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Weighted 3-Wise 2-Intersecting Families

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Let n and r be positive integers. Suppose that a family $\mathscr{F} \subset 2^{[n]}$ satisfies $|F_1 \cap F_2 \cap F_3| \geqslant 2$ for all $F_1, F_2, F_3 \in \mathscr{F}$. We prove that if $w < 0.5018$, then $\sum_{F \in \mathscr{F}} w^{|F|}(1-w)^{n-|F|} \leqslant w^2$.
Key Words: intersecting family; Sperner family.

1. INTRODUCTION

Let n, r and t be positive integers. A family $\mathscr{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geqslant t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. For a real $w \in (0, 1)$, let us define the weighted size $W_w(\mathscr{F})$ of $\mathscr{F}$ by

$$W_w(\mathscr{F}) := \sum_{F \in \mathscr{F}} w^{|F|}(1-w)^{n-|F|}.$$

Note that $W_{1/2}(\mathscr{F}) = |\mathscr{F}|/2^n$. Further, define

$$f_{w,r,t}(n) := \max\{W_w(\mathscr{F}) \colon \mathscr{F} \subset 2^{[n]} \text{ is } r\text{-wise } t\text{-intersecting}\}.$$

Let us check

$$f_{w,r,t}(n) \geqslant w^t. \tag{1}$$

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Set $\mathscr{F}_0 := \{F \subset [n]\colon [t] \subset F\}$. Then $\mathscr{F}_0$ is r-wise t-intersecting for every r, and

$$W_w(\mathscr{F}_0) = w^t \sum_{F \subset [t+1,n]} w^{|F|}(1-w)^{n-t-|F|}$$
$$= w^t \sum_{i=0}^{n-t} \binom{n-t}{i} w^i (1-w)^{n-t-i} = w^t.$$

Problem 1. Does $f_{w,r,t}(n) = w^t$ hold if $w \leqslant w(r,t)$ and $t \leqslant 2^r - r - 1$?

For 1-intersecting families, the authors proved the following in [8].

Theorem 1. *$f_{w,r,1}(n) = w$ if $w \leqslant (r-1)/r$.*

On the other hand, for all $t \geqslant 1$ one has

$$\lim_{n\to\infty} f_{w,r,t}(n) = 1 \qquad \text{if } w > (r-1)/r.$$

To obtain an exact formula for $f_{w,r,2}(n)$ seems to be much harder. In this paper, we shall prove

Theorem 2. *$f_{w,3,2}(n) = w^2$ if $w < 0.5018$.*

This implies $f_{w,r,2}(n) = w^2$ if $r \geqslant 3$ and $w < 0.5018$, since $w^t \leqslant f_{w,r+1,t}(n) \leqslant f_{w,r,t}(n)$. Using Theorem 2, the following variation of the Erdős–Ko–Rado theorem is deduced.

Theorem 3. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $k/n < 0.501$. Then $|\mathscr{F}| \leqslant (1+o(1))\binom{n-2}{k-2}$.*

A family $\mathscr{F} \subset 2^{[n]}$ is called a Sperner family if $F \not\subset G$ holds for all distinct $F, G \in \mathscr{F}$. The maximum size of 2-wise t-intersecting Sperner families was determined by Milner [18], it is given by the simple formula $\binom{n}{\lceil (n+t)/2 \rceil}$. For 3-wise t-intersecting families, the situation becomes more complicated. For 3-wise 1-intersecting families, it was the subject of several papers of Frankl [3] and Gronau [10, 11, 12, 13] and it is known that for $n \geqslant 53$ the only optimal families are

$$\mathscr{F} = \begin{cases} \left\{F \cup \{n\}\colon F \in \binom{[n-1]}{n/2}\right\} \cup \{[n-1]\}, & n \text{ even}, \\ \left\{F \cup \{n\}\colon F \in \binom{[n-1]}{(n-1)/2}\right\}, & n \text{ odd}. \end{cases}$$

This motivates the following conjecture.

Conjecture 1. Let $\mathscr{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then,

$$|\mathscr{F}| \leqslant \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even,} \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd} \end{cases}$$

holds for $n \geqslant n_0$. The corresponding families are

$$\mathscr{F} = \begin{cases} \left\{ F \cup \{n-1, n\} \colon F \in \binom{[n-2]}{(n-2)/2} \right\}, & n \text{ even,} \\ \left\{ F \cup \{n-1, n\} \colon F \in \binom{[n-2]}{(n-1)/2} \right\} \cup \{[n-1]\} \cup \{[n] - \{n-1\}\}, & \\ & n \text{ odd.} \end{cases}$$

Since $\mathscr{F} = \binom{[8]}{6}$ is 3-wise 2-intersecting Sperner and $|\mathscr{F}| = \binom{8}{6} > \binom{6}{3}$, we need the condition $n > n_0$ in the above conjecture.

