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A note on the jumping constant conjecture of Erdős

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Abstract

Let $r \geqslant 2$ be an integer. The real number $\alpha \in [0,1]$ is a jump for r if there exists $c > 0$ such that for every positive ϵ and every integer $m \geqslant r$, every r-uniform graph with $n > n_0(\epsilon, m)$ vertices and at least $(\alpha+\epsilon)\binom{n}{r}$ edges contains a subgraph with m vertices and at least $(\alpha+c)\binom{m}{r}$ edges. A result of Erdős, Stone and Simonovits implies that every $\alpha \in [0,1)$ is a jump for $r=2$. For $r \geqslant 3$, Erdős asked whether the same is true and showed that every $\alpha \in [0, \frac{r!}{r^r})$ is a jump. Frankl and Rödl gave a negative answer by showing that $1-\frac{1}{l^{r-1}}$ is not a jump for r if $r \geqslant 3$ and $l > 2r$. Another well-known question of Erdős is whether $\frac{r!}{r^r}$ is a jump for $r \geqslant 3$ and what is the smallest non-jumping number. In this paper we prove that $\frac{5}{2}\frac{r!}{r^r}$ is not a jump for $r \geqslant 3$. We also describe an infinite sequence of non-jumping numbers for $r = 3$.

Keywords: Extremal hypergraph problems

1. Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. We call $G=(V,E)$ an *r-uniform graph* if $E \subseteq \binom{V}{r}$. The density of G is defined by $d(G)=\frac{|E|}{|\binom{V}{r}|}$.

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Let $\mathcal{S}=\{G_n\}_{n=1}^{\infty}$, $G_n=(V_n,E_n)$, be a sequence of r-uniform graphs with the property that $|V_n|\to\infty$ as $n\to\infty$. For $k\geqslant r$ we define

$$\sigma_k(\mathcal{S})=\max_n \max_{V\in\binom{V_n}{k}} \frac{|E_n\cap\binom{V}{r}|}{\binom{k}{r}}. \tag{1}$$

An averaging argument yields (cf. [5]): $\sigma_k(\mathcal{S})\geqslant\sigma_{k+1}(\mathcal{S})$. Hence $\lim_{k\to\infty}\sigma_k(\mathcal{S})$ exists. We denote this limit by $\bar{d}(\mathcal{S})=\lim_{k\to\infty}\sigma_k(\mathcal{S})$ and call $\bar{d}(\mathcal{S})$ the *upper density* of $\mathcal{S}$.

Definition 1.1. For $0\leqslant\alpha<1$ define $\Delta_r(\alpha)=\sup\{\delta\colon\ \bar{d}(\mathcal{S})>\alpha$ implies $\bar{d}(\mathcal{S})\geqslant\alpha+\delta$ for all sequences of r-uniform graphs $\mathcal{S}=\{G_n\}_{n=1}^{\infty}$, $G_n=(V_n,E_n)$, with the property that $|V_n|\to\infty$ as $n\to\infty\}$. We call α a *jump* for r if $\Delta_r(\alpha)>0$.

Erdős, Stone, Simonovits [2] proved that the only possible values of $\bar{d}(\mathcal{S})$, for $r=2$, are $1-\frac{1}{l}$ ($l=1,2,3,\ldots$) and 1, therefore every $\alpha\in[0,1)$ is a jump for $r=2$. This result follows easily from the following theorem.

Theorem 1.1. [3] *For every $\epsilon>0$ and positive integers l,m, there exists $n_0(l,m,\epsilon)$ such that every graph G on $n>n_0(l,m,\epsilon)$ vertices with density $d(G)\geqslant 1-\frac{1}{l}+\epsilon$ contains a copy of the complete $(l+1)$-partite subgraph with partition class of size m (i.e., there exist $l+1$ pairwise disjoint subsets $V_1,\ldots,V_{l+1}$ such that $\{x_i,x_j\}$ is an edge of G whenever $x_i\in V_i$, $x_j\in V_j$ and $i\neq j$ hold).*

For $r\geqslant 3$, Erdős proved that every $0\leqslant\alpha<r!/r^r$ is a jump. This result directly follows from the following theorem.

Theorem 1.2. [1] *For every $c>0$ and positive integer m, there exists $n_0(c,m)$ such that every r-uniform graph G on $n>n_0(c,m)$ vertices with density $d(G)\geqslant c$ contains a copy of the complete r-partite r-uniform graph with partition class of size m (i.e., there exist r pairwise disjoint subsets $V_1,\ldots,V_r$ such that $\{x_1,x_2,\ldots,x_r\}$ is an edge whenever $x_i\in V_i$, $1\leqslant i\leqslant r$).*

Furthermore, Erdős proposed the following jumping constant conjecture.

Conjecture 1.3. *Every $\alpha\in[0,1)$ is a jump for every $r\geqslant 2$.*

In [4], Frankl and Rödl disproved this conjecture by showing the following result.

Theorem 1.4. [4] *Suppose $r\geqslant 3$ and $l>2r$, then $1-\frac{1}{l^{r-1}}$ is not a jump for r.*

It follows from Theorem 1.2 that every number in $[0,\frac{r!}{r^r})$ is a jump for $r\geqslant 3$. To decide whether $\alpha=\frac{r!}{r^r}$ is a jump for $r\geqslant 3$ is a well-known problem of Erdős. It seems that the analogous problem for $\alpha\in(\frac{r!}{r^r},1)$ gets harder if α is small (that is close to $\frac{r!}{r^r}$). Therefore finding α 'as small as possible' which is not a jump seems to be a problem of interest. The smallest known value of a non-jumping number for $r\geqslant 3$, given by Theorem 1.4 [4], is $1-\frac{1}{(2r+1)^{r-1}}$. In this paper we 'improve' on this by showing that $\frac{5}{2}\frac{r!}{r^r}$ is not a jump for $r\geqslant 3$.

