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The Maximum Size of 3-Wise Intersecting and 3-Wise Union Families

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Abstract. Let $\mathcal{F}$ be an n-uniform hypergraph on $2n$ vertices. Suppose that $|F_1 \cap F_2 \cap F_3| \geq 1$ and $|F_1 \cup F_2 \cup F_3| \leq 2n - 1$ holds for all $F_1, F_2, F_3 \in \mathcal{F}$. We prove that the size of $\mathcal{F}$ is at most $\binom{2n-2}{n-1}$.

1. Introduction

A family $\mathcal{F} \subset 2^X$ is called r-wise intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. A family $\mathcal{F} \subset 2^X$ is called r-wise union if $F_1 \cup \cdots \cup F_r \neq X$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. The Erdős–Ko–Rado theorem [2] states that if $n \geq 2k$ and $\mathcal{F} \subset \binom{n}{k}$ is 2-wise intersecting then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. By considering the complement, the theorem can be restated as follows: if $n \leq 2k$ and $\mathcal{F} \subset \binom{n}{k}$ is 2-wise union then $|\mathcal{F}| \leq \binom{n-1}{k}$.

We can extend the Erdős–Ko–Rado theorem for r-wise intersecting families as follows.

Theorem 1 [3]. *If $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise intersecting and $(r-1)n \geq rk$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. If $r \geq 3$ then equality holds iff $\mathcal{F} = \{F \in \binom{[n]}{k} : i \in F\}$ holds for some $i \in [n]$.*

The equivalent complement version is the following. If $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise union and $rk \geq n$ then $|\mathcal{F}| \leq \binom{n-1}{k}$.

Gronau [6], and Engel and Gronau [1] proved the following.

Theorem 2. *Let $r \geq 4$, $s \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$. Suppose that $\mathcal{F}$ is r-wise intersecting and s-wise union, and*

$$\frac{n-1}{s} + 1 \leq k \leq \frac{r-1}{r}(n-1).$$

Then we have $|\mathcal{F}| \leq \binom{n-2}{k-1}$.

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In this note we prove the following.

Theorem 3. *Let $\mathcal{F} \subset \binom{[2n]}{n}$ be a 3-wise intersecting and 3-wise union family. Then we have $|\mathcal{F}| \le \binom{2n-2}{n-1}$. Equality holds iff $\mathcal{F} = \{F \in \binom{[2n]-\{j\}}{n} : i \in F\}$ holds for some $i, j \in [2n]$.*

2. Proof of Theorem 3

We can prove the theorem for $n \le 3$ easily, so we assume that $n \ge 4$. Let $\mathcal{F} \subset \binom{[2n]}{n}$ be a 3-wise intersecting and 3-wise union family. If $\mathcal{F} \subset \binom{[2n]-\{j\}}{n}$ holds for some $j \in [2n]$ then Theorem 1 implies that $|\mathcal{F}| \le \binom{2n-2}{n-1}$ and equality holds iff there exists some $i \in [2n]$ such that $i \in F$ holds for all $F \in \mathcal{F}$, which verifies the theorem. From now on we assume that there is no such j, in other words, we assume that

$$\bigcup_{F \in \mathcal{F}} F = [2n]. \tag{1}$$

Considering the complement, we may assume that

$$\bigcap_{F \in \mathcal{F}} F = \emptyset. \tag{2}$$

Now suppose that

$$|\mathcal{F}| \ge \binom{2n-2}{n-1} \tag{3}$$

and we shall prove that there is no such $\mathcal{F}$.

For $A \in \binom{[2n]}{n}$, we define the corresponding walk on $\mathbb{Z}^2$, denoted by walk(A), in the following way. The walk is from $(0,0)$ to (n,n) with $2n$ steps, and if $i \in A$ (resp. $i \notin A$) then we move one unit up (resp. one unit to the right) at the i-th step. Let us define

$$\mathcal{A}_i := \{A \in \binom{[2n]}{n} : |A \cap [1+3\ell]| \ge 1+2\ell \text{ first holds at } \ell = i\},$$

$$\mathcal{A}_{\bar{j}} := \{A \in \binom{[2n]}{n} : |A \cap [2n-3\ell, 2n]| \le \ell \text{ first holds at } \ell = j\}.$$

If $A \in \mathcal{A}_i$ then, after starting from the origin, walk(A) touches the line $y = 2x+1$ at $(i, 2i+1)$ for the first time. If $A \in \mathcal{A}_{\bar{j}}$ then walk(A) touches the line $y = \frac{1}{2}(x-(n-1))+n$ at $(n-2j-1, n-j)$ and after passing this point this walk never touches the line again. Set $\mathcal{A}_{i\bar{j}} := \mathcal{A}_i \cap \mathcal{A}_{\bar{j}}$, and

$$a_i := |\mathcal{A}_i|/\binom{2n-2}{n-1}, \quad a_{\bar{j}} := |\mathcal{A}_{\bar{j}}|/\binom{2n-2}{n-1}, \quad a_{i\bar{j}} := |\mathcal{A}_{i\bar{j}}|/\binom{2n-2}{n-1}.$$