As an application of Theorem 3, we prove the following weaker result, conjectured in [3].

THEOREM 4. *Let* $\mathscr{F} \subset 2^{[n]}$ *be a* 3-*wise* 2-*intersecting Sperner family. Then,* $|\mathscr{F}| \leqslant (1 + o(1)) \binom{n-2}{\lceil (n-2)/2 \rceil}$.

Using the same technique, we can remove the above $o(1)$ term for 4-wise case as follows:

THEOREM 5. *Let* $\mathscr{F} \subset 2^{[n]}$ *be a* 4-*wise* 2-*intersecting Sperner family. Then,* $|\mathscr{F}| \leqslant \binom{n-2}{\lceil (n-2)/2 \rceil}$ holds for $n > n_0$.

Note that the same upper bound is valid for r-wise 2-intersecting Sperner families if $r \geqslant 4$.

2. TOOLS

2.1. Shifting

For integers $1 \leqslant i < j \leqslant n$ and a family $\mathscr{F} \subset 2^{[n]}$, define the (i,j)-shift S_{ij} as follows:

$$S_{ij}(\mathscr{F}) := \{S_{ij}(F) \colon F \in \mathscr{F}\},$$

where

$$S_{ij}(F) := \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ (F - \{j\}) \cup \{i\} \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathscr{F} \subset 2^{[n]}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $1 \leqslant i < j \leqslant n$. We call $\mathscr{F}$ co-complex if $G \supset F \in \mathscr{F}$ implies $G \in \mathscr{F}$. It is not difficult to check that $f_{w,r,t}(n)$ (the maximal weighted size of r-wise t-intersecting families) is attained by a shifted co-complex. See [6] for details.

Let us introduce a partial order in $2^{[n]}$ by using shifting. Let $A, B \subset [n]$. Define $A \succ B$ if there exists $A' \subset [n]$ such that $A \subset A'$ and B is obtained by repeating a shifting to A'. The following fact is trivial but useful.

Fact 1. *Let $\mathscr{F} \subset 2^{[n]}$ be a shifted co-complex. If $A \in \mathscr{F}$ and $A \succ B$, then $B \in \mathscr{F}$.*

Let us see how to apply the above fact.

Fact 2. *Let $\mathscr{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting shifted co-complex. Set $G_0 := \{1, 3, 4, 6, 7, \ldots, 3i, 3i+1, \ldots\} \cap [n]$. Then, $G_0 \notin \mathscr{F}$.*

Proof. Let us define G_1 and G_2 from G_0 by applying a shifting, i.e., $G_1 := \{1, 2, 4, 5, 7, \ldots, 3i-1, 3i+1, \ldots\} \cap [n]$, and $G_2 := \{1, 2, 3, 5, 6, \ldots, 3i-1, 3i, \ldots\} \cap [n]$. More visually,

$$G_0 = \{1 \cdot 34 \cdot 67 \cdot 9 \ldots\},$$
$$G_1 = \{12 \cdot 45 \cdot 78 \cdot \ldots\},$$
$$G_2 = \{123 \cdot 56 \cdot 89 \cdot \ldots\}.$$

Then, $G_0 \cap G_1 \cap G_2 = \{1\}$ and $G_0 \succ G_1 \succ G_2$. If $G_0 \in \mathscr{F}$, then $G_1, G_2 \in \mathscr{F}$ must hold by Fact 1. But this is impossible because $\mathscr{F}$ is 3-wise 2-intersecting. ∎

In the same reason, an r-wise t-intersecting shifted co-complex cannot contain the set $[n] - \{t, t+r, t+2r, \ldots\}$.

2.2. Random Walk

Let $w \in (0, 2/3)$ be a fixed real number, and let $\alpha \in (0,1)$ be the root of the equation $(1-w)x^3 - x + w = 0$, more explicitly, $\alpha = \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$. Consider the infinite random walk, starting from the origin, in which at each step we move one unit up with probability w or move one unit right with probability $1-w$. Then, the probability that we ever hit the line $y = 2x + s$ is given by α^s (see [4] or [6] for details).

