Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Combinatorial Theory, Series A 109 (2005) 121–134
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Random walks and multiply intersecting families

Peter Frankl[a], Norihide Tokushige[b]

[a] *CNRS, ER 175 Combinatoire, 2 Place Jussieu, 75005 Paris, France*
[b] *College of Education, Ryukyu University, 1 Nishihara, Okinawa 903-0213, Japan*

Received 12 May 2003
Available online 30 September 2004

Abstract

Let $\mathscr{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. It is proved that

$$|\mathscr{F}| \leqslant \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even,} \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd} \end{cases}$$

holds for $n \geqslant n_0$. The unique extremal configuration is determined as well.

Keywords: Intersecting family; Sperner family; Random walk

1. Introduction

Let n, r and t be positive integers. A family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geqslant t$ holds for all $F_1, \ldots, F_r \in \mathcal{F}$. An r-wise t-intersecting family $\mathcal{F}$ is called trivial if $|\bigcap_{F \in \mathcal{F}} F| \geqslant t$ holds. For a real $w \in (0, 1)$ let us define the weighted size $W_w(\mathcal{F})$ of $\mathcal{F}$ by

$$W_w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w^{|F|}(1-w)^{n-|F|}.$$

Some basic results concerning the maximum weighted size of multiply intersecting families can be found in [6–8]. Among others, the following is proved in [7].

E-mail addresses: Peter111F@aol.com (P. Frankl), hide@edu.u-ryukyu.ac.jp (N. Tokushige).
URL: http://www.cc.u-ryukyu.ac.jp/∼ hide.

0097-3165/$ - see front matter
doi:10.1016/j.jcta.2004.08.001

Theorem 1. *Let $\mathcal{F}$ be a* 3*-wise* 2*-intersecting family. Then* $W_w(\mathcal{F}) \leqslant w^2$ *if* $w < 0.5018$.

Moreover if $W_w(\mathcal{F}) \geqslant 0.999w^2$, then $\mathcal{F}$ contains a certain configuration, which we will explain later (see Theorem 10 in Section 4). Using this result, the following variation of the Erdős–Ko–Rado theorem [2,1] is deduced.

Theorem 2. *Let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a* 3*-wise* 2*-intersecting family with* $k/n \leqslant 0.501$, $n > n_0$. *Then* $|\mathcal{F}| \leqslant \binom{n-2}{k-2}$, *and equality holds only if $\mathcal{F}$ is trivial.*

For the proof of the above result, we use the "random walk method." The main tool is Theorem 6 described in the next section.

A family $\mathcal{F} \subset 2^{[n]}$ is called a Sperner family if $F \not\subset G$ holds for all distinct $F, G \in \mathcal{F}$. As an application of Theorem 2, we prove the following result.

Theorem 3. *Let* $\mathcal{F} \subset 2^{[n]}$ *be a* 3*-wise* 2*-intersecting Sperner family. Then,*

$$|\mathcal{F}| \leqslant \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even},\\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd}, \end{cases}$$

holds for $n \geqslant n_0$. *The extremal configurations are*

$$\begin{aligned} \mathcal{F} &= \{\{1,2\} \cup F : F \in \tbinom{[3,n]}{(n-2)/2}\} && n \text{ even},\\ \mathcal{F} &= \{\{1,2\} \cup F : F \in \tbinom{[3,n]}{(n-1)/2}\} \cup \{[n]-\{1\}\} \cup \{[n]-\{2\}\} && n \text{ odd}. \end{aligned}$$

Since $\mathcal{F} = \binom{[8]}{6}$ is 3-wise 2-intersecting Sperner and $|\mathcal{F}| = \binom{8}{6} > \binom{6}{3}$, the condition $n > n_0$ in the above theorem can not be omitted completely. It is an interesting but difficult problem to determine how small n_0 can be.

Other results concerning the maximum size of r-wise t-intersecting Sperner families can be found in [16] for the case $r = 2$, and in [3,9–12] for the case $r \geqslant 3$ and $t = 1$.

2. Tools

2.1. Shifting

For integers $1 \leqslant i < j \leqslant n$ and a family $\mathcal{F} \subset 2^{[n]}$, define the (i, j)-shift S_{ij} as follows.

$$S_{ij}(\mathcal{F}) := \{S_{ij}(F) : F \in \mathcal{F}\},$$

where

$$S_{ij}(F) := \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ (F-\{j\}) \cup \{i\} \notin \mathcal{F},\\ F & \text{otherwise.} \end{cases}$$

A family $\mathcal{F} \subset 2^{[n]}$ is called shifted if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leqslant i < j \leqslant n$. We call $\mathcal{F}$ a co-complex if $G \supset F \in \mathcal{F}$ implies $G \in \mathcal{F}$.

Let us introduce a partial order in $2^{[n]}$ by using shifting. Let $A, B \subset [n]$. Define $A \succ B$ if there exists $A' \subset [n]$ such that $A \subset A'$ and B is obtained by repeating a shifting to A'. The following fact is trivial but useful.

Fact 4. *Let $\mathcal{F} \subset 2^{[n]}$ be a shifted co-complex. If $A \in \mathcal{F}$ and $A \succ B$, then $B \in \mathcal{F}$.*

2.2. Random walk

Let $w \in (0, 2/3]$ be a fixed real number, and let $\alpha \in (0, 1)$ be the root of the equation $(1-w)x^3 - x + w = 0$, more explicitly, $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Note that $\alpha = \alpha(w)$ is an increasing function of w and $\alpha(0) = 0$, $\alpha(2/3) = 1$. Consider the infinite random walk, starting from the origin, in which at each step we move one unit up with probability w or move one unit right with probability $1 - w$. Then the probability that we ever hit the line $y = 2x + s$ is given by α^s where s is a non-negative integer. (See [4] for details.)

Let $F \in \mathcal{F} \subset 2^{[n]}$. We define the corresponding (finite) walk to F, denoted by walk(F), in the following way. If $i \in F$ (resp. $i \notin F$) then we move one unit up (resp. one unit right) at the ith step. Note that $F \succ G$ means walk(G) is in the area to the upper left of walk(F). The following fact shows how to use random walks to estimate the weighted size of a family.

