COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **26** (1) (2006) 37–46

WEIGHTED NON-TRIVIAL MULTIPLY INTERSECTING FAMILIES

PETER FRANKL, NORIHIDE TOKUSHIGE

Received July 13, 2001
Revised February 11, 2005

Let n and r be positive integers. Suppose that a family $\mathcal{F} \subset 2^{[n]}$ satisfies $F_1 \cap \cdots \cap F_r \neq \emptyset$ for all $F_1, \ldots, F_r \in \mathcal{F}$ and $\bigcap_{F \in \mathcal{F}} F = \emptyset$. We prove that there exists $\epsilon = \epsilon(r) > 0$ such that $\sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|} \leq w^r(r+1-rw)$ holds for $1/2 \leq w \leq 1/2 + \epsilon$ if $r \geq 13$.

1. Introduction

Let n, r and t be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. An r-wise 1-intersecting family is also called an r-wise intersecting family for short. An r-wise t-intersecting family $\mathcal{F}$ is called non-trivial if $|\bigcap_{F \in \mathcal{F}} F| < t$.

Let us define the Brace–Daykin structure as follows.

$$\mathcal{F}^r_{BD} = \{F \subset [n] : |F \cap [r+1]| \geq r\}.$$

Then $\mathcal{F}^r_{BD}$ is a non-trivial r-wise intersecting family. Brace and Daykin proved the following.

Theorem 1 **([1]).** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is a non-trivial r-wise intersecting family. Then $|\mathcal{F}| \leq |\mathcal{F}^r_{BD}|$.*

For a real $w \in (0,1)$ let us define the weighted size (or simply weight) $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Mathematics Subject Classification (2000): 05D05

Note that $W_{1/2}(\mathcal{F})=|\mathcal{F}|/2^n$. See [3] for the maximum weighted size of intersecting families, and see [2,4] for applications of weighted size to Erdős–Ko–Rado and Sperner type results concerning multiply intersecting families. In this note, we consider the maximum weighted size of non-trivial intersecting families and extend Theorem 1. The weight of the Brace–Daykin family is calculated as follows:

$$W_w(\mathcal{F}^r_{BD}) = (r+1)w^r(1-w) + w^{r+1} = w^r(r+1-rw).$$

Let us define

$$g_n(w,r,t) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\},$$

$$g(w,r,t) := \lim_{n\to\infty} g_n(w,r,t).$$

Then the Brace–Daykin theorem states that $g_n(1/2,r,1)=W_{1/2}(\mathcal{F}^r_{BD})$ and thus $g(1/2,r,1) = (r+2)(1/2)^{r+1}$. Can we expect the same thing for $w = 1/2+\epsilon$? The answer is "yes" for $r \geq 13$, and "no" for $r \leq 5$.

Theorem 2. *Let $r \geq 13$. Then there exists $\epsilon=\epsilon(r)>0$ such that $g(w,r,1)= W_w(\mathcal{F}^r_{BD})=w^r(r+1-rw)$ holds for $1/2 \leq w \leq 1/2+\epsilon$.*

In the last section, we shall construct non-trivial r-wise intersecting families with weights larger than $W_w(\mathcal{F}^r_{BD})$ for $r \leq 5$. The cases $6 \leq r \leq 12$ remain open.

Conjecture 1. Theorem 2 is true for $r \geq 6$.

2. Tools

In this section we summarize some results on the maximum weight of (not necessarily non-trivial) r-wise t-intersecting families. Let us define

$$f_n(w,r,t) := \max\{W_w(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\},$$

$$f(w,r,t) := \lim_{n\to\infty} f_n(w,r,t).$$

If $\mathcal{F} \subset 2^{[n]}$ satisfies $f_n(w,r,t)=W_w(\mathcal{F})$ then $\mathcal{F}' := \mathcal{F} \cup \{F \cup \{n+1\} : F \in \mathcal{F}\} \subset 2^{[n+1]}$ satisfies $W_w(\mathcal{F}') = W_w(\mathcal{F}) = f_n(w,r,t)$, which implies $f_{n+1}(w,r,t) \geq f_n(w,r,t)$. Since $\mathcal{F} = \{F \subset [n] : [t] \subset F\}$ is r-wise t-intersecting and $W_w(\mathcal{F}) = w^t$, it follows that $f(w,r,t) \geq f_n(w,r,t) \geq w^t$.