Let $F \in \mathscr{F} \subset 2^{[n]}$. We define the corresponding (finite) walk to F, denoted by walk(F), in the following way. If $i \in F$ (resp. $i \notin F$), then we move one unit up (resp. one unit right) at the ith step. Note that $F \succ G$ means walk(G) is in the upper left area than walk(F). (Draw walks corresponding to G_0, G_1, G_2 in Fact 2 then one may see the situation visually. This visualization will be helpful to understand the computations in the proof of Theorem 2.)

The following example shows how to use the random walk to bound the weighted size of families.

FACT 3. *Let $\mathscr{F} \subset 2^{[n]}$ be 3-wise 2-intersecting shifted co-complex. Then, $W_w(\mathscr{F}) \leqslant \alpha^2$.*

Proof. Set $G_0 := \{1, 3, 4, 6, 7, \ldots, 3i, 3i+1, \ldots\} \cap [n]$. Note that walk$(G_0)$ is the maximal walk which does not touch the line ℓ: $y = 2x + 2$. We know that $G_0 \notin \mathscr{F}$ by Fact 2. Thus, if $G \succ G_0$ then $G \notin \mathscr{F}$ by Fact 1. In other words, for every $F \in \mathscr{F}$, walk(F) must touch the line ℓ. Therefore,

$$W_w(\mathscr{F}) \leqslant \text{Prob(a random walk of } n\text{-steps touches the line } \ell) \leqslant \alpha^2. \quad \blacksquare$$

For an r-wise t-intersecting family, we consider the equation $(1-w)x^r - x + w = 0$, its root $\alpha_r \in (0,1)$, and the line $y = (r-1)x + t$. Then the weight of the family is at most α_r^t.

2.3. Shadow

For a family $\mathscr{F} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓth shadow of $\mathscr{F}$, denoted by $\Delta_\ell(\mathscr{F})$, as follows:

$$\Delta_\ell(\mathscr{F}) := \left\{ G \in \binom{[n]}{\ell} : G \subset \exists F \in \mathscr{F} \right\}.$$

Suppose that $\mathscr{F} \subset \binom{[n]}{k}$ and $|\mathscr{F}| = \binom{m}{k} + \binom{x}{k-1}$ where $m \in \mathbf{N}$, $x \in \mathbf{R}$, $x \leqslant m-1$. Then, by the Kruskal–Katona theorem [15, 16] and its Lovász version [17], it follows that

$$|\Delta_\ell(\mathscr{F})| \geqslant \binom{m}{\ell} + \binom{x}{\ell-1}.$$

We shall use the above inequality to prove Theorem 3.

Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 2-wise t-intersecting family. Katona [14] found the following bound for the ℓth shadow ($t \leqslant \ell < k$):

$$\frac{|\Delta_\ell(\mathscr{F})|}{|\mathscr{F}|} \geqslant \frac{\binom{2k-t}{\ell}}{\binom{2k-t}{k}}.$$

We need the above inequality to prove Theorem 4. See [6] or [7] for the detail of inequalities concerning the size of shadows.

3. PROOF OF THEOREM 2

Let $\mathscr{F} \subset 2^{[n]}$ be 3-wise 2-intersecting. Further, we assume that $\mathscr{F}$ is shifted co-complex. Fix a constant w, $0 < w < 0.5018$. In this section, we write $W(\mathscr{F})$ instead of $W_w(\mathscr{F})$. Set $\alpha := \frac{1}{2}(\sqrt{\frac{1+3w}{1-w}} - 1)$, $v := 1 - w$.

Let us define the following:

$$\begin{aligned} *(i) &:= \{i, i+1, i+3, i+4, i+6, i+7, \ldots\} \cap [n] \\ &= [n] - \left([i-1] \cup \left\{ i + 3j + 2 : 0 \leqslant j \leqslant \left\lfloor \frac{n-i-2}{3} \right\rfloor \right\}\right), \end{aligned}$$

$$P_i := \{1,2\} \cup *(i+4),$$

$$Q_i := \{1,2,i+4\} \cup *(i+6),$$

$$\mathscr{F}_{12} := \{F \in \mathscr{F} \colon \{1,2\} \subset F\},$$

$$\mathscr{F}_{1\bar{2}} := \{F \in \mathscr{F} \colon 1 \in F, 2 \notin F\},$$

$$\mathscr{F}_{\bar{1}2} := \{F \in \mathscr{F} \colon 1 \notin F, 2 \in F\},$$

$$\mathscr{F}_{\bar{1}\bar{2}} := \{F \in \mathscr{F} \colon 1 \notin F, 2 \notin F\}.$$