Fact 5. *Let $\mathcal{F} \subset 2^{[n]}$, and suppose that, for all $F \in \mathcal{F}$, walk(F) touches the line $y = 2x+s$. Then $W_w(\mathcal{F}) \leqslant \alpha^s$.*

Now we give a variation of the above fact for the size of a uniform family, which we will use to prove Theorem 2.

Theorem 6. *Let $w \in \mathbb{R}$, $d \in \mathbb{Q}$, $s \in \mathbb{N}$ be fixed constants with $0 < d \leqslant w \leqslant 2/3$, and set $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Let $\mathcal{F} \subset \binom{[n]}{k}$ with $d = k/n$, $k > s$. Suppose that, for all $F \in \mathcal{F}$, walk(F) touches the line $y = 2x + s$. Then we have the following.*

(i) *For every $\varepsilon > 0$, $|\mathcal{F}|/\binom{n}{k} \leqslant (1+\varepsilon)\alpha^s$ holds for $n > n_0(\varepsilon)$.*
(ii) *If $w \leqslant 0.51$ then $|\mathcal{F}|/\binom{n}{k} \leqslant \alpha^s$ for $n > n_0$.*

Conjecture 7. *Theorem 6* (i) *is true for $\varepsilon = 0$ (or equivalently,* (ii) *is true for all $w \leqslant 2/3$).*

2.3. Shadow

For a family $\mathcal{F} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th shadow of $\mathcal{F}$, denoted by $\Delta_\ell(\mathcal{F})$, as follows.

$$\Delta_\ell(\mathcal{F}) := \{G \in \binom{[n]}{\ell} : G \subset \exists F \in \mathcal{F}\}.$$

We use the following version of the Kruskal–Katona theorems [15,14,5]:

Proposition 8. *Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *and* $|\mathcal{F}| \leqslant \binom{m}{k}$*. Then,*

$$|\Delta_\ell(\mathcal{F})| \geqslant |\mathcal{F}| \binom{m}{\ell} / \binom{m}{k}.$$

Equality holds only if $\mathcal{F} = \binom{Y}{k}$, $|Y| = m$.

We also use the following Katona's shadow theorem for t-intersecting families [13].

Proposition 9. *Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *is* 2*-wise* t*-intersecting, and* $n \geqslant 2k-t$, $k > l \geqslant k-t$*. Then,*

$$|\Delta_\ell(\mathcal{F})| \geqslant |\mathcal{F}| \binom{2k-t}{\ell} / \binom{2k-t}{k}.$$

Equality holds only if $\mathcal{F} = \binom{Y}{k}$, $|Y| = 2k - t$.

3. Proof of Theorem 6

If $w = 2/3$ then $\alpha = 1$ and the theorem is trivial in this case. So we assume that $w < 2/3$. Since the theorem clearly holds for $s = 0$ also, we may assume that $s \geqslant 1$. For each $i = 0, 1, \dots, \lfloor \frac{k-s}{2} \rfloor$ let a_i be the number of walks of length $3i + s$, which attain the line L: $y = 2x + s$ at $(i, 2i + s)$ for the first time. Then the total number of walks from $(0, 0)$ to $(n - k, k)$ that attain L is

$$\sum_{i=0}^{\lfloor \frac{k-s}{2} \rfloor} a_i \binom{n-3i-s}{k-2i-s}. \tag{1}$$

To obtain the probability that a walk attains the line, we have to divide (1) by $\binom{n}{k}$.

Next consider a walk where each step is chosen independently and randomly with probability w for one step up and probability $1 - w$ for one step right. Then the probability for this random walk to attain the line by n steps is

$$\sum_{i=0}^{\lfloor \frac{k-s}{2} \rfloor} a_i w^{2i+s} (1-w)^i. \tag{2}$$

Recall that the above probability is less than α^s, where $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$.

Comparing (1) and (2), Theorem 6 (i) will be proved as soon as we establish the following inequality for all $0 \leqslant i \leqslant \lfloor \frac{k-s}{2} \rfloor$, $n > n_0(\varepsilon)$:

$$\binom{n-3i-s}{k-2i-s} / \binom{n}{k} \leqslant (1+\varepsilon) w^s \{w^2(1-w)\}^i.$$

This is certainly true for $i = 0$ (even if $\varepsilon = 0$) because $\binom{n-s}{k-s}/\binom{n}{k} \leqslant (k/n)^s \leqslant w^s$. Note that $\binom{n-3i-s}{k-2i-s}/(\binom{n}{k}w^s)$ is a decreasing function of s. So it suffices to prove the above inequality for $s = 1$, that is,

$$\frac{k}{n}\prod_{j=0}^{i-1}\frac{(k-2j-1)(k-2j-2)(n-k-j)}{(n-3j-1)(n-3j-2)(n-3j-3)} \leqslant (1+\varepsilon)w\,\{w^2(1-w)\}^i$$

for $1 \leqslant i \leqslant \lfloor \frac{k-s}{2} \rfloor$, $n > n_0(\varepsilon)$. Since $d \leqslant w$ and $w^2(1-w)$ is an increasing function of w for $0 \leqslant w \leqslant 2/3$, we have $d(d^2(1-d))^i \leqslant w(w^2(1-w))^i$. Thus, it is sufficient to prove the case $d = w$, that is

$$\prod_{j=0}^{i-1} f(j) \leqslant (1+\varepsilon)\{d^2(1-d)\}^i, \tag{3}$$

where

$$f(j) = \frac{(dn-2j-1)(dn-2j-2)(n-dn-j)}{(n-3j-1)(n-3j-2)(n-3j-3)}.$$

Here let us check that $f(j)$ is a decreasing function of j for $0 \leqslant j \leqslant i-1 \leqslant \frac{k-1}{2} - 1 = \frac{dn-3}{2}$. Set $g(j) = f'(j)(n-3j-1)^2(n-3j-2)^2(n-3j-3)^2$, and $g'(j) = 2(n-3j-2)h(j)$. Then $h(j) = -36j^2 + O(j)$, $h(0) = (2-3d)^2(1+3d)n^3 + O(n^2) > 0$ and $h(dn/2) = (1/2)(2-3d)^3n^3 + O(n^2) > 0$. Note that $h(j)$ is a concave parabola as a function of j, and the both ends ($j = 0, dn/2$) have positive value. This means $h(j) > 0$ and $g'(j) > 0$ for $0 \leqslant j \leqslant dn/2$. Then $g(\frac{dn-3}{2}) = -\frac{3}{8}(2-3d)^4n^4 + O(n^3) < 0$ implies $g(j) < 0$ and so $f'(j) < 0$ for $0 \leqslant j \leqslant \frac{dn-3}{2}$.