Let $\alpha_{w,r} \in (1/2,1)$ be the unique root of the equation $(1-w)x^r - x + w = 0$. The following inequality is not sharp but it is very useful (see Fact 3 on page 98 of [2]).

Lemma 1. $f(w,r,t) \le \alpha_{w,r}^t$.

For the case $t=1$, we proved the following in [3].

Lemma 2. *$f(w,r,1)=w$ if $w \le \frac{r-1}{r}$, and $f(w,r,1)=1$ if $w > \frac{r-1}{r}$.*

For the case $r=3$, we proved the following in [2] (see Proposition 2 on page 104).

Lemma 3. *$f(w,3,t) \le w^2 \alpha_{w,3}^{t-2}$ if $t \ge 2$ and $w < 0.5018$.*

We also use the following simple fact.

Lemma 4. *If $\alpha_{w,r-1}^{t+1} \le w^t$ then $f(w,r,t)=w^t$.*

Proof. Suppose that $\mathcal{F}$ is an r-wise t-intersecting family with $W_w(\mathcal{F}) = f(w,r,t) \ge w^t$. If $\mathcal{F}$ has $(r-1)$ edges $F_1,\dots,F_{r-1}$ with $|F_1 \cap \cdots \cap F_{r-1}| = t$ then all edges in $\mathcal{F}$ must contain this t-subset, which proves $W_w(\mathcal{F}) \le w^t$. Thus we may assume that $\mathcal{F}$ is $(r-1)$-wise $(t+1)$-intersecting. By Lemma 1, we have $W_w(\mathcal{F}) \le f(w,r-1,t+1) \le \alpha_{w,r-1}^{t+1} \le w^t$. ∎

Using above lemmas, we have the following.

Lemma 5. *There exists $\epsilon = \epsilon(r)$ such that $f(w,r,t)=w^t$ holds for $1/2 \le w \le 1/2+\epsilon$ in the following cases: $r=3$ and $t \le 2$, $r=4$ and $t \le 2$, $r=5$ and $t \le 7$.*

Proof. The case $t=1$ follows from Lemma 2. The case $r=3$ and $t=2$ follows from Lemma 3.

Let us consider the case $r=4$ and $t=2$. Since $\alpha_{\frac{1}{2},3} = \frac{\sqrt{5}-1}{2} \approx 0.618$, we have $\alpha_{\frac{1}{2},3}^3 < (\frac{1}{2})^2$. Then, by the continuity, $\alpha_{\frac{1}{2}+\epsilon,3}^3 < (\frac{1}{2}+\epsilon)^2$ holds for sufficiently small $\epsilon > 0$. Thus $f(w,4,2) \le w^2$ for $\frac{1}{2} \le w \le \frac{1}{2}+\epsilon$ follows from Lemma 4. One can prove the case $r=5$ and $2 \le t \le 7$ similarly. ∎

Note also that

$$\alpha_{\frac{1}{2},r-1}^{t+1} < \left(\frac{1}{2} + \frac{1}{2^{r-1}}\right)^{t+1} = \left(\frac{1}{2}\right)^{t+1} \left(1 + \frac{1}{2^{r-2}}\right)^{t+1} < \left(\frac{1}{2}\right)^{t+1} \exp\left(\frac{t+1}{2^{r-2}}\right),$$

which is smaller than $(1/2)^t$ if $t+1 \le 2^{r-2}\log 2$. This means that $f(w,r,t)=w^t$ holds for $w=1/2+\epsilon(r)$ if $t \le 2^{r-2}\log 2 - 1$. We shall use the following weaker version later.

Proposition 1. *Let $\mathcal{F} \subset 2^{[n]}$ be an r-wise r-intersecting family. If $r \ge 5$, then there exists $\epsilon = \epsilon(r) > 0$ such that $W_w(\mathcal{F}) \le w^r$ holds for $\frac{1}{2} \le w \le \frac{1}{2}+\epsilon$.*

3. Proof of Theorem 2

Proof. We prove Theorem 2 by induction on r. First we prove the initial step $r=13$.