Set also

$$\mathcal{F}_i := \mathcal{A}_i \cap \mathcal{F}, \quad \mathcal{F}_{\bar{j}} := \mathcal{A}_{\bar{j}} \cap \mathcal{F}, \quad \mathcal{F}_{i\bar{j}} := \mathcal{A}_{i\bar{j}} \cap \mathcal{F},$$

$$f_i := |\mathcal{F}_i|/\binom{2n-2}{n-1}, \quad f_{\bar{j}} := |\mathcal{F}_{\bar{j}}|/\binom{2n-2}{n-1}, \quad f_{i\bar{j}} := |\mathcal{F}_{i\bar{j}}|/\binom{2n-2}{n-1},$$

and

$$\mathcal{G}_{i\bar{j}} := \{F \cap [3i+2, 2n-3j-1] : F \in \mathcal{F}_{i\bar{j}}\}.$$

Note that $|\mathcal{G}_{i\bar{j}}| \le |\mathcal{F}_{i\bar{j}}|$ and equality holds if both of i and j are at most 1.

We also use the following basic facts about shifting. (See e.g., [8, 4, 5] for the details.) We may assume that $\mathcal{F} \subset \binom{[2n]}{n}$ is shifted, i.e., for all $F \in \mathcal{F}$ and $1 \le i < j \le 2n$, if $i \notin F$ and $j \in F$ then $(F - \{j\}) \cup \{i\} \in \mathcal{F}$. It follows then for all $F \in \mathcal{F}$, walk (F) must touch the line $y = 2x + 1$ because $\mathcal{F}$ is a shifted 3-wise 1-intersecting family. In the same way, walk (F) must touch the line $y = \frac{1}{2}(x - (n-1)) + n$ because $\mathcal{F}$ is a shifted 3-wise 1-union family.

Claim 1. *$\mathcal{G}_{0\bar{0}} \subset \binom{[2,2n-1]}{n-1}$ is 2-wise intersecting.*

Proof. Otherwise we have $A, B \in \mathcal{F}_{0\bar{0}}$ such that $A \cap B = \{1\}$. This forces $\bigcap_{F \in \mathcal{F}} F = \{1\}$, contradicting (2). □

By Claim 1 and the Erdős–Ko–Rado theorem, we have $|\mathcal{F}_{0\bar{0}}| = |\mathcal{G}_{0\bar{0}}| \le \binom{2n-3}{n-2}$ and

$$f_{0\bar{0}} \le \binom{2n-3}{n-2} \Big/ \binom{2n-2}{n-1} = \frac{1}{2}. \tag{4}$$

Claim 2. *$\mathcal{G}_{1\bar{0}} \subset \binom{[5,2n-1]}{n-3}$ is 2-wise intersecting.*

Proof. Suppose on the contrary that there exist $A, B \in \mathcal{G}_{1\bar{0}}$ such that $A \cap B = \emptyset$. Then $\{2,3,4\} \cup A$, $\{2,3,4\} \cup B \in \mathcal{F}_{1\bar{0}}$. Since $\mathcal{F}$ is shifted we also have $\{1,3,4\} \cup B \in \mathcal{F}_{1\bar{0}}$. If there is $F \in \mathcal{F}$ such that $|F \cap [4]| \le 2$ then we may assume that $F \cap [2] = \{1, 2\}$ by the shiftedness of $\mathcal{F}$. But this is impossible because $(\{2,3,4\} \cup A) \cap (\{1,3,4\} \cup B) \cap F = \emptyset$.

Thus we may assume that $|F \cap [4]| \ge 3$ holds for all $F \in \mathcal{F}$. Let

$$\mathcal{F}(\bar{1}234) := \{F \cap [5, 2n] : F \in \mathcal{F},\ F \cap [4] = \{2,3,4\}\} \subset \binom{[5,2n]}{n-3},$$

$$\mathcal{F}(1\bar{2}34) := \{F \cap [5, 2n] : F \in \mathcal{F},\ F \cap [4] = \{1,3,4\}\} \subset \binom{[5,2n]}{n-3},$$

$$\mathcal{F}(12\bar{3}4) := \{F \cap [5, 2n] : F \in \mathcal{F},\ F \cap [4] = \{1,2,4\}\} \subset \binom{[5,2n]}{n-3}.$$