Thus, we have $\prod_{j=0}^{i-1} f(j) \leqslant f(0)^i$. If $d \leqslant 1/2$ then one can check $f(0) < d^2(1-d)$ for n sufficiently large, and so $\prod_{j=0}^{i-1} f(j) < (d^2(1-d))^i$ follows. This is stronger than (3). Now we may assume that $d > 1/2$.

If $j \geqslant \sqrt{n}$ then for $n > n_0$ we have

$$f(j) \leqslant d^2(1-d). \tag{4}$$

In fact, for $j = \sqrt{n}$, we have

$$d^2(1-d)D - N = d(2-3d)^2n^{5/2} + O(n^2) > 0,$$

where D and N stand for the denominator and the numerator of $f(j)$.

Since

$$\lim_{n\to\infty}\left(\frac{f(0)}{d^2(1-d)}\right)^{\sqrt{n}} = 1,$$

we have

$$\prod_{j=0}^{\sqrt{n}-1} f(j) \leqslant f(0)^{\sqrt{n}} < (1+\varepsilon)\{d^2(1-d)\}^{\sqrt{n}}. \tag{5}$$

If $i > \sqrt{n}$ then by (4) and (5) we have

$$\prod_{j=0}^{i-1} f(j) = \left(\prod_{j=0}^{\sqrt{n}-1} f(j)\right)\left(\prod_{j=\sqrt{n}}^{i-1} f(j)\right) \leqslant (1+\varepsilon)\{d^2(1-d)\}^i.$$

So we may assume that $i \leqslant \sqrt{n}$. Since $d > 1/2$ and $n > n_0$, we have $f(0) > d^2(1-d)$ and

$$\left(\frac{f(0)}{d^2(1-d)}\right)^i \leqslant \left(\frac{f(0)}{d^2(1-d)}\right)^{\sqrt{n}} < 1+\varepsilon.$$

Therefore, $\prod_{j=0}^{i-1} f(j) \leqslant f(0)^i < (1+\varepsilon)(d^2(1-d))^i$ follows. This completes the proof of (i).

Now we prove (ii). For $d \leqslant 1/2$, we have proved $f(0) < d^2(1-d)$ and this implies the desired inequality. So we assume $d > 1/2$. Then $f(0) > d^2(1-d)$. However, for $j \geqslant 1$ and $d < 0.547$, we still have $f(j) \leqslant d^2(1-d)$ because

$$d^2(1-d) - f(1) = \{d(15d^2 - 21d + 7)n^2 + O(n)\}/\{n^3 + O(n^2)\}.$$

In the same way, one can prove $f(0)f(1) \leqslant \{d^2(1-d)\}^2$ for $d < 0.529$ because

$$\{d^2(1-d)\}^2 - f(0)f(1) = \frac{d^3(1-d)(21d^2 - 30d + 10)n^5 + O(n^4)}{n^6 + O(n^5)}.$$

Therefore, we have

$$\prod_{j=0}^{i-1} f(j) \leqslant \{d^2(1-d)\}^i \tag{6}$$

for $i \geqslant 2$. Our goal is to prove

$$\sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} a_i \prod_{j=0}^{i-1} f(j) \leqslant \sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} a_i \{d^2(1-d)\}^i. \tag{7}$$

To deal with the case $i = 1$, we show the following for $d < 0.515$:

$$a_1 f(0) + a_2 f(0) f(1) \leqslant a_1 d^2(1-d) + a_2 d^4 (1-d)^2. \tag{8}$$

Since $a_1 = 1,\ a_2 = 3$, the above inequality follows from the fact that RHS–LHS is

$$\{3d(1-d)^2(1-d+9d^2-21d^3)n^5 + O(n^4)\}/\{n^6 + O(n^5)\}.$$

Finally (7) follows from (6) and (8). This completes the proof of (ii).

In principle, one can verify whether

$$\sum_{i=1}^{p} a_i \prod_{j=0}^{i-1} f(j) \leqslant \sum_{i=1}^{p} a_i \{d^2(1-d)\}^i \tag{9}$$

is true or not for any concrete p, and (8) is the case $p = 2$. The larger p we take, the better bound for d we can get if (9) is true. For example, taking $p = 42$ we can verify (9) (with the aid of computer) for $d \leqslant 0.6$, this shows that Conjecture 7 is true for $d \leqslant 0.6$.

4. Proof of Theorem 2

Let us define the following.

$$\begin{aligned} *(i) &:= \{i, i+1, i+3, i+4, i+6, i+7, \ldots\} \cap [n] \\ &= [n] - ([i-1] \cup \{i+3j+2 : 0 \leqslant j \leqslant \lfloor \frac{n-i-2}{3} \rfloor\}), \\ P_i &:= \{1, 2\} \cup *(i+4). \end{aligned}$$

Note that $*(i) \cap *(i+1) \cap *(i+2) = \emptyset$, and $P_i \cap P_{i+1} \cap P_{i+2} = \{1, 2\}$. In [7] the following is proved (see the first paragraph of the proof of Proposition 4 on p. 111 in [7]).

Theorem 10. *Let $\mathcal{G} \subset 2^{[n]}$ be a 3-wise non-trivial 2-intersecting shifted co-complex. If $W_w(\mathcal{G}) \geqslant 0.999w^2$ and $w \leqslant 0.5015$ then, for some $i \geqslant 1$, $\mathcal{G}$ contains $P_0, P_1, \ldots, P_i$ but does not contain P_{i+1}.*

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family. If $\mathcal{F}$ fixes a 2-element set, then $|\mathcal{F}| \leqslant \binom{n-2}{k-2}$. So we may assume that $\mathcal{F}$ is non-trivial. We shall prove that $|\mathcal{F}| < \binom{n-2}{k-2}$. Suppose that $|\mathcal{F}| \geqslant 0.999\binom{n-2}{k-2}$, and set $w := 0.5015$. Define $\mathcal{F}^c := \{[n] - F : F \in \mathcal{F}\}$ and

$$\mathcal{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathcal{F}^c))^c \quad (\subset \bigcup_{i=k}^{n} \binom{[n]}{i}).$$

Clearly $\mathcal{G}$ is a non-trivial 3-wise 2-intersecting family. Let us show that $W_w(\mathcal{G}) > 0.999w^2$ if n is sufficient large.