By definition, it follows that $P_{i+1} \succ Q_i \succ P_i$, $|\mathscr{F}| = |\mathscr{F}_{12}| + |\mathscr{F}_{1\bar{2}}| + |\mathscr{F}_{\bar{1}2}| + |\mathscr{F}_{\bar{1}\bar{2}}|$. If $\{1,2\} = P_{n-3} \in \mathscr{F}$, then $\mathscr{F} = \{F \subset [n]\colon \{1,2\} \subset F\}$ and $W(\mathscr{F}) = w^2$. From now on, we assume $P_{n-3} \notin \mathscr{F}$ and we shall prove $W(\mathscr{F}) < w^2$.

Case 1: $P_0 \notin \mathscr{F}$. Note that walk(P_0) is the maximal walk which does not touch the line $\ell\colon y = 2x + 3$, because

$$P_0 = *(1) = \{12 \cdot 45 \cdot 78 \cdot \ldots\}.$$

Since $P_0 \notin \mathscr{F}$, walk(F) must meet the line ℓ if $F \in \mathscr{F}$.

If $F \in \mathscr{F}_{12}$, walk(F) starts with "up, up," and then from $(0,2)$ the walk must meet the line ℓ. Thus,

$$\begin{aligned} W_w(\mathscr{F}_{12}) &\leqslant w^2 \, \text{Prob(a random walk of } n-2 \text{ steps starting from} \\ &\qquad\qquad \text{the origin, which touches the line } y = 2x+1) \\ &\leqslant w^2\alpha. \end{aligned}$$

Since $\{1,3\} \cup *(4) \succ *(1)$, we have $\{1,3\} \cup *(4) \notin \mathscr{F}$. The corresponding walk to this set starts with "up, right," then from $(1,1)$ this is the maximal walk which does not touch the line ℓ. So if $F \in \mathscr{F}_{1\bar{2}}$, walk$(F)$ starts with "up, right," then from $(1,1)$ the walk must meet ℓ. Thus, $W(\mathscr{F}_{1\bar{2}}) \leqslant wv\alpha^4$.

In the same way, since $\{2,3\} \cup *(4) \succ *(1)$, we have $\{2,3\} \cup *(4) \notin \mathscr{F}$ and $W(\mathscr{F}_{\bar{1}2}) \leqslant vw\alpha^4$. For the last case, we have $\{3,4,5,6\} \cup *(7) \notin \mathscr{F}$ and $W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^7$.

Therefore,

$$\begin{aligned} W(\mathscr{F}) &= W(\mathscr{F}_{12}) + W(\mathscr{F}_{1\bar{2}}) + W(\mathscr{F}_{\bar{1}2}) + W(\mathscr{F}_{\bar{1}\bar{2}}) \\ &\leqslant w^2\alpha + 2wv\alpha^4 + v^2\alpha^7 < w^2. \end{aligned}$$

Case 2: $P_i \in \mathscr{F}$, $P_{i+1} \notin \mathscr{F}$, $i \geqslant 1$.

Case 2.1: $Q_i \notin \mathscr{F}$. Observe that walk(Q_i) starts with "up, up," and $i+1$ "right," then from $(i+1, 2)$ this walk is the maximal walk which does not touch the line $\ell\colon y = 2(x - (i+1)) + 4$.

Let $F \in \mathscr{F}_{12}$, then walk(F) starts with "up, up." If walk(F) passes the point $(i+1, 2)$, then this walk must meet the line ℓ after passing $(i+1, 2)$. This happens with probability at most $w^2 v^{i+1} \alpha^2$. Otherwise walk(F) must go through one of $(0, i+3), (1, i+2), \ldots, (i, 3)$. This happens with probability $w^2(1 - v^{i+1})$. Thus, we have

$$\begin{aligned} W(\mathscr{F}_{12}) &\leqslant w^2(v^{i+1}\alpha^2 + (1 - v^{i+1})) \\ &= w^2(1 - v^{i+1}(1 - \alpha^2)). \end{aligned}$$

Set

$$F := [1, i+3] \cup \{i+6, i+9, i+12, \ldots, 4i, 4i+3\} \cup *(4i+5),$$
$$G := \{1\} \cup [3, 4i+4] \cup *(4i+6).$$