The paper is organized as follows: in Section 2, we introduce the Lagrange function and some other tools used in the proof. In Section 3, we focus on the case $r=3$ and prove the following result.

Theorem 1.5. *The number $\frac{5}{9}$ is not a jump for $r = 3$.*

In Section 4 we extend Theorem 1.5 to arbitrary $r \geqslant 3$ and show that $\frac{5}{2}\frac{r!}{r^r}$ is not a jump for $r \geqslant 3$.

In Section 5 we restrict our attention to $r = 3$ again and describe an infinite sequence of non-jumping numbers.

We should emphasize that our method of proof is similar to that of [4]. In order to determine whether or not $\frac{r!}{r^r}$ is a jump for $r \geqslant 3$ we are likely to require an essentially new approach.

2. The Lagrange function of an r-uniform hypergraph

In this section we give a definition of the Lagrange function, $\lambda(G)$, which has proved to be a helpful tool in calculating the upper density of certain sequences of r-uniform graphs (cf. [4]).

Definition 2.1. For an r-uniform graph G with vertex set $V = \{1, 2, \dots, n\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$\lambda(G, \vec{x}) = \sum_{\{i_1,\dots,i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Definition 2.2. Let $S = \{\vec{x} = (x_1, x_2, \dots, x_n)\colon \sum_{i=1}^n x_i = 1,\ x_i \geqslant 0 \text{ for } i = 1, 2, \dots, n\}$. The Lagrange function of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\bigl\{\lambda(G, \vec{x})\colon \vec{x} \in S\bigr\}.$$

Fact 2.1. Let G_1, G_2 be r-uniform graphs and $G_1 \subset G_2$. Then

$$\lambda(G_1) \leqslant \lambda(G_2).$$

We call two vertices i, j of G *equivalent* if for all $f \in \binom{V(G)-\{i,j\}}{r-1}$, $f \cup \{j\} \in E(G)$ if and only if $f \cup \{i\} \in E(G)$. We denote this by $i \sim j$ and note that it is an equivalence relation. For an r-uniform graph G and $i \in V(G)$ we define G_i to be the $(r-1)$-uniform graph on $V - \{i\}$ with edge set $E(G_i)$ given by $e \in E(G_i)$ if and only if $e \cup \{i\} \in E(G)$. Similarly for $i, j \in V(G)$ we define G_{ij} to be the $(r-2)$-uniform graph on $V - \{i, j\}$ with edge set given by $e \in E(G_{ij})$ if and only if $e \cup \{i, j\} \in E(G)$.

An r-uniform graph G is said to be *covering* if for every $i, j \in V(G)$ there is an edge $e \in E(G)$ such that $i, j \in e$ (that is every pair of vertices is covered by an edge).

The following simple lemma will be useful when calculating the Lagrange function of certain graphs.

Lemma 2.2. (Cf. [4].) *Let G be an r-uniform graph of order n.*

(a) *There exists a covering subgraph H of G such that $\lambda(G) = \lambda(H)$.*
(b) *Suppose $\vec{y} \in S$ satisfies $\lambda(G) = \lambda(G, \vec{y})$ and $v_1, \dots, v_t \in V(G)$ are all pairwise equivalent. If $\vec{z} \in S$ is obtained from $\vec{y}$ by setting the weights of the vertices $v_1, \dots, v_t$ to be equal while leaving the other weights unchanged then $\lambda(G) = \lambda(G, \vec{z})$.*
(c) *If $\vec{y} \in S$ satisfies $\lambda(G) = \lambda(G, \vec{y})$ and $y_i > 0$ then $r\lambda(G) = \lambda(G_i, \vec{y})$.*

Proof. Let $\vec{y}$ satisfy $\lambda(G) = \lambda(G, \vec{y})$. Let K be the induced subgraph consisting of those vertices v such that $y_v > 0$. By Fact 2.1, $\lambda(K) = \lambda(K, \vec{y}) = \lambda(G)$. If $i, j \in V(K)$ and $\lambda(K_{ij}, \vec{y}) = 0$ then w.l.o.g. $\lambda(K_i, \vec{y}) \geqslant \lambda(K_j, \vec{y})$. Defining $\vec{z} \in S$ by $z_i = y_i + y_j$, $z_j = 0$ and $z_l = y_l$ otherwise we have

$$\lambda(K, \vec{z}) - \lambda(K, \vec{y}) = y_j\big(\lambda(K_i, \vec{y}) - \lambda(K_j, \vec{y})\big) \geqslant 0.$$

Hence if H is the induced subgraph with vertex set $V(K) - \{j\}$ then $\lambda(G) = \lambda(K) = \lambda(H)$. Repeating this process yields a covering subgraph satisfying (a).

For (b) let $\vec{y} \in S$ be as above and suppose that $v_1, \ldots, v_t \in V(G)$ are all pairwise equivalent. If vertex v_i receives weight y_i then we may suppose that there are $1 \leqslant i, j \leqslant t$ such that $y_i > \mu > y_j$, where $\mu = \sum_{i=1}^{t} y_i/t$ (otherwise $\vec{y}$ already has the desired properties). If $\lambda(G_{ij}, \vec{y}) > 0$ then taking $0 < \delta < y_i - y_j$ and defining $\vec{z} \in S$ by $z_i = y_i - \delta$, $z_j = y_j + \delta$ and $z_l = y_l$ otherwise we have

$$\lambda(G, \vec{z}) - \lambda(G, \vec{y}) = \delta\lambda(G_{ij}, \vec{y})(y_i - y_j - \delta) > 0,$$

but this is impossible, hence $\lambda(G_{ij}, \vec{y}) = 0$. Now defining $\vec{z} \in S$ by $z_i = \mu$, $z_j = y_i + y_j - \mu$ and $z_l = y_l$ otherwise we have $\lambda(G, \vec{z}) = \lambda(G, \vec{y}) = \lambda(G)$. Repeating this process we obtain $\vec{z} \in S$ with the desired properties after at most $t - 1$ iterations.