Choose $\varepsilon > 0$ sufficiently small so that

$$0.9998(1-\varepsilon)^4 > 0.999, \tag{10}$$

$$0.501 < (1-\varepsilon)w. \tag{11}$$

Define an open interval $I := ((1-\varepsilon)wn, (1+\varepsilon)wn)$. Set $v = 1 - w$ and choose $n_0 = n_0(\varepsilon)$ sufficiently large so that

$$\sum_{i \in I} \binom{n}{i} w^i v^{n-i} > 1 - \varepsilon \quad \text{for all } n > n_0, \tag{12}$$

$$(((1-\varepsilon)wn - 1)/n)^2 > (1-\varepsilon)^3 w^2 \quad \text{for all } n > n_0. \tag{13}$$

By our assumption on k/n and (11), we have $k \leqslant 0.501n < (1-\varepsilon)wn$, and

$$W_w(\mathcal{G}) = \sum_{i=k}^{n} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i} \geqslant \sum_{i \in I} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

It follows from the Kruskal–Katona theorem that $|\Delta_{n-i}(\mathcal{F}^c)| \geqslant 0.9998\binom{n-2}{n-i}$ for $i \in I$. (This is Lemma 7 on p. 112 in [7].) Therefore,

$$\begin{aligned} W_w(\mathcal{G}) &\geqslant 0.9998 \sum_{i\in I} \binom{n-2}{n-i} w^i v^{n-i} \\ &= 0.9998 \sum_{i\in I} \frac{i}{n} \cdot \frac{i-1}{n-1} \binom{n}{i} w^i v^{n-i} \\ &> 0.9998(1-\varepsilon)^3 w^2 \sum_{i\in I} \binom{n}{i} w^i v^{n-i} \quad \text{(by (13))} \\ &> 0.9998(1-\varepsilon)^4 w^2 \quad \text{(by (12))} \\ &> 0.999 w^2 \quad \text{(by (10))}. \end{aligned}$$

This completes the proof of $W_w(\mathcal{G}) > 0.999w^2$.

So by Theorem 10 we may assume that $P_i \in \mathcal{G}$, $P_{i+1} \notin \mathcal{G}$, for some $i \geqslant 1$. Let us define the following.

$$\begin{aligned} Q_i &:= \{1, 2, i+4\} \cup *(i+6), \\ \mathcal{F}_{12} &:= \{F \in \mathcal{F} : \{1, 2\} \subset F\}, \\ \mathcal{F}_{1\bar{2}} &:= \{F \in \mathcal{F} : 1 \in F, 2 \notin F\}, \\ \mathcal{F}_{\bar{1}2} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \in F\}, \\ \mathcal{F}_{\bar{1}\bar{2}} &:= \{F \in \mathcal{F} : 1 \notin F, 2 \notin F\}. \end{aligned}$$

By definition, it follows that $P_{i+1} \succ Q_i \succ P_i$, $|\mathcal{F}| = |\mathcal{F}_{12}| + |\mathcal{F}_{1\bar{2}}| + |\mathcal{F}_{\bar{1}2}| + |\mathcal{F}_{\bar{1}\bar{2}}|$. Set $d = k/n$ ($d \leqslant 0.501$), and $\alpha = \frac{1}{2}(\sqrt{\frac{1+3d}{1-d}} - 1)$. (Redefine $w := d$.)

Case 1: $Q_i \notin \mathcal{G}$.

If $4i+4 \geqslant n$ then we have $R = [i+2] \cup \{i+3, i+6, i+9, \dots\} \in \mathcal{G}$ because $\mathcal{G} \ni P_i \succ R$. But this is impossible because $P_i \cap R = \{1, 2\}$ implies $\mathcal{G}$ is trivial. So we may assume that $n \geqslant 4i + 5$.

Observe that walk(Q_i) starts with “up, up,” and $i+1$ “right,” then from $(i+1, 2)$ this walk is the maximal walk which does not touch the line L: $y = 2(x - (i+1)) + 4$.

Let $F \in \mathcal{F}_{12}$, then walk(F) starts with “up, up.” If walk(F) goes through the point $(i+1, 2)$, then this walk must meet the line L after passing $(i+1, 2)$. To apply Theorem 6, it is convenient to neglect the first $i+3$ moves (up, up, and then $i+1$ times right) from walk(F), in other words, we shift the origin to $(i+1, 2)$. Then the modified walk corresponding to $F - \{1, 2\} \subset \binom{[3,n]}{k-2}$, starting from the new origin, must touch the line $y = 2x + 2$. Therefore, by Theorem 6 (ii), the number of walks of this type is at most $\alpha^2\binom{n-i-3}{k-2}$. Otherwise walk$(F)$ must go through one of $(0, i+3), (1, i+2), \dots, (i, 3)$, and the number of corresponding walks is $\binom{n-2}{k-2} - \binom{n-i-3}{k-2}$. Thus, we have

$$\begin{aligned} |\mathcal{F}_{12}| &\leqslant \binom{n-2}{k-2} - \binom{n-i-3}{k-2} + \alpha^2\binom{n-i-3}{k-2} \\ &= \binom{n-2}{k-2}\{1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha^2)\}. \end{aligned}$$

To obtain an upper bound for $|\mathcal{F}_{1\bar{2}}|$, let us set

$$F_0 := [1, i+3] \cup \{i+6, i+9, i+12, \ldots, 4i, 4i+3\} \cup *(4i+5),$$
$$G := \{1\} \cup [3, 4i+4] \cup *(4i+6).$$