Then $|\mathcal{F}(\bar{1}234)| + |\mathcal{F}(1\bar{2}34)| + |\mathcal{F}(12\bar{3}4)| \le 3\binom{2n-4}{n-3}$. Let

$$\mathcal{F}(123) := \{F \cap [4, 2n] : \{1,2,3\} \subset F \in \mathcal{F}\} \subset \binom{[4,2n]}{n-3}.$$

Then $\mathcal{F}(123)$ is 3-wise union and it follows from the complement version of Theorem 1 that $|\mathcal{F}(123)| \le \binom{2n-4}{n-3}$. Therefore we have

$$|\mathcal{F}| = |\mathcal{F}(\bar{1}234)| + |\mathcal{F}(1\bar{2}34)| + |\mathcal{F}(12\bar{3}4)| + |\mathcal{F}(123)| \le 4\binom{2n-4}{n-3} < \binom{2n-2}{n-1},$$

which contradicts (3). □

By Claim 2 and the Erdős–Ko–Rado theorem, we have $|\mathcal{F}_{1\bar{0}}| = |\mathcal{G}_{1\bar{0}}| \le \binom{2n-6}{n-4}$ and

$$f_{1\bar{0}} \le \binom{2n-6}{n-4} \Big/ \binom{2n-2}{n-1} = \frac{(n-1)(n-3)}{4(2n-3)(2n-5)}.$$

Considering the complement, we have the same estimation for $f_{0\bar{1}}$. Therefore we have

$$f_{1\bar{0}} + f_{0\bar{1}} \le \frac{(n-1)(n-3)}{2(2n-3)(2n-5)}. \tag{5}$$

Claim 3. *$\mathcal{G}_{1\bar{1}} \subset \binom{[5,2n-4]}{n-4}$ is 2-wise intersecting.*

Proof. Suppose that there are $A, B \in \mathcal{G}_{1\bar{1}}$ such that $A \cap B = \emptyset$. Then we have $F_1 := \{2, 3, 4, 2n\} \cup A \in \mathcal{F}$. Since $\mathcal{F}$ is shifted and $\{2, 3, 4, 2n\} \cup B \in \mathcal{F}$, we also have $F_2 := \{1, 3, 4, 2n-1\} \cup B \in \mathcal{F}$. If $|F \cap [4]| \ge 3$ holds for all $F \in \mathcal{F}$ then we are done as we saw in the proof of Claim 2. So there is $G \in \mathcal{F}$ such that $|G \cap [4]| \le 2$ and by the shiftedness we may assume that $G \cap [4] = \{1, 2\}$. Then $F_1 \cap F_2 \cap G = \emptyset$, which is a contradiction. □

By Claim 3 and the Erdős–Ko–Rado theorem, we have $|\mathcal{F}_{1\bar{1}}| = |\mathcal{G}_{1\bar{1}}| \le \binom{2n-9}{n-5}$ and

$$f_{1\bar{1}} \le \binom{2n-9}{n-5} \Big/ \binom{2n-2}{n-1} = \frac{(n-1)(n-2)(n-3)}{16(2n-3)(2n-5)(2n-7)}. \tag{6}$$

By (4), (5) and (6), we have the following.

Claim 4. *$f_{0\bar{0}} + f_{1\bar{0}} + f_{0\bar{1}} + f_{1\bar{1}} \le H_1$, where*

$$H_1 := \frac{1}{2} + \frac{(n-1)(n-3)}{2(2n-3)(2n-5)} + \frac{(n-1)(n-2)(n-3)}{16(2n-3)(2n-5)(2n-7)}.$$

Next we consider $f_{i\bar{j}}$ where $\max\{i, j\} = 2$. Let c_i be the number of walks from $(0, 0)$ to $(i, 2i+1)$ which touch the line $y = 2x+1$ only at $(i, 2i+1)$. Then it follows that $c_i = \frac{1}{3i+1}\binom{3i+1}{i}$ (see e.g. Fact 3 in [7]).

If $A \in \mathcal{A}_{i\bar{j}}$ then walk(A) goes through the two points $P = (i, 2i+1)$ and $Q = (n-2j-1, n-j)$. Since the number of walks from P to Q is $\binom{2n-(3i+3j+2)}{n-(i+2j+1)}$, we get the following simple estimation.

$$f_{i\bar{j}} \le a_{i\bar{j}} = c_i c_j \binom{2n-(3i+3j+2)}{n-(i+2j+1)} \Big/ \binom{2n-2}{n-1} =: g(i, j).$$

Thus we have

$$(f_{2\bar{0}} + f_{0\bar{2}}) + (f_{2\bar{1}} + f_{1\bar{2}}) + f_{2\bar{2}} \le 2(g(2, 0) + g(2, 1)) + g(2, 2) =: H_2. \tag{7}$$

Finally we consider $f_i, f_{\bar{i}}$ for $i \ge 3$. We use the following fact which we prove in the next section.

