of length 9 formed by vertices $v_{4(i-1)+1}, \dots, v_{5(i-1)+10}$. The neighborhood of w_{i+1} is also a path of length 9, which begins at $v_{4(i-1)+5}$, so the neighborhood of w_i and w_{i+1} have six common vertices and the neighborhood of w_i and w_{i+2} have two common vertices (except maybe

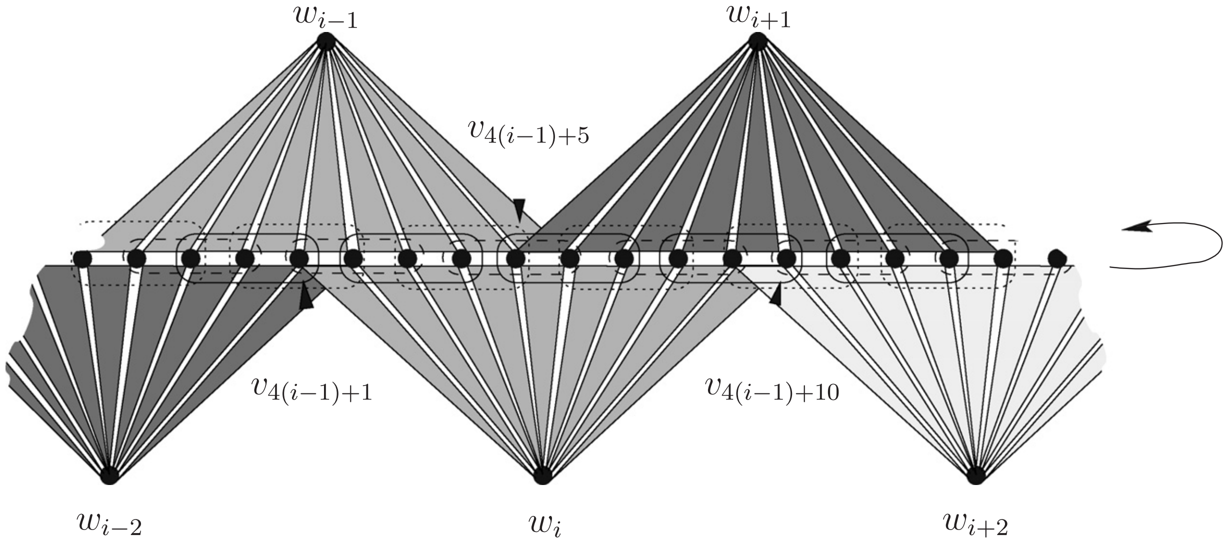

Fig. 5. 2-Edge-hamiltonian 3-uniform hypergraph.

at the "end" where the overlap is larger if $4\nmid n$). Let $\mathscr{E}(\mathscr{H}) := \mathscr{E}_1(\mathscr{H}) \cup \mathscr{E}_2(\mathscr{H})$, then it is clear that $|\mathscr{E}(\mathscr{H})| = q + 9p = n + 9\lceil n/4\rceil = 13n/4 + \mathrm{o}(n)$ (see Fig. 5).

Using the method described in the proof of Theorem 5 it can be easily proven, that $\mathscr{H}$ is 1-edge-hamiltonian. It is also clear that $\mathscr{H}$ remains hamiltonian if the two removed edges are "far" from each other, namely if no w_i for which its neighborhood intersects both removed edges.

If both edges contains w_i then we can still insert w_i in a similar way as in Fig. 2, since there are nine edges containing w_i, so after the removal of 2, we still have three consecutive.

The other cases can be also proved one by one, the reader may verify this with the help of a few examples in Fig. 6. □

In order to obtain a lower bound for general k, one should know the minimum number of edges in a graph which contains a P_4 after removing any k edges of the graph. We will call such graphs *k-stable* and denote the minimum number of edges in a k-stable graph by $S(k)$.

A trivial upper bound is obtained for $S(k)$ in the following way.

Observe that the maximum number of edges on n vertices in a P_4-free graph is $n-1$ if n is not divisible by 3 and n if n is divisible by 3. The extremal graphs are union of at least one star and some (possibly zero) triangles in the first case, and union of triangles in the second case. Taking the densest graph on n vertices, a complete graph or an almost complete graph will give the desired bound. By the above observation if $e(G) > n - 1 + k$ if $3\nmid n$ and $e(G) > n + k$ if $3 \mid n$ then G is k-stable. However, for three values of k there are constructions which give bounds smaller by 1.

