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Note

The game of n-times nim

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Abstract

The following game is considered. The first player can take any number of stones, but not all the stones, from a single pile of stones. After that, each player can take at most n-times as many as the previous one. The player first unable to move loses and his opponent wins. Let $f_1, f_2, \ldots$ be an initial sequence of stones in increasing order, such that the second player has a winning strategy when play begins from a pile of size f_i. It is proved that there exist constants $c = c(n)$ and $k_0 = k_0(n)$ such that $f_{k+1} = f_k + f_{k-c}$ for all $k > k_0$, and $\lim_{n\to\infty} c(n)/(n\log n) = 1$.

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Let us consider the following game which we call *n-times nim*. The first player can take any number of stones, but not all the stones, from a single pile of stones. After that, each player can take at most n-times as many as the previous one. The player first unable to move loses and his opponent wins. Usually this game is considered for a positive integer n, but throughout this paper we only assume that $n \geqslant 1$, i.e., n can be a real number.

Let $F(n) := \{f_1, f_2, \ldots\}$ be the sequence of initial numbers of stones in increasing order such that the second player has a winning strategy when the first player begins moving from a pile of size f_i. Clearly $f_1 = 1$, since the first player has no move, so the second wins. Then, obviously,

$$f_i = i \quad \text{holds for } i \leqslant \lfloor n+1 \rfloor, \tag{1}$$

also $f_{\lfloor n+2 \rfloor} = \lfloor n+3 \rfloor$.

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Whinihan (who ascribes “Fibonacci nim” to Gaskell, see [2]) found that 2-times nim satisfies $f_{k+1}=f_k+f_{k-1}$, that is, $F(2)$ is the Fibonacci sequence. Then Schwenk [1] proved that in n-times nim there exist constants $c=c(n)$ and $k_0=k_0(n)$ such that $f_{k+1}=f_k+f_{k-c}$ for all $k>k_0$. He asked to determine the behavior of $c=c(n)$. We are going to prove the result of Schwenk in a different way, and answer his question.

Theorem 1. *Let n be a fixed positive real at least* 1.

(i) *For every* $k\geqslant 1$ *there exists an* $r=r(k)$ *such that* $f_{k+1}=f_k+f_r$ *holds.*
(ii) $r(k)$ *can be computed by* $r(k)=\min\{r\colon\ nf_r\geqslant f_k\}$.
(iii) $((n+1)/n)f_k\leqslant f_{k+1}$ *holds for all* $k\geqslant 1$.
(iv) $r(k)\leqslant r(k+1)\leqslant r(k)+1$, *i.e., the function* $r(k)$ *is continuous in the discrete sense.*
(v) *There is a constant* $c=c(n)$ *such that* $r(k)=k-c$ *holds for all* $k>k_0$.

Proof. We prove all these statements simultaneously, applying induction on k. The cases $k\leqslant n$ are trivial, with $r(k)=1$.

Suppose now that all statements are proved for $k'<k$ and consider k. Let $r(k)$ be defined via (ii). We first prove that the first player has a winning strategy for s stones as long as

$$f_k<s<f_k+f_{r(k)}. \tag{2}$$

If $n(s-f_k)<f_k$ holds then the first player can remove $s-f_k$ stones and win, as f_k is a second player win. From now on, we suppose

$$n(s-f_k)\geqslant f_k,\quad \text{i.e.,}\quad s\geqslant\frac{n+1}{n}f_k. \tag{3}$$

Let us show that

$s-f_k$ is a first player win.

Suppose the contrary, then $s-f_k=f_q$ holds for some q. Since $f_q=s-f_k<f_{r(k)}$ by (2), the definition of $r(k)$ implies $nf_q<f_k$, and $n(s-f_k)=nf_q<f_k$, contradicting (3).

Now let the first player play according to the winning strategy for $s-f_k$ stones. This will enable him a finite number of moves to reduce the number of remaining stones to exactly f_k.

For convenience, we make this strategy even more clear, by requiring him to remove all the “extra stones”, i.e., reduce the number of remaining stones to f_k only if he has no other winning moves for his “mind game” of $s-f_k$ stones. This makes sure that when he reduces the number of stones to exactly f_k, the number of stones, say t, that he is taking is a *second player win*. That is

$$t=f_l\quad\text{for some } l<r(k), \tag{4}$$

implying, by the definition of $r(k)$ that

$$nt < f_k \tag{5}$$

and thus completing the proof that this is a winning strategy for the first player.

Now, to complete the proof of (i), we must show that

$f_k + f_{r(k)}$ is a second player win.

If the first player removes $f_{r(k)}$ or more stones, then the second can remove all the remaining and win. Otherwise let the second player play the "mind game" for $f_{r(k)}$ stones, by delaying his ultimate move (as above) as long as he can. Then, the number of stones (say t) which he removes finally to reduce the number of remaining stones to f_k will satisfy (4) and thus (5) too, proving that this is a correct winning strategy. This concludes the proof of (i) and (ii). Then (iii) follows directly from (i) and (ii).

The proof of (iv). From (ii), $r(k) \leqslant r(k+1)$ is clear. Using (i) and (ii), $n f_{r(k)+1} \geqslant n (f_{r(k)} + f_{r(r(k))}) \geqslant f_k + f_{r(k)} = f_{k+1}$ follows, proving $r(k+1) \leqslant r(k) + 1$.