Since $P_i \in \mathscr{F}$ and $P_i = \{1,2\} \cup *(i+4) \succ F$, we have $F \in \mathscr{F}$. Note that $P_i \cap F \cap G = \{1\}$. Thus, $G \notin \mathscr{F}$ follows from the assumption that $\mathscr{F}$ is 3-wise 2-intersecting. Therefore,

$$W(\mathscr{F}_{1\bar{2}}) \leqslant wv\alpha^{4i+3}.$$

In the same way, we have $W(\mathscr{F}_{\bar{1}2}) \leqslant wv\alpha^{4i+3}$. Next, set $H := [3, 4i+7] \cup *(4i+9)$. Since $P_i \cap F \cap H = \{4i+5\}$, we have $H \notin \mathscr{F}$, which implies

$$W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^{4i+6}.$$

Therefore,

$$W(\mathscr{F}) \leqslant w^2(1 - v^{i+1}(1-\alpha^2)) + 2wv\alpha^{4i+3} + v^2\alpha^{4i+6} < w^2. \tag{2}$$

(This is equivalent to $\left(\frac{\alpha^4}{v}\right)^i < \frac{v(1-\alpha^2)}{\alpha^3(2(v/w)+(v/w)^2\alpha^3)}$.)

Case 2.2: $Q_i \in \mathscr{F}$. Since $P_{i+1} \notin \mathscr{F}$, we have

$$W(\mathscr{F}_{12}) \leqslant w^2(v^{i+1}\alpha + (1 - v^{i+1})) = w^2(1 - v^{i+1}(1-\alpha)).$$

Set

$$F := [1, i+3] \cup \{i+5, i+8, i+11, \ldots, 4i+5\} \cup *(4i+7),$$
$$G := \{1\} \cup [3, 4i+6] \cup *(4i+8).$$

Since $Q_i \in \mathscr{F}$ and $Q_i \succ F$, we have $F \in \mathscr{F}$. Note that $Q_i \cap F \cap G = \{1\}$. Thus, $G \notin \mathscr{F}$ follows from the assumption that $\mathscr{F}$ is 3-wise 2-intersecting. Therefore,

$$W(\mathscr{F}_{1\bar{2}}) \leqslant wv\alpha^{4i+5}.$$

In the same way, we have $W(\mathscr{F}_{\bar{1}2}) \leqslant wv\alpha^{4i+5}$. Set $H := [3, 4i+9] \cup *(4i+11)$. Since $Q_i \cap F \cap H = \{4i+7\}$, we have $H \notin \mathscr{F}$, which implies

$$W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^{4i+8}.$$

Therefore,

$$W(\mathscr{F}) \leqslant w^2(1 - v^{i+1}(1-\alpha)) + 2wv\alpha^{4i+5} + v^2\alpha^{4i+8} < w^2. \tag{3}$$

(This is equivalent to $(\frac{\alpha^4}{v})^i < \frac{v(1-\alpha)}{\alpha^5(2(v/w)+(v/w)^2\alpha^3)}$.)

Now we may assume that $P_0 \in \mathscr{F}$ and $P_1 \notin \mathscr{F}$.

Case 3: $P_1 \notin \mathscr{F}$, $Q_0 \in \mathscr{F}$. Set $F := \{1,2,3,5\} \cup *(7)$, $G := \{1,3,4,5,6\} \cup *(8)$. Since $Q_0 \in \mathscr{F}$ and $Q_0 \succ F$, we have $F \in \mathscr{F}$. Note that $Q_0 \cap F \cap G = \{1\}$. Thus, $G \notin \mathscr{F}$, and

$$W(\mathscr{F}_{1\bar{2}}) \leqslant wv\alpha^5.$$

In the same way, we have $W(\mathscr{F}_{\bar{1}2}) \leqslant wv\alpha^5$. Next set $H := \{3,4,5,6,7\} \cup *(8)$. Since $Q_0 \cap F \cap H = \{7\}$, we have $H \notin \mathscr{F}$, which implies

$$W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^8.$$

Case 3.1: $S_1 := \{1,2,5\} \cup *(6) \notin \mathscr{F}$. Since $S_1 \notin \mathscr{F}$, we have

$$W(\mathscr{F}_{12}) \leqslant w^2(w^2 + 2wv + v^2\alpha^4) =: W_{31}.$$

Therefore, we have

$$W(\mathscr{F}) \leqslant W_{31} + 2wv\alpha^5 + v^2\alpha^8 < w^2.$$

Case 3.2: $S_2 := \{1,2,5,6,8,9\} \cup *(10) \notin \mathscr{F}$. Since $S_2 \notin \mathscr{F}$, we have

$$\begin{aligned} W(\mathscr{F}_{12}) &\leqslant w^2(w^5 + 5w^4v + 10w^3v^2 + w^2v^3(7+3\alpha^5) \\ &\quad + wv^4(2+3\alpha^8) + v^5\alpha^{11}) := W_{32}. \end{aligned}$$

Therefore, we have

$$W(\mathscr{F}) \leqslant W_{32} + 2wv\alpha^5 + v^2\alpha^8 < w^2.$$

This is the hardest case and the above inequality fails if $w \geqslant 0.5019$.