For (c) let $\vec{y}$ be as above with $y_i > 0$ for $1 \leqslant i \leqslant k$ and $y_j = 0$ for $k + 1 \leqslant j \leqslant n$. If $y_a, y_b > 0$ and $\lambda(G_a, \vec{y}) > \lambda(G_b, \vec{y})$ then taking $0 < \delta < y_b$ sufficiently small and defining $\vec{z} \in S$ by $z_a = y_a + \delta$, $z_b = y_b - \delta$ and $z_l = y_l$ otherwise we have

$$\lambda(G, \vec{z}) - \lambda(G, \vec{y}) = \delta\big(\lambda(G_a, \vec{y}) - \lambda(G_b, \vec{y})\big) - O\big(\delta^2\big) > 0$$

which is impossible. Hence $\lambda(G_i, \vec{y})$ is constant for $1 \leqslant i \leqslant k$. So if $y_i > 0$ then

$$r\lambda(G) = \sum_{l=1}^{n} y_l \lambda(G_l, \vec{y}) = \lambda(G_i, \vec{y}) \sum_{l=1}^{k} y_l = \lambda(G_i, \vec{y}). \qquad \square$$

The blow-up of an r-uniform graph will play an important role in the proof of Theorem 1.5.

Definition 2.3. Let G be an r-uniform graph with n vertices and $(m_1, \ldots, m_n)$ be a non-negative integer vector. Define the $(m_1, \ldots, m_n)$ blow-up of G, $(m_1, \ldots, m_n) \otimes G$ to be the n-partite r-uniform graph with vertex set $V_1 \cup \cdots \cup V_n$, $|V_i| = m_i$, $1 \leqslant i \leqslant n$, and edge set $E((m_1, \ldots, m_n) \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}\colon\ v_i \in V_i,\ \{i_1, i_2, \ldots, i_r\} \in E(G)\}$.

For an integer $m \geqslant 1$ and an r-uniform graph G, we simply write $(m, m, \ldots, m) \otimes G$ as $\vec{m} \otimes G$.

The Lagrange function of an r-uniform graph G is closely related to the upper density of a certain sequence of r-uniform graphs, as described in the following claim.

Claim 2.3. *Let $m \geqslant 1$ be an integer and G be an r-uniform graph. Then $\bar{d}(\{\vec{m} \otimes G\}_{m=1}^{\infty}) = r!\lambda(G)$ holds.*

Proof. Suppose G has n vertices and $\vec{y} = (y_1, \ldots, y_n) \in S$ satisfies $\lambda(G) = \lambda(G, \vec{y})$. For a positive integer m, take the subgraph $H_m = (\lfloor my_1 \rfloor, \lfloor my_2 \rfloor, \ldots, \lfloor my_n \rfloor) \otimes G$ of $\vec{m} \otimes G$. It is easy to verify that for every $\epsilon > 0$, there exists $m_0(\epsilon)$ such that $d(H_m) \geqslant r!\lambda(G) - \epsilon$ if $m \geqslant m_0$. Hence $\bar{d}(\{\vec{m} \otimes G\}_{m=1}^{\infty}) \geqslant r!\lambda(G)$.

On the other hand, by the definition of $\bar{d}(\{\vec{m}\otimes G\}_{m=1}^{\infty})$, for every $\epsilon>0$, there exists k_0 such that for every $k\geqslant k_0$, there exist an integer m and a subgraph H of $\vec{m}\otimes G$ with $|V(H)|=k$ satisfying $d(H)>\bar{d}(\{\vec{m}\otimes G\}_{m=1}^{\infty})-\epsilon/2$. Suppose $V(H)=\bigcup_{i=1}^{n}V_i$, where $V_i, 1\leqslant i\leqslant n$, are the corresponding color classes of the n-partite r-uniform graph H. If $\vec{y}=(y_1,\ldots,y_n)$, where $y_i=|V_i|/\sum_{i=1}^{n}|V_i|$, then it is easy to verify that $r!\lambda(G,\vec{y})\geqslant d(H)-\epsilon/2$. Consequently, for any $\epsilon>0$, we are able to find $\vec{y}$ such that

$$r!\lambda(G,\vec{y})\geqslant \bar{d}\big(\{\vec{m}\otimes G\}_{m=1}^{\infty}\big)-\epsilon.$$

Therefore $\bar{d}(\{\vec{m}\otimes G\}_{m=1}^{\infty})\leqslant r!\lambda(G)$. □

Lemma 2.2(a) implies that the following holds.

Fact 2.4. For every r-uniform graph G and every integer m, $\lambda(\vec{m}\otimes G)=\lambda(G)$.

3. The proof of Theorem 1.5

We require the following definition.

Definition 3.1. If $\mathcal{F}$ is a family of r-uniform graphs and $\alpha\in[0,1]$ then we say that α is a threshold for $\mathcal{F}$ if for every $\epsilon>0$ there exists $n_0=n_0(\epsilon,\alpha,r,\mathcal{F})$ such that every r-uniform graph G with $d(G)\geqslant\alpha+\epsilon$ and $|V(G)|>n_0$ contains some member of $\mathcal{F}$ as a subgraph. We denote this fact by $\alpha\to\mathcal{F}$.

Our proof of Theorem 1.5 relies on the following result.

Lemma 3.1. (Cf. [4].) *The following two properties are equivalent*:

(1) *α is a jump for r*;
(2) *$\alpha\to\mathcal{F}$ for some finite family $\mathcal{F}$ of r-uniform graphs satisfying* $\min_{F\in\mathcal{F}}\lambda(F)>\frac{\alpha}{r!}$.

The proof of this lemma was given in [4] and we omit it here.