Proposition 2. *Suppose that $\mathcal{F}\subset 2^{[n]}$ is a non-trivial 13-wise intersecting family. Then there exists $\epsilon>0$ such that $W_w(\mathcal{F})\leq W_w(\mathcal{F}_{BD}^{13})$ holds for $\frac{1}{2}\leq w\leq\frac{1}{2}+\epsilon$.*

Proof. Let $\mathcal{F}\subset 2^{[n]}$ be a non-trivial 13-wise intersecting family. We assume that $\mathcal{F}$ is shifted and (size) maximal. (Recall that $\mathcal{F}$ is called shifted iff $(F-\{j\})\cup\{i\}\in\mathcal{F}$ holds for all $1\leq i<j\leq n$ and for all $F\in\mathcal{F}$ which satisfies $F\cap\{i,j\}=\{j\}$. See [2] for more about shifting.) Note also that if $F\in\mathcal{F}$ and $F\subset G$ then $G\in\mathcal{F}$ because $\mathcal{F}$ is maximal.

Let

$$k:=\max\{i:\forall F\in\mathcal{F},\ |F\cap[i+1]|\geq i\}.$$

We can find such k, for $|F\cap[1]|\geq 0$ (i.e., the case $i=0$) is evident. If $k\geq 13$ then $\mathcal{F}\subset\mathcal{F}_{BD}^{13}$. So we may assume that $k\leq 12$. Let $t(\ell):=\max\{t:\mathcal{F}$ is ℓ-wise t-intersecting$\}$. Then $1\leq t(13)<t(12)<\cdots<t(6)<\cdots$. This implies $8\leq t(6)<t(5)<t(4)$.

Since $\alpha_{1/2,4}\approx 0.543689$, the weight of 4-wise 12-intersecting family is, by Lemma 1, at most $\alpha_{1/2,4}^{12}\approx 0.000667124$. On the other hand, $W_{1/2}(\mathcal{F}_{BD}^{13})=15(1/2)^{14}\approx 0.000915527$. Thus for sufficiently small $\epsilon>0$ we have $\alpha_{\frac{1}{2}+\epsilon,4}^{12}<W_{\frac{1}{2}+\epsilon}(\mathcal{F}_{BD}^{13})$, because these functions of both sides are continuous with respect to $w=\frac{1}{2}+\epsilon$. This means $W_w(\mathcal{F})<W_w(\mathcal{F}_{BD}^{13})$ holds for $\frac{1}{2}\leq w\leq\frac{1}{2}+\epsilon$ if $\mathcal{F}$ is 4-wise 12-intersecting. So we may assume that $\mathcal{F}$ is not 4-wise 12-intersecting, that is, $t(4)\leq 11$. Consequently we have $8\leq t(6)<t(5)<t(4)\leq 11$, and so $t(6)+1=t(5)$ or $t(5)+1=t(4)$.

Lemma 6. *If $t(\ell+1)+1=t(\ell)$ then $k\geq t(\ell+1)$.*

Proof. Set $t:=t(\ell+1)$. If $t(\ell)=t+1$ then $\mathcal{F}$ is ℓ-wise $(t+1)$-intersecting, but $\mathcal{F}$ is not ℓ-wise $(t+2)$-intersecting. So there exist $F_1,\ldots,F_\ell\in\mathcal{F}$ such that $|F_1\cap\cdots\cap F_\ell|=t+1$. Since $\mathcal{F}$ is shifted, we may assume that $F_1\cap\cdots\cap F_\ell=[t+1]$. If there exists $F\in\mathcal{F}$ such that $|F\cap[t+1]|\leq t-1$, then $|F\cap F_1\cap\cdots\cap F_\ell|\leq t-1$ and this means $\mathcal{F}$ is not $(\ell+1)$-wise t-intersecting. Thus we must have $|F\cap[t+1]|\geq t$ for all $F\in\mathcal{F}$ and this proves $k\geq t=t(\ell+1)$. ∎

Using the lemma we have $k\geq t(6)$ if $t(6)+1=t(5)$, or $k\geq t(5)>t(6)$ if $t(5)+1=t(4)$. In either case we have $8\leq t(6)\leq k\leq 12$. For $1\leq i\leq k+1$ define

$$\mathcal{F}(i):=\{F\in\mathcal{F}:F\cap[k+1]=([k+1]\setminus\{i\})\},$$

and for $i=0$ define $\mathcal{F}(0):=\{F\in\mathcal{F}:[k+1]\subset F\}$, and set

$$\mathcal{G}(i) := \{F \cap [k+2,n] : F \in \mathcal{F}(i)\}$$

for $0\le i\le k+1$. Since $\mathcal{F}$ is non-trivial intersecting, shifted and maximal, we have

$$\emptyset \neq \mathcal{G}(1) \subset \mathcal{G}(2) \subset \cdots \subset \mathcal{G}(k+1) \subset \mathcal{G}(0). \tag{1}$$