Since $P_i \in \mathcal{G}$ and $P_i = \{1,2\} \cup *(i+4) = \{1,2\} \cup \{i+4, i+5, i+7, i+8, \ldots, 4i+1, 4i+2\} \cup *(4i+4) \succ F_0$, we have $F_0 \in \mathcal{G}$. Note that $P_i \cap F_0 \cap G = \{1\}$. Thus $G \notin \mathcal{G}$ follows from the assumption that $\mathcal{G}$ is 3-wise 2-intersecting. Now let us look at walk(G). This walk starts with "up, right," then from $(1,1)$ this is the maximal walk which does not touch the line L: $y = 2(x-1) + (4i+4)$. Since $G \notin \mathcal{G}$, for every $F \in \mathcal{F}_{1\bar{2}}$, walk$(F)$ must touch the line L. To apply Theorem 6, we neglect the first two moves (up, right) from walk(F), or equivalently, we shift the origin to $(1,1)$. Then the modified walk corresponding to $F - \{1\} \subset \binom{[3,n]}{k-1}$, starting from the new origin, must touch the line $y = 2x + (4i+3)$. Then due to Theorem 6 (ii), we have

$$|\mathcal{F}_{1\bar{2}}| \leqslant \binom{n-2}{k-1} \alpha^{4i+3} = \binom{n-2}{k-2} \frac{n-k}{k-1} \alpha^{4i+3}.$$

The same estimation is valid for $|\mathcal{F}_{\bar{1}2}|$. From now on, we will use the above trick (shifting the origin) without mentioning when we apply Theorem 6.

Next, set $H := [3, 4i+7] \cup *(4i+9)$. Since $P_i \cap F_0 \cap H = \{4i+5\}$, we have $H \notin \mathcal{G}$, which implies

$$|\mathcal{F}_{\bar{1}\bar{2}}| \leqslant \binom{n-2}{k} \alpha^{4i+6} = \binom{n-2}{k-2} \frac{(n-k)(n-k-1)}{k(k-1)} \alpha^{4i+6}.$$

Therefore, $|\mathcal{F}| \leqslant c\binom{n-2}{k-2}$ where

$$c = 1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha^2) + \frac{2(n-k)}{k-1}\alpha^{4i+3} \frac{(n-k)(n-k-1)}{k(k-1)} \alpha^{4i+6}.$$

Let us check $c < 1$ for $n > n_0$. The target inequality can be rewritten as

$$2\alpha^3 + \frac{(1-d)n-1}{dn}\alpha^6 < (1-\alpha^2)\frac{dn-1}{n-2} \prod_{j=1}^{i} \frac{(1-d)n-j}{(n-j-2)\alpha^4}. \tag{14}$$

Since $d \leqslant 0.501$ and $j \leqslant i \leqslant \frac{n-5}{4}$, we have $\frac{(1-d)n-j}{(n-j-2)\alpha^4} > 1$. So the RHS of (14) is minimal when $i = 1$, and to prove the inequality for $n > n_0$ it suffices to show

$$2\alpha^3 + \frac{1-d}{d}\alpha^6 < \frac{(1-\alpha^2)\, d\, (1-d)}{\alpha^4}$$

and this is true for $d \leqslant 0.528$. (To verify this, reduce $f(d) := d\alpha^4$(RHS–LHS) by using $(1-d)\alpha^3 - \alpha + d = 0$. Then one can check that $g(d) := f(d)(1-d)^3$ has two real zeros, i.e., $d = 0$ and $d = 0.528\ldots$, and moreover $g(d) > 0$ inside this interval.)

Case 2: $Q_i \in \mathcal{G}$.

If $4i+6 \geqslant n$ then we have $R = [i+3] \cup \{i+5, i+8, i+11, \ldots\} \in \mathcal{G}$ because $\mathcal{G} \ni Q_i \succ R$. But this is impossible because $Q_i \cap R = \{1,2\}$ implies $\mathcal{G}$ is trivial. So we may assume that $n \geqslant 4i+7$.

Since $P_{i+1} \notin \mathcal{G}$, we have

$$|\mathcal{F}_{12}| \leqslant \binom{n-2}{k-2} - \binom{n-i-3}{k-2} + \alpha\binom{n-i-3}{k-2}$$
$$= \binom{n-2}{k-2}\{1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha)\}.$$

Set

$$F := [1, i+3] \cup \{i+5, i+8, i+11, \ldots, 4i+5\} \cup *(4i+7),$$
$$G := \{1\} \cup [3, 4i+6] \cup *(4i+8).$$

Since $Q_i \in \mathcal{G}$ and $Q_i = \{1, 2\} \cup \{i+4, i+6, i+7, \ldots, 4i+3, 4i+4\} \cup *(4i+6) \succ F$, we have $F \in \mathcal{G}$. Note that $Q_i \cap F \cap G = \{1\}$. Thus $G \notin \mathcal{G}$ follows from the assumption that $\mathcal{G}$ is 3-wise 2-intersecting. Therefore,

$$|\mathcal{F}_{1\bar{2}}| \leqslant \binom{n-2}{k-1}\alpha^{4i+5}.$$

The same estimation is valid for $|\mathcal{F}_{\bar{1}2}|$. Set $H := [3, 4i+9] \cup *(4i+11)$. Since $Q_i \cap F \cap H = \{4i+7\}$, we have $H \notin \mathcal{G}$, which implies

$$|\mathcal{F}_{\bar{1}\bar{2}}| \leqslant \binom{n-2}{k}\alpha^{4i+8}.$$

Therefore, $|\mathcal{F}| \leqslant c\binom{n-2}{k-2}$ where

$$c = 1 - \frac{\binom{n-i-3}{k-2}}{\binom{n-2}{k-2}}(1-\alpha) + \frac{2(n-k)}{k-1}\alpha^{4i+5}\frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+8}.$$

One can check that $c < 1$ for $n > n_0$. Indeed, this time it suffices to show

$$2\alpha^5 + \frac{1-d}{d}\alpha^8 < \frac{(1-\alpha)\, d\, (1-d)}{\alpha^4},$$

and this is true for $d \leqslant 0.536$. This completes the proof of Theorem 2.