For an integer $t\geqslant 2$ let $G(t)=(V,E)$ be the 3-uniform graph defined as follows. The vertex set $V=V_1\cup V_2\cup V_3$, where $|V_1|=|V_2|=|V_3|=t$ and V_1, V_2, V_3 are pairwise disjoint. The edge set E consists of all triples of the form $\{\{a,b,c\}:\ a\in V_1,\ b\in V_2,\ \text{and } c\in V_3\}$ and all triples of the form $\{\{a,b,c\}:\ a\in V_i \text{ and } b,c\in V_j,\ \text{where } j-i=1 \bmod 3\}$.

By taking the vector $\vec{y}=(y_1,\ldots,y_{3t})$, where $y_i=1/3t$ for each i, $1\leqslant i\leqslant 3t$, it is easy to see that

$$\lambda\big(G(t)\big)\geqslant\frac{1}{3!}\left(\frac{5}{9}-\frac{1}{3t}\right). \tag{2}$$

Consider the sequence $\mathcal{S}=\{\vec{m}\otimes G(t)\}_{m=1}^{\infty}$. Inequality (2) and Claim 2.3 imply that $\bar{d}(\mathcal{S})\geqslant\frac{5}{9}-\frac{1}{3t}$. Our plan is to add $3ct^2$ edges to $G(t)$ and hence obtain a new graph $G^*(t)$ satisfying

$$\bar{d}\big(\big\{\vec{m}\otimes G^*(t)\big\}_{m=1}^{\infty}\big)=3!\lambda\big(G^*(t)\big)>\frac{5}{9}$$

while $\lambda(F)\leqslant\frac{5}{9}\frac{1}{3!}$ for any small subgraph $F\subset\vec{m}\otimes G^*(t)$. Lemma 3.1 then implies that 5/9 cannot be a jump for $r=3$.

The next lemma allows us to construct $G^*(t)$.

Lemma 3.2. [4] *Let k be any fixed integer and $c \geqslant 0$ be any fixed real number. Then there exists $t_0(k,c)$ such that for every $t > t_0(k,c)$, there exists a 3-uniform graph A satisfying*:

(i) $|V(A)| = t$;
(ii) $|E(A)| \geqslant ct^2$;
(iii) *for all* $V_0 \subset V(A)$, $3 \leqslant |V_0| \leqslant k$ *we have* $|E(A) \cap \binom{V_0}{3}| \leqslant |V_0| - 2$.

The proof of Lemma 3.2, based on a simple random construction, was given in [4]. We omit the proof here.

For k, c fixed and $t > t_0(k,c)$ let A be a 3-uniform graph satisfying the conditions of Lemma 3.2. We construct the graph $G^*(t,k,c)$ from $G(t)$ by adding a copy of $E(A)$ into each vertex class of $G(t)$. (So now $E(V_i) = E(A)$, for $i = 1, 2, 3$.)

The proof of Theorem 1.5 is based on the following lemma.

Lemma 3.3. *For any integer $k \geqslant 1$, real number $c > 0$ and $t > t_0(k,c)$ given in Lemma 3.2 if M is a subgraph of $G^*(t,k,c)$ and $|V(M)| \leqslant k$, then*

$$\lambda(M) \leqslant \frac{1}{3!} \cdot \frac{5}{9}. \tag{3}$$

Assuming this result for the moment we may complete the proof of Theorem 1.5 as follows.

Proof of Theorem 1.5. Suppose that $\frac{5}{9}$ is a jump. In view of Lemma 3.1, there exists a finite collection $\mathcal{F}$ of 3-uniform graphs satisfying the following two conditions:

(i) $\lambda(F) > \frac{1}{3!}\frac{5}{9}$ for all $F \in \mathcal{F}$;
(ii) $\frac{5}{9}$ is a threshold for $\mathcal{F}$.

Set $k = \max_{F \in \mathcal{F}} |V(F)|$ and $c = 1$. Take $t > t_0(k,c)$ as given by Lemma 3.2 and let $G^*(t) = G^*(t,k,c)$. If $\vec{y} = (y_1, \ldots, y_{3t})$, where $y_i = 1/3t$ for each i, $1 \leqslant i \leqslant 3t$, then

$$3!\lambda\big(G^*(t)\big) \geqslant \frac{6|E(G^*(t))|}{(3t)^3} \geqslant \frac{2}{9t^3}\left(t^3 + 3\binom{t}{2}t + 3t^2\right) \geqslant \frac{5}{9} + \frac{1}{3t}.$$

Hence, by Claim 2.3, we have

$$\bar{d}\big(\{\vec{m} \otimes G^*(t)\}_{m=1}^{\infty}\big) \geqslant \frac{5}{9} + \frac{1}{3t}. \tag{4}$$

Now condition (ii) above, the definition of 'threshold' and inequality (4) imply that some member F of $\mathcal{F}$ is a subgraph of $\vec{m} \otimes G^*(t)$ for $m \geqslant m_0(k,t)$. For such $F \in \mathcal{F}$, there exists a subgraph M of $G^*(t)$ with $|V(M)| \leqslant k$ satisfying $F \subset \vec{m} \otimes M \subset \vec{m} \otimes G^*(t)$.

By Facts 2.1, 2.4 and Lemma 3.3, we have

$$\lambda(F) \leqslant \lambda(\vec{m} \otimes M) = \lambda(M) \leqslant \frac{1}{3!} \cdot \frac{5}{9}$$

which contradicts condition (i) above that $\lambda(F) > \frac{1}{3!}\frac{5}{9}$ for all $F \in \mathcal{F}$. This completes the proof of Theorem 1.5. □

It remains to prove Lemma 3.3.

Proof of Lemma 3.3. By Fact 2.1, we may assume that M is an induced subgraph of $G^*(t)$. Let

$$U_i = V(M) \cap V_i = \{v_1^i, v_2^i, \dots, v_{k_i}^i\}.$$

So $k = k_1 + k_2 + k_3$.