Note also that

$$W_w(\mathcal{F}) = w^k(1-w)\sum_{i=1}^{k+1} W_w(\mathcal{G}(i)) + w^{k+1}W_w(\mathcal{G}(0)). \tag{2}$$

By the definition of k, there exists $F\in\mathcal{F}$ such that $|F\cap[k+2]|\le k$. Since $\mathcal{F}$ is shifted and maximal, it follows that $E_1 := [n]-\{k+1,k+2\}\in\mathcal{F}$. By shifting E_1, we have $E_i:=[n]-\{k+i,k+i+1\}\in\mathcal{F}$ for $1\le i\le n-k-1$. Set $s:=r-k=13-k$. We will only use the fact that there exist $\mathcal{F}\ni E_1,\dots,E_{2s}$ such that

$$k+i, k+i+1 \notin E_i \text{ for } i=1,\dots,2s.$$

Note that $E_1\cap E_3\cap\cdots\cap E_{2j-1}\cap[k+1,k+2j]=\emptyset$, and $E_2\cap E_4\cap\cdots\cap E_{2j}\cap[k+2,k+2j+1]=\emptyset$.

Lemma 7. *$\mathcal{G}(i)$ is $(k+1-i)$-wise $2s$-intersecting for $i=1,\dots,k-2$.*

Proof. Suppose, on the contrary, that $\mathcal{G}(i)$ is not $(k+1-i)$-wise $2s$-intersecting. Then we can find $G_i,G_{i+1},\dots,G_k\in\mathcal{G}(i)$ such that $|G_i\cap\cdots\cap G_k|\le 2s-1$. By the shiftedness, we may assume that $G_i\cap\cdots\cap G_k=[k+2,k+2s]$. For $i\le j\le k$, let $F_j:=([k+1]-\{i\})\cup G_j\in\mathcal{F}(i)$. Applying (i,j)-shift to F_j we have

$$F'_j := (F_j \setminus \{j\}) \cup \{i\} \in \mathcal{F}(j) \text{ for } i<j\le k.$$

Set $F'_i := F_i$ and choose $F_j\in\mathcal{F}(j)$ for $j=1,\dots,i-1$ arbitrarily. Then $F_1\cap\cdots\cap F_{i-1}\cap F'_i\cap\cdots\cap F'_k\subset[k+2,k+2s]$ and so $F_1\cap\cdots\cap F_{i-1}\cap F'_i\cap\cdots\cap F'_k\cap E_1\cap E_3\cap\cdots\cap E_{2s-1}=\emptyset$. This means that we have $k+s=r$ edges in $\mathcal{F}$ whose intersection is empty and this is a contradiction. ∎

Lemma 8. *$\mathcal{G}(k-1)$ is 3-wise $(2s-1)$-intersecting if $s\ge 1$.*
$\mathcal{G}(k)$ is 3-wise $(2s-3)$-intersecting if $s\ge 2$.
$\mathcal{G}(k+1)$ is 3-wise $(2s-5)$-intersecting if $s\ge 3$.
$\mathcal{G}(0)$ is 3-wise $(2s-6)$-intersecting if $s\ge 4$.

Proof. The proof is similar to the previous lemma. For example, suppose that $\mathcal{G}(k-1)$ is not 3-wise $(2s-1)$-intersecting. Then there exist $G_{k-1}, G_k, G_{k+1} \in \mathcal{G}(k-1)$ such that $G_{k-1} \cap G_k \cap G_{k+1} = [k+2, k+2s-1]$. Set $F'_j := ([k+1] - \{j\}) \cup G_j \in \mathcal{F}(j)$ for $j = k-1, k, k+1$ and choose $F_j \in \mathcal{F}(j)$ for $j = 1, \ldots, k-2$ arbitrarily. Then $F_1 \cap \cdots \cap F_{k-2} \cap F'_{k-1} \cap F'_k \cap F'_{k+1} \cap E_2 \cap E_4 \cap \cdots \cap E_{2s-2} = \emptyset$, which is a contradiction. The remaining statements can be proved in the same way. ∎

Recall that $8 \leq k \leq 12$ and so $1 \leq s \leq 5$. Let us deal with the hardest case $k = 10$ $(s = 3)$ first.