An other remark is that $S(k)$ is strictly monotone, since by removing an edge from a k-stable graph results in a $(k-1)$-stable graph.

The following lemma shows, that to prove $S(k) > m$ it is enough to prove, that none of the graphs on exactly m edges are k-stable, so it is not possible that there is a k-stable graph with $m-2$ edges for example.

Lemma 8. *If for any graph G with $e(G) = m$ the graph is not k-stable then $S(k) > m$.*

Proof. Let G' be a graph with $e(G') < m$. We will prove that G' cannot be k-stable. Construct G from G' by adding $m - e(G')$ independent edges. Suppose indirectly that G' is k-stable. Since $e(G) = m$, if we remove k edges from G then we removed $\leqslant k$ edges from G' so it will contain a P_4, thus G is k-stable, a contradiction. □

The next lemma will help us to handle some easy extremal cases.

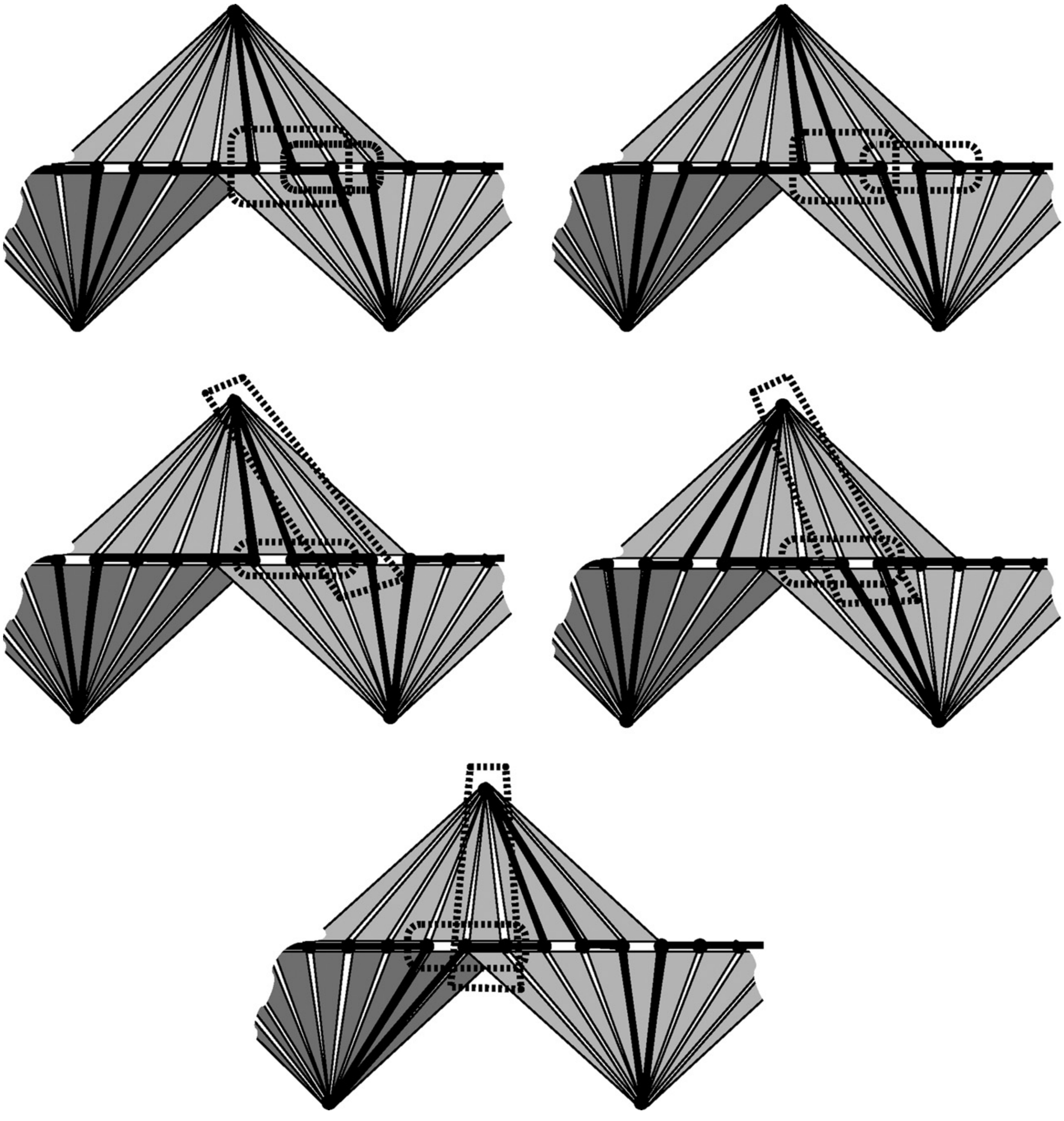

Fig. 6. Examples of the more complicated cases.