Claim 3.4. (Cf. [4].) *If N is the 3-uniform graph formed from M by removing the edges contained in each U_i and inserting the edges $\{\{v_1^i, v_2^i, v_j^i\}\colon\ 1 \leqslant i \leqslant 3,\ 3 \leqslant j \leqslant k_i\}$ then $\lambda(M) \leqslant \lambda(N)$.*

Proof. Let $M_i = (U_i, E(M) \cap \binom{U_i}{3})$, $N_i = (U_i, E(N) \cap \binom{U_i}{3})$ and $x_1 \geqslant x_2 \geqslant \dots \geqslant x_{k_i} \geqslant 0$. It is sufficient to prove that $\lambda(M_i, \vec{x}) \leqslant \lambda(N_i, \vec{x})$.

Let the edges of M_i in decreasing order be $e_1, e_2, \dots, e_s$, i.e., $\prod_{v \in e_p} x_v \geqslant \prod_{v \in e_q} x_v$ for $p < q$. By the construction of $G^*(t)$ (Lemma 3.2(iii)) we have $s \leqslant k_i - 2$. We will prove that $\prod_{v \in e_p} x_v \leqslant x_1 x_2 x_{2+p}$ for all $1 \leqslant p \leqslant s$. By Lemma 3.2(iii) we have $|e_1 \cup e_2 \cup \dots \cup e_p| \geqslant 2 + p$ for $p = 1, 2, \dots, s$, so at least one of the edges from $e_1, e_2, \dots, e_p$ contains some v_j^i with $j \geqslant 2 + p$ and thus, by monotonicity, $\prod_{v \in e_p} x_v \leqslant x_1 x_2 x_{2+p}$. Thus $\lambda(M_i, \vec{x}) \leqslant \lambda(N_i, \vec{x})$. □

By Claim 3.4 the proof of Lemma 3.3 will be complete if we show that $\lambda(N) \leqslant 5/54$. Since $v_1^i \sim v_2^i$ and $v_3^i, v_4^i, \dots, v_{k_i}^i$ are all pairwise equivalent we can use Lemma 2.2(b) to obtain $\vec{z} \in S$ satisfying $\lambda(N, \vec{z}) = \lambda(N)$ such that

$$z_1^i = z_2^i = a_i, \qquad z_3^i = z_4^i = \dots = z_{k_i}^i = b_i,$$

where a_i, b_i $(i = 1, 2, 3)$ are constants.

Let $w_i = 2a_i + (k_i - 2)b_i$ (so $w_1 + w_2 + w_3 = 1$). If $P = \{i\colon\ w_i > 0\}$ and $p = |P|$ then we may suppose that $p \geqslant 2$ (since otherwise Lemma 2.2(a) allows us to reduce M to a single edge with $\lambda(M) = 1/27$). So suppose that $2 \leqslant p \leqslant 3$.

For each $i \in P$ take a vertex $u_i \in U_i$ as follows: if $b_i > 0$ then $u_i = v_3^i$ otherwise $u_i = v_1^i$. The vertex u_i receives non-zero weight so by Lemma 2.2(c) we have $3\lambda(N) = \lambda(N_{u_i}, \vec{z})$. Moreover, by considering the edges containing vertex u_i we have

$$\lambda(N_{u_i}, \vec{z}) \leqslant a_i^2 + w_i w_{i+2} + w_{i+1} w_{i+2} + \sum_{\{c,d\} \in \binom{U_{i+1}}{2}} z_c z_d, \tag{5}$$

where all subscripts are modulo 3.

Now, since $\sum_{\{c,d\} \in \binom{U_{i+1}}{2}} z_c z_d$ is zero if $w_{i+1} = 0$, so (5) implies that

$$3p\lambda(N) = \sum_{i \in P} \lambda(N_{u_i}, \vec{z}) \leqslant \sum_{i \in P} \left(a_i^2 + w_{i+2}(1 - w_{i+2}) + \sum_{\{c,d\} \in \binom{U_i}{2}} z_c z_d \right). \tag{6}$$

We claim that the following holds for $i = 1, 2, 3$:

$$a_i^2 + \sum_{\{c,d\} \in \binom{U_i}{2}} z_c z_d \leqslant \frac{w_i^2}{2}. \tag{7}$$

We have $w_i = 2a_i + (k_i - 2)b_i$.

Hence

$$a_i^2 + \sum_{\{c,d\}\in\binom{U_i}{2}} z_c z_d = 2a_i^2 + 2(k_i-2)a_i b_i + \binom{k_i-2}{2} b_i^2 \leqslant \frac{w_i^2}{2}.$$

Combining (6) and (7) we obtain

$$3p\lambda(N) \leqslant \sum_{i\in P}\left(\frac{w_i^2}{2} + w_{i+2}(1-w_{i+2})\right).$$

Now, using $w_1+w_2+w_3=1$, if $p=3$ we have

$$9\lambda(N) \leqslant 1 - \frac{w_1^2+w_2^2+w_3^2}{2} \leqslant \frac{5}{6}.$$

While if $p=2$ (so w.l.o.g. $w_3=0$) then we have

$$6\lambda(N) \leqslant \frac{(w_1+w_2)^2}{2} = \frac{1}{2}.$$

Hence $\lambda(N) \leqslant 5/54$ as required. □

4. An extension of Theorem 1.5

In this section we extend Theorem 1.5 to arbitrary $r \geqslant 3$ and prove the following result.

Theorem 4.1. *Let $r \geqslant 3$ be an integer. Then $\frac{5}{2}\cdot\frac{r!}{r^r}$ is not a jump for r.*

Proof. We assume that $r \geqslant 4$ and $\frac{5}{2}\cdot\frac{r!}{r^r}$ is a jump for r. In view of Lemma 3.1, there exists a finite collection $\mathcal{F}$ of r-uniform graphs satisfying the following:

(i) $\lambda(F) > \frac{5}{2}\cdot\frac{1}{r^r}$ for all $F\in\mathcal{F}$, and
(ii) $\frac{5}{2}\cdot\frac{r!}{r^r}$ is a threshold for $\mathcal{F}$.