Case 1. $k = 10$ $(s = 3)$.

By Lemma 7 and Lemma 8, we get a table representing the ℓ-wise t-intersecting property for $\mathcal{G}(i)$ as follows:

$\mathcal{G}(i)$	$\mathcal{G}(6)$	$\mathcal{G}(7)$	$\mathcal{G}(8)$	$\mathcal{G}(9)$	$\mathcal{G}(10)$	$\mathcal{G}(11)$
ℓ-wise	5	4	3	3	3	3
t-int.	6	6	6	5	3	1

By Lemma 5 we have $W_w(\mathcal{G}(6)) \leq w^6$. Using (1) we have

$$W_w(\mathcal{G}(1)) + \cdots + W_w(\mathcal{G}(6)) \leq 6W_w(\mathcal{G}(6)) \leq 6w^6.$$

By Lemma 1, $W_w(\mathcal{G}(7)) \leq \alpha_{w,4}^6$ follows. By Lemma 3 we have

$$W_w(\mathcal{G}(8)) + W_w(\mathcal{G}(9)) + W_w(\mathcal{G}(10)) \leq w^2\alpha_{w,3}^4 + w^2\alpha_{w,3}^3 + w^2\alpha_{w,3}.$$

By Lemma 2, $W_w(\mathcal{G}(11)) \leq w$. For $\mathcal{G}(0)$ we use the trivial bound $W_w(\mathcal{G}(0)) \leq 1$. Therefore using (kUz2) we have

$$W_w(\mathcal{F}) \leq w^{10}(1-w)\{6w^6 + \alpha_{w,4}^6 + w^2\alpha_{w,3}^4 + w^2\alpha_{w,3}^3 + w^2\alpha_{w,3} + w\} + w^{11}. \tag{3}$$

Since $\alpha_{1/2,3} \approx 0.618033$ and $\alpha_{1/2,4} \approx 0.543689$, we have

$$W_{1/2}(\mathcal{F}) \leq 0.00091288 < W_{1/2}(\mathcal{F}_{BD}^{13}) \approx 0.000915527.$$

So we can conclude that $W_w(\mathcal{F}) < W_w(\mathcal{F}_{BD}^{13})$ for $w = 1/2 + \epsilon$ because both the RHS of (3) and $W_w(\mathcal{F}_{BD}^{13}) = (14 - 13w)w^{13}$ are continuous with respect to w.

The proof for the cases $k = 12, 11, 9, 8$ is similar (and easier). We give a sketchy proof here.

Case 2. $k = 12$ $(s = 1)$.

By Lemma 7 and Lemma 8, we have the following table.

$\mathcal{G}(i)$	$\mathcal{G}(10)$	$\mathcal{G}(11)$
ℓ-wise	3	3
t-int.	2	1

Therefore, we have

$$W_w(\mathcal{F}) \le w^{12}(1-w)\{10w^2 + w + 1 + 1\} + w^{13},$$

and $W_{1/2}(\mathcal{F}) \le 0.000732422 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

Case 3. $k=11$ $(s=2)$.

By Lemma 7 and Lemma 8, we have the following table.

$\mathcal{G}(i)$	$\mathcal{G}(9)$	$\mathcal{G}(10)$	$\mathcal{G}(11)$
ℓ-wise	3	3	3
t-int.	4	3	1

Therefore, we have

$$W_w(\mathcal{F}) \le w^{11}(1-w)\{9w^2\alpha_{w,3}^2 + w^2\alpha_{w,3} + w + 1\} + w^{12},$$

and $W_{1/2}(\mathcal{F}) \le 0.000857893 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

Case 4. $k=9$ $(s=4)$.

By Lemma 7 and Lemma 8, we have the following table.

$\mathcal{G}(i)$	$\mathcal{G}(9)$	$\mathcal{G}(10)$	$\mathcal{G}(0)$
ℓ-wise	3	3	3
t-int.	5	3	2

Therefore, we have

$$W_w(\mathcal{F}) \le w^9(1-w)\{9w^2\alpha_{w,3}^3 + w^2\alpha_{w,3}\} + w^{10}\cdot w^2,$$

and $W_{1/2}(\mathcal{F}) \le 0.000913729 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

Case 5. $k=8$ $(s=5)$.