Lemma 9. *If the maximum degree in G is* 2, *then G contains a* P_4*-free subgraph with at least* $\lceil e(G)/2 \rceil$.

Proof. It is enough to prove the claim for connected graphs, because otherwise taking the union of the subgraphs found in each component will prove the claim.

If the graph is an even cycle, then take every other edge for the desired subgraph. If the graph is an odd cycle, then first take two consecutive edges, then every other edges. Similarly is the graph is an odd path, then take every other edge starting with the first edge. If the graph is an even path, then take the first two consecutive edges and then every other edges. □

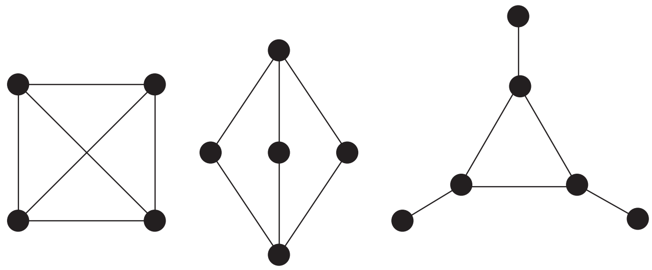

Fig. 7. 2-Stable graphs with six edges.

Theorem 10. $S(1)=4$, $S(2)=6$.

Proof. For $k=1$ the proof is trivial.

If $k=2$ then suppose that there exists a 2-stable graph G_2 with five edges. It is clear that removing any edge of G_2 gives a 1-stable graph with four edges, so it must be C_4. One can easily verify that there is no such G_2. Note that there are three different 2-stable graphs with six edges (see Fig. 7).

On the other hand $S(k) \leqslant 6$ since K_4 is 2-stable. If we remove two edges from it four edges remain on four vertices, so it must contain a P_4. $\square$

Theorem 11. $S(3)=8$, $S(4)=9$, $S(5)=10$.

Proof. Since $S(k)$ is strictly monotone, it is enough to prove that $S(3) \geqslant 8$ and $S(5) \leqslant 10$.

To prove the second claim we show that K_5 is 5-stable. If five edges are removed from K_5 then five edges remain on five vertices, so it must contain a P_4.

To prove $S(3) \geqslant 8$ suppose indirectly that there exists a 3-stable graph G_3 with $|E(G_3)|=7$.

(a) There exists a vertex of degree $\geqslant 4$ in G_3: Four edges incident to a vertex does not contain a P_4 so if we remove the rest of the edges, no P_4 remains. Thus we may suppose that the maximum degree in G_3 is at most 3.
(b) There exists a triangle in G_3: If there is an edge independent from the triangle, then these four edges do not contain a P_4, so by removing the rest of the edges our claim is proved. Otherwise, all other edges have one end common with the triangle, moreover by case (a) there is at most 1 such edge at each vertex of the triangle. This implies that $|E(G_3)| \leqslant 6$, so we may suppose that there is no triangle.
(c) There exist a vertex v_1 with degree 3: Let v_2, v_3, v_4 be its neighbors. Since there is no triangle in the graph, there are no edges between v_2, v_3, v_4. Since there are four more edges, there must be two of them which are not adjacent to two vertices of v_2, v_3, v_4, say to v_2, v_3. So these two edges and (v_1, v_2) and (v_1, v_3) forms a P_4-free subgraph. Thus we may suppose that the maximum degree is 2. Applying Lemma 9 we complete the proof. $\square$

Theorem 12. $S(6)=12$, $S(7)=13$, $S(8)=14$.

Proof. Since $S(k)$ is strictly monotone, it is enough to prove that $S(6) \geqslant 12$ and $S(8) \leqslant 14$.

To prove the second claim we show that the graph G_8 in Fig. 8 is 8-stable. If eight edges are removed from G_8 then six edges remain on seven vertices. There are only three ways for these edges to form a P_4-free graph. (1) two independent triangles; (2) a triangle and a star with three edges; (3) a star with six edges.

(1) is not possible, since any triangle contains at least two vertices from the a_i vertices. (2) is also not possible, since all stars with three edges in G_8 contain at least two vertices from the a_i vertices, and the same holds for any triangle. (3) is not possible since there is no vertex with degree 6 in G_8.

To prove $S(6) \geqslant 12$ suppose indirectly that there exists a 6-stable graph G_6 with $|E(G_6)|=11$.