Lemma 1. *We have*

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} |\mathcal{A}_i| \le \alpha \binom{2n}{n}$$

for all $n \ge 1$ *where* $\alpha = \frac{\sqrt{5}-1}{2}$.

We also use the following trivial estimation.

$$\max\{f_i, f_{\bar{i}}\} \le a_i = a_{\bar{i}} = c_i \binom{2n-3i-1}{n-i} \Big/ \binom{2n-2}{n-1}.$$

Then this together with Lemma 1 implies

$$\sum_{i>2} f_i \le \sum_{i>2} a_i \le \alpha \binom{2n}{n} \Big/ \binom{2n-2}{n-1} - \sum_{i=0}^{2} a_i =: H_3. \tag{8}$$

By Claim 4, (7) and (8), we have

$$|\mathcal{F}| \Big/ \binom{2n-2}{n-1} \le \sum_{0 \le i \le 2,\, 0 \le j \le 2} f_{i\bar{j}} + \sum_{i>2} f_i + \sum_{j>2} f_{\bar{j}} \le H_1 + H_2 + 2H_3 =: H_4(n),$$

where

$$H_4(n) = 4\sqrt{5} - \frac{32551}{4096} - \frac{2(\sqrt{5}-2)}{n} + \frac{1}{2^{20}} \left(\frac{6237}{2n-13} + \frac{2835}{2n-11} + \frac{28770}{2n-9} - \frac{156090}{2n-7} + \frac{923313}{2n-5} + \frac{298295}{2n-3} \right).$$

Note that $\lim_{n\to\infty} H_4(n) = 4\sqrt{5} - \frac{32551}{4096} = 0.997\ldots$. In fact one can check that $H_4(n) < 1$ for $n \ge 34$. For the remainder cases $4 \le n \le 33$, one can directly check that

$$|\mathcal{F}| \Big/ \binom{2n-2}{n-1} \le H_1 + H_2 + 2 \sum_{i=3}^{\lfloor \frac{n-1}{2} \rfloor} a_i < 1.$$

Consequently we showed that $|\mathcal{F}| < \binom{2n-2}{n-1}$ for all $n \ge 4$ and this contradicts (3). This completes the proof of Theorem 3. $\square$

3. Proof of Lemma 1

Since $|\mathcal{A}_i| = c_i\binom{2n-3i-1}{n-i}$ we need to prove that

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} c_i\binom{2n-3i-1}{n-i}\Big/\binom{2n}{n} \le \alpha.$$

We use the following fact (cf. (6) in [7]):

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} c_i\left(\frac{1}{2}\right)^{3i+1} \le \sum_{i=0}^{\infty} c_i\left(\frac{1}{2}\right)^{3i+1} = \alpha.$$

Thus to prove the lemma, it suffices to show that

$$\binom{2n-3i-1}{n-i}\Big/\binom{2n}{n} \le \left(\frac{1}{2}\right)^{3i+1} \tag{9}$$

for $0 \le i \le \lfloor\frac{n-1}{2}\rfloor$. We prove this inequality by induction on i. For the case $i = 0$, one can check that the equality holds in (9). Now let $i > 0$ and we assume (9) for i and we show the case $i + 1$, that is,

$$\binom{2n-3i-4}{n-i-1}\Big/\binom{2n}{n} \le \left(\frac{1}{2}\right)^{3i+4},$$

or equivalently,

$$\binom{2n}{n} \ge 2^{3i+4}\binom{2n-3i-4}{n-i-1}.$$

By the induction hypothesis, we have

$$\binom{2n}{n} \ge 2^{3i+1}\binom{2n-3i-1}{n-i},$$

and so it suffices to show that

$$2^{3i+1}\binom{2n-3i-1}{n-i} \ge 2^{3i+4}\binom{2n-3i-4}{n-i-1},$$

or equivalently,

$$f(i) := 5i^3 - (10n+6)i^2 + (4n^2-17)i + 6n - 6 \ge 0.$$

Since $f''(i) = -2(10n - 15i + 6) < 0$, the function $f(i)$ is concave on the domain $0 \le i \le \lfloor\frac{n-1}{2}\rfloor$. Thus it suffices to check that $f(0) \ge 0$ and $f(\lfloor\frac{n-1}{2}\rfloor) \ge 0$. Indeed, $f(0) = 6(n-1) \ge 0$, and $f(\lfloor\frac{n-1}{2}\rfloor) \ge \min\{f(\frac{n-1}{2}), f(\frac{n-2}{2})\} = f(\frac{n-1}{2}) = \frac{1}{8}(n+1)(n-1)(n-3) \ge 0$ if $n \ge 3$. For the case $n \le 2$, we only have $0 \le i \le \lfloor\frac{1}{2}\rfloor = 0$, that is, $i = 0$ and we already checked this case. $\square$

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