In Cases 1 and 2, we proved $c = |\mathcal{F}|/\binom{n-2}{k-2} < 1$. On the other hand, we can construct a series of non-trivial 3-wise 2-intersecting k-uniform families $\mathcal{F}^{(n)}$ on n vertices with $k = (\frac{1}{2} + \varepsilon)n$ which satisfies $\lim_{n\to\infty} \mathcal{F}^{(n)}/\binom{n-2}{k-2} = 1$ as follows:

$$\mathcal{F}_{12}^{(n)} = \left\{\{1, 2\} \cup G : |G \cap [3, k+2]| \geqslant \frac{k+2}{2}\right\},$$

$$\mathcal{F}_{1\bar{2}}^{(n)} = \mathcal{F}_{\bar{1}2}^{(n)} = \emptyset,\ \ \mathcal{F}_{\bar{1}\bar{2}}^{(n)} = \{[3, k+2]\}.$$

The maximal i such that $P_i \in \mathcal{F}^{(n)}$ is given by $i = \lfloor \frac{k}{4} \rfloor - 2$ for k odd, and $i = \lceil \frac{k}{4} \rceil - 2$ for k even.

5. Proof of Theorem 3

For a family $\mathcal{F} \subset 2^{[n]}$, set $\mathcal{F}_i := \mathcal{F} \cap \binom{[n]}{i}$. First we prove the following inequality.

Proposition 11. *Let $\mathcal{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family with $k/n \leqslant 0.501$, $n > n_0$. Then $\sum_{i=1}^{k} |\mathcal{F}_i|/\binom{n-2}{i-2} \leqslant 1$.*

Proof. We prove $\sum_{i=1}^{k} |\mathcal{F}_i|/\binom{n-2}{i-2} \leqslant 1$ for $n > n_0$ by induction on the number of nonzero $|\mathcal{F}_i|$'s.

If this number is one then the inequality follows from Theorem 2. If it is not the case then let p be the smallest and r the second-smallest index for which $|\mathcal{F}_i| \neq 0$. Set $\mathcal{F}_p^c := \{[n] - F : F \in \mathcal{F}_p\} \subset \binom{[n]}{n-p}$. Since $\mathcal{F}_p$ is 3-wise 2-intersecting, it follows from Theorem 2 that $|\mathcal{F}_p| = |\mathcal{F}_p^c| \leqslant \binom{n-2}{p-2} = \binom{n-2}{n-p}$. Then by Proposition 8, we have

$$\frac{|\Delta_{n-r}(\mathcal{F}_p^c)|}{|\mathcal{F}_p^c|} \geqslant \frac{\binom{n-2}{n-r}}{\binom{n-2}{n-p}} = \frac{\binom{n-2}{r-2}}{\binom{n-2}{p-2}}. \tag{15}$$

Set $\mathcal{G}_r := \{G \in \binom{[n]}{r} : G \supset \exists F \in \mathcal{F}_p\}$. Due to (15) and the fact $\mathcal{G}_r = (\Delta_{n-r}(\mathcal{F}_p^c))^c$, we have $|\mathcal{G}_r|/\binom{n-2}{r-2} \geqslant |\mathcal{F}_p|/\binom{n-2}{p-2}$. Since $\mathcal{F}$ is Sperner, $\mathcal{F}_r \cap \mathcal{G}_r = \emptyset$ and $\mathcal{H} := (\mathcal{F} - \mathcal{F}_p) \cup \mathcal{G}_r$ is a 3-wise 2-intersecting Sperner family. Moreover, the number of nonzero $|\mathcal{H}_i|$'s is one less than that of $|\mathcal{F}_i|$'s. Therefore, by the induction hypothesis and the fact that $\mathcal{F} \triangle \mathcal{H} = \mathcal{F}_p \cup \mathcal{G}_r$, we have

$$\sum_{i=1}^{k} \frac{|\mathcal{F}_i|}{\binom{n-2}{i-2}} \leqslant \sum_{i=1}^{k} \frac{|\mathcal{H}_i|}{\binom{n-2}{i-2}} \leqslant 1,$$

which completes the proof of the proposition. □

By (15), we have $|\Delta_{n-r}(\mathcal{F}_p^c)| \geqslant |\mathcal{F}_p^c|$ (and so $|\mathcal{F}| \leqslant |\mathcal{H}|$) if $n \geqslant p + r - 2$. Replace $\mathcal{F}$ by $\mathcal{H}$ (and find new p and r) and continue the same procedure as long as $n \geqslant p + r - 2$. In the end, we have at most one index $p < \lceil \frac{n+2}{2} \rceil$ such that $\mathcal{F}_p \neq \emptyset$. If we have such p, then set $r := \lceil \frac{n+2}{2} \rceil$ even though $\mathcal{F}_r = \emptyset$ may happen only in this last step, and replace $\mathcal{F}_p$ by $\mathcal{G}_r$ and obtain $\mathcal{H}$ from $\mathcal{F}$. In this way, we can construct a 3-wise 2-intersecting Sperner family $\mathcal{H}$ with $|\mathcal{H}| \geqslant |\mathcal{F}|$ and $\mathcal{H}_i = \emptyset$ for all $i < \lceil \frac{n+2}{2} \rceil$. In this process, $|\mathcal{H}| = |\mathcal{F}|$ happens only if $n = p + r - 2$ and $\mathcal{F}_p^c = \binom{Y}{n-p}$, $|Y| = n - 2$ (cf. Proposition 8), that is,

$$\mathcal{F}_p \cong \left\{ \{a, b\} \cup G : G \in \binom{Y}{p-2} \right\}.$$

But then we can find $A, B \in \mathcal{F}_p$ with $A \cap B = \{a, b\}$ because $|Y| = n - 2 = (p - 2) + (r - 2) \geqslant 2(p - 2)$. In this case, all members in $\mathcal{F}$ must contain $\{a, b\}$ and we can easily verify Theorem 3. Therefore, for the proof of Theorem 3, we may assume that $\mathcal{F}_i = \emptyset$ for $i < \lceil \frac{n+2}{2} \rceil$ from the beginning (otherwise replace $\mathcal{F}$ by $\mathcal{H}$). This remark is needed because we claim the uniqueness of the extremal configuration.

Let us now prove Theorem 3. Suppose that $\mathcal{F} \subset 2^{[n]}$ is a 3-wise 2-intersecting Sperner family of maximal size. We may assume that $\mathcal{F}_i = \emptyset$ for all $i < \lceil \frac{n+2}{2} \rceil =: m$. Set $k = \lfloor 0.501n \rfloor$ and $r_i = |\mathcal{F}_i| / \binom{n-2}{i-2}$ ($r_1 = \cdots = r_{m-1} = 0$).