By Lemma 7 and Lemma 8, we have the following table.

$\mathcal{G}(i)$	$\mathcal{G}(8)$	$\mathcal{G}(9)$	$\mathcal{G}(0)$
ℓ-wise	3	3	3
t-int.	7	5	4

Therefore, we have

$$W_w(\mathcal{F}) \leq w^8(1-w)\{8w^2\alpha_{w,3}^5 + w^2\alpha_{w,3}^3\} + w^9 \cdot w^2\alpha_{w,3}^2,$$

and $W_{1/2}(\mathcal{F}) \leq 0.000653997 < W_{1/2}(\mathcal{F}_{BD}^{13})$.

This completes the proof of Proposition 2. ∎

Now we are going back to the proof of the theorem. Let $\mathcal{F}$ be a non-trivial r-wise intersecting family. To apply induction, we suppose $r > 13$. We also suppose that $\mathcal{F}$ is shifted and maximal. Let us define

$$\mathcal{F}(1) := \{F - \{1\} : 1 \in F \in \mathcal{F}\}, \quad \mathcal{F}(\bar{1}) := \{F \in \mathcal{F} : 1 \notin F\}.$$

Since $\mathcal{F}$ is non-trivial intersecting and maximal, we have $[2,n] \in \mathcal{F}(\bar{1})$. By shifting $[2,n]$, we have $[n] - \{i\} \in \mathcal{F}$ for $1 \leq i \leq n$. Thus $\bigcap_{F \in \mathcal{F}(1)} F = \emptyset$. Since $\mathcal{F}$ is r-wise intersecting and $[2,n] \in \mathcal{F}$, it follows that $\mathcal{F}(1)$ is a non-trivial $(r-1)$-wise intersecting family. Thus using the induction hypothesis we have $W_w(\mathcal{F}(1)) \leq W_w(\mathcal{F}_{BD}^{r-1}) = w^{r-1}(r-(r-1)w)$.

On the other hand, $\mathcal{F}(\bar{1})$ is r-wise r-intersecting. To see this fact, suppose on the contrary that there exist $F_1, \ldots, F_r \in \mathcal{F}(\bar{1})$ such that $|F_1 \cap \cdots \cap F_r| < r$. Since $\mathcal{F}$ is shifted, we may assume that $F_1 \cap \cdots \cap F_r = [2,r]$. Then $F_i' := (F_i - \{i\} \cup \{1\}) \in \mathcal{F}$ for $2 \leq i \leq r$, and $F_1 \cap F_2' \cap \cdots \cap F_r' = \emptyset$, a contradiction. Therefore $\mathcal{F}(\bar{1})$ is r-wise r-intersecting and using Proposition 1 we have $W_w(\mathcal{F}(\bar{1})) \leq w^r$. Consequently it follows that

$$\begin{aligned} W_w(\mathcal{F}) &= wW_w(\mathcal{F}(1)) + (1-w)W_w(\mathcal{F}(\bar{1})) \\ &\leq w(w^{r-1}(r-(r-1)w)) + (1-w)w^r \\ &= w^r(r+1-rw) = W_w(\mathcal{F}_{BD}^r). \end{aligned}$$

This completes the proof of Theorem 2. ∎

4. Constructions

First we check that Theorem 2 fails if $r=5$. Recall that $W_w(\mathcal{F}^r_{BD})=(r+1-rw)w^r$.

Example 1. We construct a non-trivial 5-wise intersecting family $\mathcal{F}\subset 2^{[n]}$ as follows:

$$\mathcal{F} := \{\{1,2,3\}\cup G : G\subset [4,n], |G|\geq \lceil \tfrac{n-2}{2}\rceil\}\cup\{F_1,F_2,F_3\},$$

$$\text{where } F_i=[n]\setminus\{i\}.$$

Then $\lim_{n\to\infty} W_w(\mathcal{F}) = w^3$ for $w>1/2$. This implies $g(w,5,1)\geq w^3 > W_w(\mathcal{F}^5_{BD})=(6-5w)w^5$ for $1/2<w<\frac{1+\sqrt{21}}{10}$.