Case 3.3: $S_1, S_2 \in \mathscr{F}$. Set $F := \{1,2,3,4,8,9\} \cup *(10)$, $G := \{1,3,4,5,6,7,8\} \cup *(11)$. Since $S_2 \in \mathscr{F}$ and $S_2 \succ F$, we have $F \in \mathscr{F}$. Note that $S_1 \cap F \cap G = \{1\}$ and $G \notin \mathscr{F}$.

A walk corresponding to an edge in $\mathscr{F}_{1\bar{2}}$ passes one of $(1,8)$, $(2,7), \ldots, (8,1)$. If the walk passes $(i, 9-i)$ $(2 \leqslant i \leqslant 8)$, then it must meet the line $y = 2x+4$ after passing $(i, 9-i)$ because $G \notin \mathscr{F}$. Thus, we have

$$\begin{aligned} W(\mathscr{F}_{1\bar{2}}) &\leqslant wv\left(w^7 + \sum_{i=1}^{7} v^i w^{7-i}\alpha^{3i-2}\right) \\ &= wv\left(w^7 + \frac{v\alpha(w^7 - v^7\alpha^{21})}{w - v\alpha^3}\right) =: W_{33}. \end{aligned}$$

In the same way, we have $W(\mathscr{F}_{\bar{1}2}) \leqslant W_{33}$. Next set $H := \{3,4,5,6,7,8,9\} \cup$

$*(11)$. Since $S_1 \cap F \cap H = \{9\}$, we have $H \notin \mathscr{F}$, which implies

$$W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^8.$$

Finally, $P_1 \notin \mathscr{F}$ implies

$$W(\mathscr{F}_{12}) \leqslant w^2(w^2 + 2wv + v^2\alpha^3).$$

Therefore, we have

$$W(\mathscr{F}) \leqslant w^2(w^2 + 2wv + v^2\alpha^3) + 2W_{33} + v^2\alpha^8 < w^2.$$

Now we may assume $P_0 \in \mathscr{F}$ and $Q_0 \notin \mathscr{F}$.

Case 4: $P_0 \in \mathscr{F}$, $Q_0 \notin \mathscr{F}$, $R := \{1,2,3\} \cup *(6) \in \mathscr{F}$. First, $Q_0 \notin \mathscr{F}$ implies

$$W(\mathscr{F}_{12}) \leqslant w^2(w + v\alpha^2).$$

Set $G := \{1,3,4\} \cup *(5)$, $H := \{3,4,5,6,7\} \cup *(8)$. Since $P_0 \cap R \cap G = \{1\}$, we have $G \notin \mathscr{F}$, which implies

$$W(\mathscr{F}_{1\bar{2}}) \leqslant wv\alpha^5.$$

In the same way, we have $W(\mathscr{F}_{\bar{1}2}) \leqslant wv\alpha^5$. Since $P_0 \cap R \cap H = \{7\}$, we have $H \notin \mathscr{F}$ and

$$W(\mathscr{F}_{\bar{1}\bar{2}}) \leqslant v^2\alpha^8.$$

Therefore, we have

$$W(\mathscr{F}) \leqslant w^2(w + v\alpha^2) + 2wv\alpha^5 + v^2\alpha^8 < w^2.$$

At this point, let us summarize what we have proved.

PROPOSITION 1. *Theorem 2 is true if* $P_0 \notin \mathscr{F}$ *or* $Q_0 \in \mathscr{F}$ *or* $\{1,2,3\} \cup *(6) \in \mathscr{F}$.