Set $k=\max_{F\in\mathcal{F}}|V(F)|$ and $c=1$. Let $t_0(k,c)$ be as in Lemma 3.2. For $t>t_0(k,c)$, take the 3-uniform graph $G^{(3)}=G^*(t,k,c)$ on vertex set $V_1\cup V_2\cup V_3$ constructed as in Section 3. Note that

$$\left|E\left(G^{(3)}\right)\right| \geqslant \frac{5t^3}{2} + \frac{3t^2}{2}.$$

Based on the 3-uniform graph $G^{(3)}$, we construct an r-uniform graph $G^{(r)}$ on r pairwise disjoint sets $V_1, V_2, V_3, V_4, \ldots, V_r$, each of order t. An r-element set $\{u_1,u_2,u_3,u_4,\ldots,u_r\}$ is an edge of $G^{(r)}$ if and only if $\{u_1,u_2,u_3\}$ is an edge in $G^{(3)}$ and for each j, $4\leqslant j\leqslant r$, $u_j\in V_j$. Notice that

$$\left|E\left(G^{(r)}\right)\right| = t^{r-3}\left|E\left(G^{(3)}\right)\right| \geqslant \frac{5t^r}{2} + \frac{3t^{r-1}}{2}.$$

We can now give a lower bound for $\lambda(G^{(r)})$. Corresponding to the rt vertices of this r-uniform graph, let us take vector $\vec{y}=(y_1,\ldots,y_{rt})$, where $y_i=\frac{1}{rt}$ for each i, $1\leqslant i\leqslant rt$.

Then

$$\lambda(G^{(r)}) \geqslant \lambda(G^{(r)}, \vec{y}) = \frac{|E(G^{(r)})|}{(rt)^r} \geqslant \left(\frac{5}{2} + \frac{3}{2t}\right)\frac{1}{r^r}.$$

Similarly as Theorem 1.5 follows from Lemma 3.3, in order to prove Theorem 4.1, it will be sufficient to prove the following lemma.

Lemma 4.2. *Let $M^{(r)}$ be a subgraph of $G^{(r)}$ with $|V(M^{(r)})| \leqslant k$. Then*

$$\lambda(M^{(r)}) \leqslant \frac{5}{2} \cdot \frac{1}{r^r} \tag{8}$$

holds.

We are going to use Lemma 3.3 to prove it.

Proof. Again, by Fact 2.1, we may assume that $M^{(r)}$ is a non-empty induced subgraph of $G^{(r)}$. Define $U_i = V(M^{(r)}) \cap V_i$ for $1 \leqslant i \leqslant r$. Let $M^{(3)}$ be the 3-uniform graph defined on $\bigcup_{i=1}^3 U_i$. The edge set of $M^{(3)}$ consists of all 3-sets of the form of $e \cap (\bigcup_{i=1}^3 U_i)$, where e is an edge in $M^{(r)}$. Let $\vec{\xi}$ be an optimal vector for $\lambda(M^{(r)})$, i.e., $\lambda(M^{(r)}, \vec{\xi}) = \lambda(M^{(r)})$. Let $\vec{\xi}^{(3)}$ be the restriction of $\vec{\xi}$ to $U_1 \cup U_2 \cup U_3$. Let w_i be the sum of all components of $\vec{\xi}$ corresponding to vertices in U_i, $1 \leqslant i \leqslant r$, respectively. In view of the relationship between $M^{(r)}$ and $M^{(3)}$, we have

$$\lambda(M^{(r)}) = \lambda(M^{(3)}, \vec{\xi}^{(3)}) \times \prod_{i=4}^{r} w_i.$$

Note that $M^{(3)}$ is a subgraph of $G^{(3)} = G^*(t, k, c)$ satisfying $|V(M^{(3)})| \leqslant |V(M^{(r)})| \leqslant k$. Also note that the summation of all components of $\vec{\xi}^{(3)}$ is $1 - \sum_{i=4}^{r} w_i$ and every term in $\lambda(M^{(3)}, \vec{\xi}^{(3)})$ has degree 3. Consequently by Lemma 3.3, we infer that

$$\lambda(M^{(3)}, \vec{\xi}^{(3)}) \leqslant \frac{5}{54}\left(1 - \sum_{i=4}^{r} w_i\right)^3.$$

Therefore,

$$\lambda(M^{(r)}) \leqslant \frac{5}{54}\left(1 - \sum_{i=4}^{r} w_i\right)^3 \prod_{i=4}^{r} w_i = \frac{5}{2}\left(\frac{1 - \sum_{i=4}^{r} w_i}{3}\right)^3 \prod_{i=4}^{r} w_i.$$

Since the geometric mean is no more than the arithmetic mean, we obtain

$$\lambda(M^{(r)}) \leqslant \frac{5}{2} \cdot \frac{1}{r^r}.$$

This completes the proof of Lemma 4.2. □

5. More non-jumping numbers

In this section, we return to the case $r = 3$. The construction used in the proof of Theorem 1.5 can be easily generalized to give the following result.

Theorem 5.1. *For any integers $s \geqslant 1$ and $l \geqslant 9s+6$ the number $1-\frac{3}{l}+\frac{3s+2}{l^2}$ is not a jump for $r=3$.*

For l, s as in the statement of Theorem 5.1 and $t \geqslant 2$ consider the 3-uniform hypergraph $G(l,s,t)$ with vertex set $V = \bigcup_{i=1}^{l} V_i$, where $|V_i| = t$ and $V_i, 1 \leqslant i \leqslant l$, are pairwise disjoint. The edge set consists of all triples of the form $\{\{a,b,c\}\colon\ a \in V_i,\ b \in V_j,\ \text{and } c \in V_k,\ \{i,j,k\} \in \binom{[l]}{3}\}$, if $l \geqslant 3$, and all triples of the form $\{\{a,b,c\}\colon\ a \in V_i \text{ and } b,c \in V_j$, with $1 \leqslant (j-i) \bmod l \leqslant s\}$. When $l=3$, $s=1$, $G(l,s,t)$ is $G(t)$.