(a) There exists a vertex of degree $\geqslant 5$ in G_6: Five edges incident to a vertex does not contain a P_4 so if we remove the rest of the edges, no P_4 remains. Thus we may suppose that the maximum degree in G_6 is at most 4.

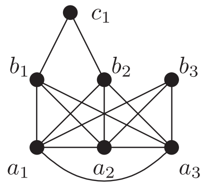

Fig. 8. G_8.

(b) There exists a triangle in G_6: If there are two independent edges from the triangle, then these five edges form a P_4-free subgraph. Otherwise, all other edges have one end common with the triangle, moreover by case (a) there is at most two such edges at each vertex of the triangle. This implies that $|E(G_6)|\leqslant 9$, so we may suppose that there is no triangle.

(c) There exist a vertex v_1 with degree 4: Let v_2, v_3, v_4, v_5 be its neighbors. Since there is no triangle in the graph, there are no edges between v_2, v_3, v_4, v_5. Since there are seven more edges, there must be two of them which are not adjacent to three vertices of v_2, v_3, v_4, v_5, say to v_2, v_3, v_4. So these two edges and (v_1, v_2), (v_1, v_3) and (v_1, v_4) form a P_4-free subgraph. Thus we may suppose that the maximum degree is 3.

(d) There exist a vertex v_1 with degree 3: Let v_2, v_3, v_4 be its neighbors. Since there is no triangle in the graph, there are no edges between v_2, v_3, v_4 and since the maximum degree is 3 there are at most two other edges incident to each of v_2, v_3, v_4. Thus there must be at least two edges which are not incident to any of v_1, v_2, v_3, v_4. So these two edges and (v_1, v_2), (v_1, v_3) and (v_1, v_4) forms a P_4-free subgraph. Thus we may suppose that the maximum degree is 2, so applying Lemma 9 we can complete the proof. □

It looks like that the extremal graph in the general case is a nearly complete graph of suitable size. This suggests the following:

Conjecture 13. The minimum number of edges in a k-stable graph is

$$S(k) = \lceil k + \sqrt{2k + \tfrac{9}{4}} + \tfrac{3}{2}\rceil + \mathrm{O}(1).$$

Following theorem gives an upper bound on the maximum number of edges. We already know that this bound is better than the trivial one if $2\leqslant k\leqslant 8$ and if Conjecture 13 is true, then we obtain a good bound for larger k values, too.

Theorem 14. *For any k-edge-hamiltonian 3-uniform hypergraph $\mathscr{H}$ on n vertices*

$$|\mathscr{E}(\mathscr{H})| \geqslant \frac{S(k)}{3}n$$

holds.

Proof. If $\mathscr{H}$ is k-hamiltonian then the neighborhood of any vertex must be k-stable, which implies that any vertex is contained in at least $S(k)$ edges. Since every edge contains exactly three vertices, the claim is proved. □

3. 1-Edge-hamiltonian hypergraphs

Theorem 15. *There exists a 1-edge-hamiltonian r-uniform hypergraph $\mathscr{H}$ on n vertices with*

$$|\mathscr{E}(\mathscr{H})| = \frac{4r-1}{2r}n + \mathrm{o}(n).$$

Proof. The idea of the construction is similar to the one in Fig. 1. Let $\mathscr{V}(\mathscr{H}) := \{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ where $p = \lceil n/2r\rceil$ and $q = n - p$. There are two types of edges in $\mathscr{H}$. The first kind of edges form a chain on $\{v_1, \ldots, v_q\}$,

$$\mathscr{E}_1(\mathscr{H}) := \{\{v_i, v_{i+1}, \ldots v_{i+r-1}\} | 1\leqslant i\leqslant q\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathscr{E}_2(\mathscr{H}) := \{\{w_i, v_{(2r-1)(i-1)+j}, \dots, v_{(2r-1)(i-1)+j+r-2}\} | 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant 2r\}.$$

This means that the neighborhood of w_i is an $(r-1)$-uniform open chain of length $2r$ formed by vertices $v_{(2r-1)(i-1)+1}, \dots, v_{(2r-1)(i-1)+3r-2}$. The neighborhood of w_{i+1} is also an open chain of length $2r$, which begins at $v_{(2r-1)(i-1)+2r}$, so

$$v_{5(i-1)+2r}, \dots, v_{5(i-1)+3r-2} \in N(w_i) \cap N(w_{i+1})$$

(except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $(2r)\nmid n$). Let $\mathscr{E}(\mathscr{H}) := \mathscr{E}_1(\mathscr{H}) \cup \mathscr{E}_2(\mathscr{H})$, then it is clear that $|\mathscr{E}(\mathscr{H})| = q + 2rp = n + (2r-1)\lceil n/2r \rceil = [(4r-1)/2r]n + \mathrm{o}(n)$.