Case 1: $n = 2m-2$.

By Proposition 11, we have $\sum_{1 \leqslant i \leqslant k} r_i = \sum_{m \leqslant i \leqslant k} r_i \leqslant 1$. Thus,

$$\sum_{m \leqslant i \leqslant k} |\mathcal{F}_i| = \sum_{m \leqslant i \leqslant k} r_i \binom{n-2}{i-2} \leqslant r_m \binom{n-2}{m-2} + (1-r_m)\binom{n-2}{m-1}$$
$$= \binom{n-2}{m-2}\left(1 - \frac{1-r_m}{m-1}\right).$$

On the other hand, by the LYM inequality, we have

$$1 \geqslant \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \geqslant \sum_{i=k+1}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathcal{F}| \leqslant \binom{n-2}{m-2} - \frac{1-r_m}{m-1}\binom{n-2}{m-2} + \binom{n}{\lfloor 0.501n \rfloor + 1}. \tag{16}$$

If $\mathcal{F}_m$ is 2-wise 3-intersecting, then $\mathcal{F}_m^c \subset \binom{[2m-2]}{m-2}$ is 2-wise 1-intersecting. By Proposition 9, we have $|\Delta_{m-3}(\mathcal{F}_m^c)| \geqslant |\mathcal{F}_m^c| = |\mathcal{F}_m|$. So we replace $\mathcal{F}$ by $(\mathcal{F} - \mathcal{F}_m) \cup (\Delta_{m-3}(\mathcal{F}_m^c))^c$, and we may assume that $\mathcal{F}_m = \emptyset$, i.e., $r_m = 0$. Then it follows $|\mathcal{F}| < \binom{n-2}{m-2}$ from (16) for $n > n_0$.

If $\mathcal{F}_m$ is not 2-wise 3-intersecting, then there exist F, F' with $|F \cap F'| = 2$. Then all members in $\mathcal{F}$ contain $F \cap F'$ and we are done.

Case 2: $n = 2m-3$.

By Proposition 11, we have $\sum_{m \leqslant i \leqslant k} r_i \leqslant 1$. Thus,

$$\sum_{m \leqslant i \leqslant k} |\mathcal{F}_i| = \sum_{m \leqslant i \leqslant k} r_i \binom{n-2}{i-2} \leqslant r_m \binom{n-2}{m-2} + (1-r_m)\binom{n-2}{m-1}$$
$$= \binom{n-2}{m-2}\left(1 - \frac{2(1-r_m)}{m-1}\right).$$

For $\mathcal{F}_i$, $i > k$, we use the LYM inequality. Then we have

$$|\mathcal{F}| \leqslant \binom{n-2}{m-2} - \frac{2(1-r_m)}{m-1}\binom{n-2}{m-2} + \binom{n}{\lfloor 0.501n \rfloor + 1}. \tag{17}$$

Now we look at $\mathcal{F}_m$ in detail.

Lemma 12. *If $\mathcal{F}_m$ is non-trivial, then $|\mathcal{F}_m| < 0.999\binom{n-2}{m-2}$ holds for $n > n_0$.*

Proof. Here we only assume that $\mathcal{F}_m \subset \binom{[2m-3]}{m}$ is shifted, non-trivial 3-wise 2-intersecting and we do not use the other $\mathcal{F}_i$, $i \neq m$. We follow the proof of Theorem 2. Suppose that $|\mathcal{F}_m| \geqslant 0.999\binom{n-2}{m-2}$ and define $\mathcal{G}$ as in the proof of Theorem 2. Then, using Theorem 10, we can conclude that $P_i \in \mathcal{G}$ and $P_{i+1} \notin \mathcal{G}$ for some $i \geqslant 1$. First we deal with the case $Q_i \notin \mathcal{G}$. We use the same estimation for the sizes of $\mathcal{F}_{1\bar{2}}, \mathcal{F}_{\bar{1}2}, \mathcal{F}_{\bar{1}\bar{2}}$ as in Case 1 of the proof of Theorem 2. Noting that $n = 2m - 3$ and $k = m$, we have

$$|\mathcal{F}_{1\bar{2}}|, |\mathcal{F}_{\bar{1}2}| \leqslant \binom{n-2}{k-2}\frac{n-k}{k-1}\alpha^{4i+3} = \binom{n-2}{m-2}\frac{m-3}{m-1}\alpha^{4i+3}, \tag{18}$$

$$|\mathcal{F}_{\bar{1}\bar{2}}| \leqslant \binom{n-2}{k-2}\frac{(n-k)(n-k-1)}{k(k-1)}\alpha^{4i+6} = \binom{n-2}{m-2}\frac{(m-3)(m-4)}{m(m-1)}\alpha^{4i+6}. \tag{19}$$

Let $\mathcal{A} = \{F \cap [3, m+1] : F \in \mathcal{F}_{12}\}$. Since $\mathcal{F}_m$ is shifted and non-trivial we may assume that $\{1\} \cup [3, m+1] \in \mathcal{F}$. So $\mathcal{A}$ is 2-wise 1-intersecting. Let $\mathcal{A}_i$ be the i-uniform subfamily of $\mathcal{A}$. Clearly $|\mathcal{A}_i| \leqslant \binom{m-1}{i}$ and if $2i \leqslant m-1$ then $|\mathcal{A}_i| \leqslant \binom{m-2}{i-1}$ follows from the Erdős–Ko–Rado theorem [2]. Thus we have

$$\begin{aligned}|\mathcal{F}_{12}| &\leqslant \sum_{i=1}^{m-2} |\mathcal{A}_i| \binom{n-(m+1)}{m-i-2} \\ &\leqslant \sum_{i \leqslant \lfloor \frac{m-1}{2} \rfloor} \binom{m-2}{i-1}\binom{m-4}{m-i-2} + \sum_{i > \lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{i}\binom{m-4}{m-i-2}.\end{aligned}$$

Set $f(i) = \binom{m-1}{i}\binom{m-4}{m-i-2}$ and $h = \lfloor \frac{m-1}{2} \rfloor$. Then, using $\binom{m-2}{i-1} = \frac{i}{m-1}\binom{m-1}{i} \leqslant \frac{1}{2}\binom{m-1}{i}$ for $i \leqslant h$, we have $|\mathcal{F}_{12}| \leqslant \frac{1}{2}\sum_{i \leqslant h} f(i) + \sum_{i > h} f(i)$. Note also that $\binom{n-2}{m-2} = \sum_{i=0}^{m-2} f(i) = \sum_{i \leqslant h} f(i) + \sum_{h > i} f(i)$, and $\lim_{m \to \infty} (\sum_{i \leqslant h} f(i)) / (\sum_{h > i} f(i)) = 1$. Therefore, we have

$$|\mathcal{F}_{12}| \leqslant \left(\frac{3}{4} + \varepsilon\right)\binom{n-2}{m-2} \tag{20}$$

for any $\varepsilon > 0$ if $n > n_0(\varepsilon)$. By (18)–(20) we have $|\mathcal{F}| \leqslant 0.76\binom{n-2}{m-2}$ for n sufficiently large. This contradicts our assumption $|\mathcal{F}_m| \geqslant 0.999\binom{n-2}{m-2}$.