Now let $k \geqslant 1$ be an integer, $c=s$ and $t \geqslant t_0(k,c)$ be as given by Lemma 3.2. We construct $G^*(l,s,t)$ from $G(l,s,t)$ by inserting into each V_i a copy of a graph A as given by Lemma 3.2. Note that

$$\lambda\big(G^*(l,s,t)\big) \geqslant \frac{|E(G^*(l,s,t))|}{(lt)^3} \geqslant \frac{\binom{l}{3}t^3 + ls\binom{t}{2}t + lst^2}{(lt)^3} = \frac{1}{6}\left(1-\frac{3}{l}+\frac{3s+2}{l^2}+\frac{3s}{l^2t}\right).$$

As with Theorem 1.5, the proof of Theorem 5.1 may be reduced to proving the following lemma.

Lemma 5.2.

$$\lambda(M) \leqslant \frac{1}{6}\left(1-\frac{3}{l}+\frac{3s+2}{l^2}\right) \tag{9}$$

holds for any subgraph M of $G^*(l,s,t)$ with $|V(M)| \leqslant k$.

Proof. An obvious analogue of Claim 3.4 holds so if N is the 3-uniform graph formed from M by replacing the edges contained in each $U_i = V_i \cap V(M)$ with the following edges: $\{\{v_1^i, v_2^i, v_j^i\}\colon\ 1 \leqslant i \leqslant l,\ 3 \leqslant j \leqslant k_i\}$ then it is sufficient to prove that

$$\lambda(N) \leqslant \frac{1}{6}\left(1-\frac{3}{l}+\frac{3s+2}{l^2}\right).$$

As before (using Lemma 2.2(b)) we may take $\vec{z} \in S$ such that $\lambda(G,\vec{z}) = \lambda(G)$ and $z_1^i = z_2^i = a_i$ and $z_3^i = z_4^i = \cdots = z_{k_i}^i = b_i$. Let $w_i = 2a_i + (k_i-2)b_i$, $P = \{1 \leqslant i \leqslant l\colon\ w_i > 0\}$ and $p = |P| \leqslant l$. For $i \in P$ define $P_i^+ = P \cap \{i+1, i+2, \ldots, i+s\}$ and $P_i^- = P \cap \{i-1, i-2, \ldots, i-s\}$. For $i \in P$ let u_i be a vertex in U_i receiving weight b_i, if $b_i > 0$, and otherwise receiving weight $a_i > 0$. Considering the edges containing u_i we have

$$\lambda(N_{u_i},\vec{z}) \leqslant a_i^2 + \sum_{j \in P_i^+} \sum_{\{c,d\} \in \binom{U_j}{2}} z_c z_d + w_i \sum_{j \in P_i^-} w_j + \sum_{\{j,k\} \in \binom{P-\{i\}}{2}} w_j w_k.$$

Using Lemma 2.2(c) we obtain

$$\begin{aligned} 3p\lambda(N) &= \sum_{i \in P} \lambda(N_{u_i},\vec{z}) \\ &\leqslant \sum_{i \in P}\left(a_i^2 + \sum_{j \in P_i^+} \sum_{\{c,d\} \in \binom{U_j}{2}} z_c z_d + w_i \sum_{j \in P_i^-} w_j + \sum_{\{j,k\} \in \binom{P-\{i\}}{2}} w_j w_k\right). \end{aligned} \tag{10}$$

Now (7) holds for $i \in P$ so

$$\sum_{i\in P}\biggl(a_i^2+\sum_{j\in P_i^+}\sum_{\{c,d\}\in\binom{U_j}{2}} z_c z_d\biggr) \leqslant \sum_{i\in P_b} a_i^2 + \sum_{i\in P}\sum_{j\in P_i^+}\frac{w_j^2}{2},$$

where $P_b=\{i\in P\colon\ P_i^-=\emptyset\}$ (so P_b contains precisely those $i\in P$ for which there is no term $\sum_{\{c,d\}\in\binom{U_i}{2}} z_c z_d$ in (10)). Using this together with $w_i w_j \leqslant (w_i^2+w_j^2)/2$ we obtain

$$3p\lambda(N) \leqslant \sum_{i\in P_b} a_i^2 + \sum_{i\in P}\biggl(\sum_{j\in P_i^+}\frac{w_j^2}{2} + \sum_{j\in P_i^-}\frac{(w_i^2+w_j^2)}{2} + \sum_{\{j,k\}\in\binom{P-\{i\}}{2}} w_j w_k\biggr).$$

Now $a_i^2 \leqslant w_i^2/2$ and $|P_i^-|, |P_i^+| \leqslant s$, so we have

$$\begin{aligned}
3p\lambda(N) &\leqslant \frac{1}{2}\biggl(\sum_{i\in P_b} w_i^2 + \sum_{i\in P}\bigl(\bigl|P_i^+\bigr| + 2\bigl|P_i^-\bigr|\bigr)w_i^2\biggr) + \sum_{\{j,k\}\in\binom{P-\{i\}}{2}} w_j w_k \\
&\leqslant \frac{1}{2}\biggl(\sum_{i\in P_b}(1+s)w_i^2 + \sum_{i\in P\setminus P_b} 3s w_i^2\biggr) + \sum_{\{j,k\}\in\binom{P-\{i\}}{2}} w_j w_k \\
&\leqslant \frac{3s}{2}\sum_{i\in P} w_i^2 + (p-2)\sum_{\{j,k\}\in\binom{P}{2}} w_j w_k.
\end{aligned}$$