In order to prove the remaining cases, we need some preparations. For a subset $S \subset [5]$, let us define

$$\mathscr{F}(S) := \{F - S\colon F \in \mathscr{F}, F \cap [5] = S\} \subset 2^{[6,n]},$$
$$f(S) := W(\mathscr{F}(S)).$$

If $S \succ S'$, the shiftedness of $\mathscr{F}$ implies $\mathscr{F}(S) \subset \mathscr{F}(S')$ (and $f(S) \leqslant f(S')$). For simplicity, we write $\mathscr{F}(123), f(123)$ instead of $\mathscr{F}(\{1,2,3\}), f(\{1,2,3\})$.

LEMMA 1. *Let* $S \subset [5]$, $|S| = 3$, *and* $F := \{1,3\} \cup *(4)$. *If* $F \in \mathscr{F}$, *then* $f(S) \leqslant \alpha^3$.

Proof. Set $G := \{1,2,4\} \cup *(5)$, $H := \{1,2,3\} \cup *(6)$. Since $F \in \mathscr{F}$ and $F \succ G$, we have $G \in \mathscr{F}$. Note that $F \cap G \cap H = \{1\}$. Thus, we have $H \notin \mathscr{F}$ and $f(123) \leqslant \alpha^3$. If $S \subset [5]$ and $|S| = 3$, then $S \succ \{1,2,3\}$. Thus, $f(S) \leqslant f(123) \leqslant \alpha^3$. ∎

LEMMA 2. *Let* $S \subset [5]$, $|S| \leqslant 3$ *and* $F := \{1,3\} \cup *(4)$. *If* $F \in \mathscr{F}$, *then* $f(S) \leqslant \alpha^{3(4-|S|)}$.

Proof. Similar as proof of Lemma 1. Use the fact that $F \in \mathscr{F}$ implies $\{1,2,6,7,8\} \cup *(9) \notin \mathscr{F}$, $\{1\} \cup [6,11] \cup *(12) \notin \mathscr{F}$, $[6,14] \cup *(15) \notin \mathscr{F}$. ∎

LEMMA 3. *Let* $S \subset [5]$ *and* $|S| = 3$. *If* $[2] \not\subset S$, *then* $\mathscr{F}(S)$ *is* 3-*wise* 3-*intersecting* (*on* $[6,n]$).

Proof. By the shiftedness of $\mathscr{F}$, it is sufficient to consider the case $S = \{1,3,4\}$. Suppose, on the contrary, that $\mathscr{F}(S)$ is not 3-wise 3-intersecting. Then, there exist $T_1, T_2, T_3 \in \mathscr{F}(S)$ such that $T_1 \cap T_2 \cap T_3 = \{x,y\}$. Set

$$F_1 := \{1,3,4\} \cup T_1,$$
$$F_2 := \{1,2,4,5\} \cup (T_2 - \{x\}),$$
$$F_3 := \{1,2,3,5\} \cup (T_3 - \{y\}).$$

Since $S \cup T_2 \succ F_2$ and $S \cup T_3 \succ F_3$, we have $F_1, F_2, F_3 \in \mathscr{F}$, but $F_1 \cap F_2 \cap F_3 = \{1\}$. This contradicts our assumption that $\mathscr{F}$ is 3-wise 2-intersecting. ∎

Using the same approach, we can extend the above lemma as follows.

LEMMA 4. *If* $[2] \not\subset S \subset [5]$ *and* $|S| \leqslant 3$, *then* $\mathscr{F}(S)$ *is* 3-*wise* $3(4-|S|)$-*intersecting* (*on* $[6,n]$).

Now, let us leave the proof of Theorem 2 aside for a while, and concentrate on the following stronger proposition.

PROPOSITION 2. *Let* $\mathscr{G} \subset 2^{[n]}$, $t \geqslant 2$, *and* $w < 0.5018$. *If* $\mathscr{G}$ *is* 3-*wise* t-*intersecting*, *then* $W(\mathscr{G}) \leqslant w^2\alpha^{t-2}$.

Note that the case $t = 2$ in Proposition 2 is exactly Theorem 2. We prove Proposition 2 by double induction on n and t.

First, let us check the cases $t \leqslant n \leqslant t+2$. Set $\mathcal{G}_0 := \{G \subset [n] : [t] \subset G\}$. It is easy to verify that if $t \leqslant n \leqslant t+2$, then

$$W(\mathcal{G}) \leqslant W(\mathcal{G}_0) = w^t \leqslant w^2 \alpha^{t-2}.$$

Another initial step of the induction is the case $t = 2$, i.e., Theorem 2. But we postpone this essential case, and check, in advance, that Theorem 2 actually implies the induction step.