Finally, we prove (v). From (iv) it follows that $k - r(k)$ is a monotone increasing, integer-valued function. Therefore, it is sufficient to prove that it is bounded from above. Actually, we shall see that

$$k - r(k) < (n+1)\log n. \tag{6}$$

To show (6), suppose the contrary. Then, using (iii), we have

$$f_{r(k)} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)}}{f_{r(k)+1}} \leqslant f_k \left(\frac{n}{n+1}\right)^{k-r(k)}$$

$$\leqslant f_k \left(\left(1 - \frac{1}{n+1}\right)^{n+1}\right)^{\log n} < f_k \mathrm{e}^{-\log n} = \frac{f_k}{n},$$

contradicting the definition of $r(k)$.

Thus the proof is complete. □

Theorem 2. *Let n be a fixed positive real at least* 1.

(vi) $(n/(n-1)) f_k > f_{k+1}$ *holds for all* $k > k_0$.
(vii) $\left\lfloor \frac{\log n}{\log n - \log(n-1)} \right\rfloor \leqslant c(n) \leqslant \left\lceil \frac{\log n}{\log(n+1) - \log n} \right\rceil$.
(viii) $\lim_{n\to\infty} \frac{c(n)}{n \log n} = 1$.

Proof. By (i) and (v), we have

$$f_{k+1} = f_k + f_{k-c} \tag{7}$$

for $k > k_0$. On the other hand, (ii) implies

$$n f_{(k+1)-c} \geqslant f_{k+1} > n f_{k-c}. \tag{8}$$

Table 1

n	L	$c(n)$	U	$\lfloor n\log n\rfloor$
2	1	1	1	1
3	2	3	3	3
4	4	5	6	5
5	7	7	8	8
6	9	10	11	10
7	12	13	14	13
8	15	16	17	16
9	18	19	20	19
10	21	22	24	23
11	25	25	27	26
12	28	29	31	29
13	32	32	34	33
14	35	37	38	36

n	$c(n)$
$1 \leqslant n < 2$	0
$2 \leqslant n < \frac{5}{2}$	1
$\frac{5}{2} \leqslant n < 3$	2
$3 \leqslant n < \frac{7}{2}$	3
$\frac{7}{2} \leqslant n < \frac{43}{11}$	4
$\frac{43}{11} \leqslant n < \frac{13}{3}$	5
$\frac{13}{3} \leqslant n < \frac{14}{3}$	6
$\frac{14}{3} \leqslant n < \frac{51}{10}$	7

By (7) and (8), we have $f_{k+1} > nf_{k-c} = n(f_{k+1} - f_k)$, i.e., $nf_k > (n-1)f_{k+1}$, which proves (vi).

Set $U := \lceil \log n/(\log(n+1) - \log n)\rceil$, then $(n/(n+1))^U \leqslant 1/n$. To show $c(n) \leqslant U$, suppose the contrary. Then using (iii), we have

$$f_{r(k)} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)}}{f_{r(k)+1}} \leqslant f_k \left(\frac{n}{n+1}\right)^{k-r(k)}$$

$$= f_k \left(\frac{n}{n+1}\right)^{c(n)} < f_k \left(\frac{n}{n+1}\right)^{U} \leqslant \frac{f_k}{n},$$

contradicting (ii).

Set

$$L := \left\lfloor \frac{\log n}{\log n - \log(n-1)} \right\rfloor, \quad \text{then} \quad \left(\frac{n-1}{n}\right)^L \geqslant \frac{1}{n}.$$

To show $c(n) \geqslant L$, suppose on the contrary that $c(n)+1 \leqslant L$. Then using (vi), we have

$$f_{r(k)-1} = f_k \cdot \frac{f_{k-1}}{f_k} \cdot \frac{f_{k-2}}{f_{k-1}} \cdots \frac{f_{r(k)-1}}{f_{r(k)}} \geqslant f_k \left(\frac{n-1}{n}\right)^{k-r(k)+1}$$

$$= f_k \left(\frac{n}{n+1}\right)^{c(n)+1} \geqslant f_k \left(\frac{n-1}{n}\right)^{L} \geqslant \frac{f_k}{n},$$

contradicting (ii).

Step (viii) follows immediately from (vii). □

Table 1 provides the numerical data concerning $c(n)$.

It is worth noting that $c(n) = c(n')$ does not necessarily imply $F(n) = F(n')$. For example, $c(n) = 4$ for $\frac{7}{2} \leqslant n < \frac{43}{11}$, but there are two winning sequences for the second player, that is,

$$F(n) = \{1, 2, 3, 4, 6, 8, 11, 15, 21, 27, 35, 46, \ldots\} \quad \text{for } \frac{7}{2} \leqslant n < \frac{11}{3},$$

$$F(n) = \{1, 2, 3, 4, 6, 8, 11, 14, 18, 24, 32, 43, \ldots\} \quad \text{for } \frac{11}{3} \leqslant n < \frac{43}{11}.$$

References

[1] A.J. Schwenk, Take-away games, Fibonacci Quart. 8 (1970) 225–234,241.
[2] M.J. Whinihan, Fibonacci nim, Fibonacci Quart. 1 (1963) 9.