Using the fact that $\binom{[n]}{k}$ is r-wise t-intersecting if $(r-1)n+(t-1)<rk$, we can extend the above construction to get a slightly general lower bound for $g(w,r,t)$ as follows.

Proposition 3. *If $\frac{r-(i+1)}{r-i}<w$ then $g(w,r,t)\geq w^{it}$, where i is a non-negative integer.*

Proof. For sufficiently small $\epsilon>0$, we may assume that $\frac{r-(i+1)}{r-i}<(1-\epsilon)w$. Moreover, for sufficiently large n, we may assume that $\frac{r-(i+1)}{r-i}+\frac{t-1}{(r-i)(n-it)}<(1-\epsilon)w$. Set an open interval $I=((1-\epsilon)wn,(1+\epsilon)wn)$ and choose an integer $k\in I$, then $(1-\epsilon)w<k/n<k/(n-it)$. Thus, $\frac{r-(i+1)}{r-i}+\frac{t-1}{(r-i)(n-it)}<\frac{k}{n-it}$, or equivalently, $(r-(i+1))(n-it)+(t-1)<(r-i)k$. This means that $\binom{[it+1,n]}{k}$ is a non-trivial $(r-i)$-wise t-intersecting family. Therefore, the family

$$\mathcal{F} := \left\{[it]\cup G : G\in\binom{[it+1,n]}{k}, k\in I\right\}\cup\{[n]-[jt+1,(j+1)t] : 0\leq j<i\}$$

is non-trivial r-wise t-intersecting, and

$$g_n(w,r,t)\geq W_w(\mathcal{F}) = w^{it}\sum_{k\in I}\binom{n-it}{k}w^k(1-w)^{n-it-k}+i(1-w)^t w^{n-t}\to w^{it}$$

as $n\to\infty$. ∎

Using the above proposition, Theorem 1 and Lemma 2, we have the following.

Example 2. $f(w,r,t)=g(w,r,t)=1$ if $w>(r-1)/r$.

$$g(w,3,1)=\begin{cases} 5/16 & \text{if } w=1/2 \\ w & \text{if } 1/2<w\le 2/3 \\ 1 & \text{if } 2/3<w\le 1. \end{cases}$$

$$g(w,4,1)=\begin{cases} 3/16 & \text{if } w=1/2 \\ \ge w^2 & \text{if } 1/2<w\le \frac{1+\sqrt{17}}{8} \\ \ge (5-4w)w^4 & \text{if } \frac{1+\sqrt{17}}{8}\le w\le 2/3 \\ w & \text{if } 2/3<w\le 3/4 \\ 1 & \text{if } 3/4<w\le 1. \end{cases}$$

$$g(w,5,1)=\begin{cases} 7/64 & \text{if } w=1/2 \\ \ge w^3 & \text{if } 1/2<w\le \frac{1+\sqrt{21}}{10} \\ \ge (6-5w)w^5 & \text{if } \frac{1+\sqrt{21}}{10}\le w\le 2/3 \\ \ge w^2 & \text{if } 2/3<w\le 3/4 \\ w & \text{if } 3/4<w\le 4/5 \\ 1 & \text{if } 4/<w\le 1. \end{cases}$$

References

[1] A. Brace and D. E. Daykin: A finite set covering theorem I, II, III; *Bull. Austral. Math. Soc.* **5** (1971), 197–202; **6** (1972) 19–24, 417–433; IV: Infinite and Finite Sets, *Colloq. Math. Soc. János Bolyai* **10** (1975), 199–203.

[2] P. Frankl and N. Tokushige: Weighted 3-wise 2-intersecting families, *J. Combin. Theory (A)* **100** (2002), 94–115.

[3] P. Frankl and N. Tokushige: Weighted multiply intersecting families, *Studia Sci. Math. Hungarica* **40** (2003), 287–291.

[4] P. Frankl and N. Tokushige: Random walks and multiply intersecting families, *J. Combin. Theory (A)* **109** (2005), 121–134.

Peter Frankl

CNRS
ER 175 Combinatoire
2 Place Jussieu
75005 Paris
France
Peter111F@aol.com

Norihide Tokushige

College of Education
Ryukyu University
Nishihara, Okinawa, 903-0213
Japan
hide@edu.u-ryukyu.ac.jp