One can prove that this hypergraph is 1-hamiltonian in the same way as in Theorem 5. □

Theorem 16. *For any* 1*-edge-hamiltonian* 4*-uniform hypergraph* $\mathscr{H}$ *on* $n \geqslant 6$ *vertices*

$$|\mathscr{E}(\mathscr{H})| \geqslant \tfrac{3}{2} n$$

holds.

Proof. Following the idea of the proof of Theorem 6 we need to know what is the minimum number of edges in a 1-stable 3-uniform hypergraph. Now 1-stable means that the hypergraph contains an open chain with four edges on six vertices $\mathscr{P}_6^{(3)}$, since the edges of a hamiltonian-chain containing a fixed vertex form such an open chain.

It is easy to see that it is impossible to create a 1-stable hypergraph by adding only one edge to $\mathscr{P}_6^{(3)}$, therefore the minimum number of edges in a 1-stable hypergraph is 6, since the 3-uniform hyperchain on six vertices, $\mathscr{C}_6^{(3)}$ is a 1-stable with six edges.

This gives that the minimum degree is 6, completing the proof. □

Note that the above bound is already better than the trivial one. On the other hand, by case analysis, we can also prove that $\mathscr{C}_6^{(3)}$ is the only 1-stable hypergraph with six vertices, which leads to a better lower bound:

$$|\mathscr{E}(\mathscr{H})| \geqslant \tfrac{11}{6} n.$$

However, the proof is too long compared with the improvement, so it is omitted.

4. An application

A natural extremal question about the hamiltonian cycle is that how many edges a graph without a hamiltonian cycle can contain. The extremal case is a complete graph on $(n-1)$ vertices completed by a vertex of degree one. The corresponding question for hypergraphs was raised in [1]. The first upper and lower bound was also given in [1]. Tuza improved the lower bound in [3] by giving a construction having $\binom{n-1}{r} + \binom{n-1}{r-2}$ edges that contains no hamiltonian-chain. We now improve the upper bound $\binom{n}{r}(1 - 1/n)$ given in [1].

Theorem 17. *If an r-uniform hypergraph* $\mathscr{H}$ *on n vertices has no hamiltonian-chain then*

$$|\mathscr{E}(\mathscr{H})| \leqslant \binom{n}{r}\left(1 - \frac{4r}{(4r-1)n}\right) \tag{1}$$

holds.

Proof. Let m denote the number of missing edges (the r-element subsets which are not edges of $\mathscr{H}$). By (1) we obtain

$$m < \frac{4r}{(4r-1)n}\binom{n}{r}.$$

Let $\mathscr{K}_n^{(k)}$ denote the complete k-uniform hypergraph on n vertices. Observe that if a hypergraph contains a 1-edge-hamiltonian subgraph then one must delete at least two edges from it to destroy all hamiltonian-chains. Therefore in $\mathscr{K}_n^{(k)}$ we count the number of occurrences of the 1-edge-hamiltonian hypergraph constructed in Theorem 15. Let $\mathscr{G}$ denote this r-uniform hypergraph on n vertices.

It is a simple matter to prove that there are $n!/|\mathrm{Aut}(\mathscr{G})|$ different $\mathscr{G}$ subhypergraphs in $\mathscr{K}_n^{(k)}$, where $\mathrm{Aut}(\mathscr{G})$ denotes the automorphism group of $\mathscr{G}$. Since every edges of $\mathscr{K}_n^{(k)}$ is contained in the same number of $\mathscr{G}$ subhypergraphs, the number of $\mathscr{G}$ subhypergraphs which contains a specified edge is

$$\frac{|\mathscr{E}(\mathscr{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\mathrm{Aut}(\mathscr{G})|}.$$

Thus the number of hamiltonian-chains in $\mathscr{H}$ is

$$\geqslant 2 \cdot \frac{n!}{|\mathrm{Aut}(\mathscr{G})|} - m\frac{|\mathscr{E}(\mathscr{G})|}{\binom{n}{r}} \cdot \frac{n!}{|\mathrm{Aut}(\mathscr{G})|} > 0,$$

our claim is proved, because by Theorem 15 $|\mathscr{E}(\mathscr{G})| = [(4r-1)/2r]n$. $\square$

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