We have one more case, that is, the case $Q_i \in \mathcal{G}$. But in this case, compared to the previous case, we can put better bounds for $\mathcal{F}_{1\bar{2}}, \mathcal{F}_{\bar{1}2}, \mathcal{F}_{\bar{1}\bar{2}}$, and the same bound for $\mathcal{F}_{12}$. This completes the proof of Lemma 12. □

If $r_m < 0.999$ then $|\mathcal{F}| < \binom{n-2}{m-2}$ follows from (17). So we may assume that $|\mathcal{F}_m| \geqslant 0.999 \binom{n-2}{m-2}$. Then Lemma 12 implies that $\mathcal{F}_m$ is trivial, i.e., all members of $\mathcal{F}_m$ contain $\{1, 2\}$.

Lemma 13. *For every j ($3 \leqslant j \leqslant n$) we can find $F, F' \in \mathcal{F}_m$ such that $F \cap F' = \{1, 2, j\}$.*

Proof. It suffices to prove the result for $j = n$. Suppose, on the contrary, that $\mathcal{C} := \{F - \{1, 2, n\} : \{1, 2, n\} \subset F \in \mathcal{F}_m\}$ is 2-wise 1-intersecting. There are $\binom{2m-6}{m-3}$ sets in $\binom{[3,n-1]}{m-3}$ and at most half of them can be in $\mathcal{C}$. This implies $|\mathcal{F}_m| \leqslant \binom{n-2}{m-2} - \frac{1}{2}\binom{2m-6}{m-3} = (1 - \frac{m-2}{2(2m-5)})\binom{n-2}{m-2}$. But this is impossible because $|\mathcal{F}_m| \geqslant 0.999\binom{n-2}{m-2}$. □

Let $i > m$ and suppose $C \in \mathcal{F}_i$. If $C \not\supset \{1, 2\}$ then, by Lemma 13, we have only two choices of C, that is, $C_1 = [n] - \{1\}$ or $C_2 = [n] - \{2\}$. Except C_1 and C_2, all the other edges in $\mathcal{F}$ contain $\{1,2\}$. Let $\mathcal{D} := \{D - \{1,2\} : \{1,2\} \subset D \in \mathcal{F}\} \subset \bigcup_{j=m}^{n} \binom{[3,n]}{j-2}$. Clearly, $\mathcal{D}$ is a Sperner family. By the Sperner theorem [17] we have $|\mathcal{D}| \leqslant |\binom{[3,n]}{m-2}|$. Equality holds only if $\mathcal{D} = \binom{[3,n]}{m-2}$ or $\mathcal{D} = \binom{[3,n]}{m-3}$, but the latter case is impossible because we have assumed $\mathcal{F}_j = \emptyset$ for $j < m$. This proves that the unique maximal configuration in Case 2 is $\mathcal{F} = \mathcal{F}_m \cup \{C_1, C_2\}$ where $\mathcal{F}_m = \{\{1,2\} \cup D : D \in \binom{[3,n]}{m-2}\}$. This completes the proof of Case 2 and so the proof of Theorem 3.

Acknowledgements

The authors thank the referees for their very careful reading.

References

[1] R. Ahlswede, L.H. Khachatrian, The complete intersection theorem for systems of finite sets, European J. Combin. 18 (1997) 125–136.
[2] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 12 (1961) 313–320.
[3] P. Frankl, On Sperner families satisfying an additional condition, J. Combin. Theory (A) 20 (1976) 1–11.
[4] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Surveys in Combinatorics 1987, LMS Lecture Note Series, vol. 123, Cambridge University Press, Cambridge, 1987, pp. 81–110.
[5] P. Frankl, N. Tokushige, The Kruskal–Katona Theorem, some of its analogues and applications, Conference on extremal problems for finite sets, Visegrád, Hungary, 1991, 92–108.
[6] P. Frankl, N. Tokushige, Weighted multiply intersecting families, Studia Sci. Math. Hungarica 40 (2003) 287–291.
[7] P. Frankl, N. Tokushige, Weighted 3-wise 2-intersecting families, J. Combin. Theory (A) 100 (2002) 94–115.
[8] P. Frankl, N. Tokushige, Weighted non-trivial multiply intersecting families, Combinatorica, to appear.
[9] H.-D.O.F. Gronau, On Sperner families in which no 3-sets have an empty intersection, Acta Cybernet. 4 (1978) 213–220.
[10] H.-D.O.F. Gronau, On Sperner families in which no k-sets have an empty intersection, J. Combin. Theory (A) 28 (1980) 54–63.
[11] H.-D.O.F. Gronau, On Sperner families in which no k-sets have an empty intersection II, J. Combin. Theory (A) 30 (1981) 298–316.
[12] H.-D.O.F. Gronau, On Sperner families in which no k-sets have an empty intersection III, Combinatorica 2 (1982) 25–36.
[13] G.O.H. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hungarica 15 (1964) 329–337.
[14] G.O.H. Katona, A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, 1966, Akademiai Kiadó, 1968, pp. 187–207, MR 45 #76.
[15] J.B. Kruskal, The number of simplices in a complex, in: Math. Opt. Techniques, University of California Press, 1963, pp. 251–278 MR 27 #4771.
[16] E.C. Milner, A combinatorial theorem on systems of sets, J. London Math. Soc. 43 (1968) 204–206.
[17] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.