Note that since $\sum_{i\in P} w_i = 1$ we have

$$\sum_{\{j,k\}\in\binom{P}{2}} w_j w_k = \frac{1}{2} - \sum_{i\in P}\frac{w_i^2}{2}. \tag{11}$$

Hence if $p \geqslant 3s+2$ then

$$3p\lambda(N) \leqslant \frac{p-2}{2} - \biggl(\frac{p-(3s+2)}{2}\biggr)\sum_{i\in P} w_i^2 \leqslant \frac{p}{2}\biggl(1-\frac{3}{p}+\frac{3s+2}{p^2}\biggr),$$

where the last inequality follows from $\sum_{i=1}^{l} w_i^2 \geqslant 1/p$. The desired bound now follows easily.

To complete the proof we need to consider the case $p \leqslant 3s+1$. In this case $l \geqslant 9s+6 \geqslant 3p+3$ and so $3/l \leqslant 1/(p+1)$. Hence it is sufficient to prove that $3\lambda(N) \leqslant \frac{1}{2}(1-\frac{1}{p+1})$.

If $p=1,2$ then $\lambda(N) \leqslant 1/12$ (see the proof of Lemma 3.3) so we may suppose that $3 \leqslant p \leqslant 3s+1$.

Choose $i \in P$ such that $w_i \geqslant 1/p$ (since $\sum_{i\in P} w_i = 1$ such an i must exist) then

$$3\lambda(N) = \lambda(N_{u_i}, \vec{z}) \leqslant a_i^2 + \sum_{j\in P_i^+}\sum_{\{c,d\}\in\binom{U_j}{2}} z_c z_d + w_i\sum_{j\in P_i^-} w_j + \sum_{\{j,k\}\in\binom{P-\{i\}}{2}} w_j w_k.$$

Since (7) holds for any $j \in P$ we have

$$\sum_{j\in P_i^+}\sum_{\{c,d\}\in\binom{U_j}{2}} z_c z_d \leqslant \sum_{j\in P_i^+}\frac{w_j^2}{2}.$$

Also $a_i^2 \leqslant w_i^2/4$ and (11) imply that

$$3\lambda(N) \leqslant \frac{1}{2} - \frac{w_i^2}{4} - \sum_{j \in C_i} \frac{w_j^2}{2} - w_i \sum_{j \in D_i} w_j,$$

where $C_i = P - (P_i^+ \cup \{i\})$ and $D_i = P - (P_i^- \cup \{i\})$. Now $l \geqslant 2s + 1$ implies that $P_i^+ \cap P_i^- = \emptyset$ and so $C_i \cup D_i \cup \{i\} = P$. Hence if $\sum_{j \in C_i} w_j = \alpha$, $\sum_{j \in D_i} w_j = \beta$ and $w_i = \gamma$ then $\alpha + \beta + \gamma \geqslant 1$. Moreover, $\gamma = w_i \geqslant 1/p$. Note that since $|C_i| \leqslant p - 1$ so $\sum_{j \in C_i} w_j^2 \geqslant \alpha^2/(p-1)$ so we have

$$3\lambda(N) \leqslant \frac{1}{2}\left(1 - \left(\frac{\gamma^2}{2} + \frac{\alpha^2}{p-1} + 2\beta\gamma\right)\right).$$

Defining

$$f(\alpha, \beta, \gamma) = \frac{\gamma^2}{2} + \frac{\alpha^2}{p-1} + 2\beta\gamma,$$

the proof will be complete if we show that for $\alpha + \beta + \gamma \geqslant 1$, $0 \leqslant \alpha, \beta \leqslant 1 - 1/p$ and $1/p \leqslant \gamma \leqslant 1$, $f(\alpha, \beta, \gamma)$ is always at least $1/(p+1)$. Now f is clearly minimized (subject to the constraints) when $\alpha + \beta + \gamma = 1$ so substituting for β we need to minimize

$$g(\alpha, \gamma) = \frac{\alpha^2}{p-1} + 2\gamma - 2\alpha\gamma - \frac{3\gamma^2}{2},$$

subject to $0 \leqslant \alpha \leqslant 1 - \gamma$, $1/p \leqslant \gamma \leqslant 1$. This function is decreasing in α so for fixed γ has minimum

$$g(1-\gamma, \gamma) = h(\gamma) = \frac{(1-\gamma)^2}{p-1} + \frac{\gamma^2}{2}.$$

Finally we minimize $h(\gamma)$ subject to $1/p \leqslant \gamma \leqslant 1$. This function has a stationary point at $2/(p+1)$ and so the constrained minimum occurs at either $\gamma = 1/p$, $\gamma = 1$ or $\gamma = 2/(p+1)$. In each case we can check that $h(\gamma) \geqslant 1/(p+1)$ (for $p \geqslant 3$). This completes the proof of Lemma 5.2 and of Theorem 5.1. □

6. Concluding remarks

We remark that if $s = 1$, then the condition $l \geqslant 15$ in Theorem 5.1 can be relaxed to $l \geqslant 2$. We also think that in general the condition $l \geqslant 9s + 6$ in Theorem 5.1 can be relaxed to $l \geqslant s + 1$ although we are not able to prove this. Since no jump in the interval $[\frac{r!}{r^r}, 1)$ has been found, we ask the following question.

Question 6.1. For $r \geqslant 3$, does there exist $\alpha_0 \in [\frac{r!}{r^r}, 1)$ such that the interval $[\alpha_0, 1]$ contains no jump?

A recent result of Mubayi and Zhao [6] answers the analogous question for the related problem of co-degree density. They showed that in this case one can take $\alpha_0 = 0$ for all $r \geqslant 3$ (see [6, Theorem 1.6]).

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