Assume that Proposition 2 is true for $t = 2$. Let $\mathcal{G} \subset 2^{[n]}$ be 3-wise t-intersecting and $t \geqslant 3$. (We also assume that $\mathcal{G}$ is shifted co-complex.) Define $\mathcal{G}_1, \mathcal{G}_{\bar{1}} \subset 2^{[2,n]}$ as follows:

$$\mathcal{G}_1 := \{G - \{1\} \colon 1 \in G \in \mathcal{G}\},$$
$$\mathcal{G}_{\bar{1}} := \{G \in \mathcal{G} \colon 1 \notin \mathcal{G}\}.$$

Note that $\mathcal{G}_1$ is 3-wise $(t-1)$-intersecting, and since $\mathcal{G}$ is shifted, $\mathcal{G}_{\bar{1}}$ is 3-wise $(t+2)$-intersecting. Using the induction hypothesis, we have

$$\begin{aligned} W(\mathcal{G}) &= wW(\mathcal{G}_1) + vW(\mathcal{G}_{\bar{1}}) \leqslant w^3\alpha^{t-3} + vw^2\alpha^t \\ &= w^2\alpha^{t-3}(v\alpha^3 + w) = w^2\alpha^{t-2}. \end{aligned}$$

(Remember that α is a root of the equation $vx^3 - x + w = 0$.) This completes the induction step for the proof of Proposition 2.

Consequently, all we have to do is to prove the case $t = 2$ (Theorem 2) by induction on n. So let us return to the proof of Theorem 2 again. But this time, we can use the induction hypothesis of Proposition 2, i.e., we assume that Proposition 2 is true for all (n', t') if $t' \leqslant n' < n$.

Lemma 5. *If* $[2] \not\subset S \subset [5]$ *and* $|S| \leqslant 3$, *then* $f(S) \leqslant w^2\alpha^{10-3|S|}$.

Proof. By Lemma 4, $\mathcal{F}(S) \subset 2^{[6,n]}$ is 3-wise $3(4-|S|)$-intersecting. Using the induction hypothesis, we have $W(\mathcal{F}(S)) \leqslant w^2\alpha^{3(4-|S|)-2} = w^2\alpha^{10-3|S|}$. ∎

Case 5: $\{1,3\} \cup *(4) \in \mathcal{F}$. Let $S \subset [5]$. Define

$$\tilde{\mathcal{F}}(S) := \{F \in \mathcal{F} \colon F \cap [5] = S\} \subset 2^{[n]}.$$

For $S = \{1,2,3\}, \{1,2,4\}, \{1,2,5\}$, we apply Lemma 1 and obtain $f(S) \leqslant \alpha^3$. For the remaining seven 3-sets S, we use Lemma 5 and obtain

$f(S) \leqslant w^2\alpha$. Thus, we have

$$\sum_{|S|=3} W(\tilde{\mathscr{F}}(S)) \leqslant (3\alpha^3 + 7w^2\alpha) \cdot w^3 v^2.$$

Similarly,

$$\sum_{|S|=2} W(\tilde{\mathscr{F}}(S)) \leqslant (\alpha^6 + 9w^2\alpha^4) \cdot w^2 v^3.$$

In this way, we have

$$\sum_{|S| \leqslant 3} W(\tilde{\mathscr{F}}(S)) \leqslant (3\alpha^3 + 7w^2\alpha) \cdot w^3 v^2 + (\alpha^6 + 9w^2\alpha^4) \cdot w^2 v^3 + 5w^2\alpha^7 \cdot wv^4 + w^2\alpha^{10} \cdot v^5 =: W_5.$$

Case 5.1: $\{2,3\} \cup *(4) \notin \mathscr{F}$. Since $f(2345) \leqslant \alpha$, we have $\sum_{|S|=4} W(\tilde{\mathscr{F}}(S)) \leqslant (4+\alpha)w^4 v$. We also use $f(12345) \leqslant 1$, i.e., $W(\tilde{\mathscr{F}}(S)) \leqslant w^5$. Therefore, we have

$$W(\mathscr{F}) = \sum_{S \subset [5]} W(\tilde{\mathscr{F}}(S)) \leqslant W_5 + (4+\alpha)w^4 v + w^5 < w^2.$$

Case 5.2: $\{2,3\} \cup *(4) \in \mathscr{F}$ and $\{1,3,4,5\} \cup *(8) \notin \mathscr{F}$. Set $\mathscr{F}_6 := \{H \in \mathscr{F}(1345)\colon 6 \in F\}$, $\mathscr{F}_{\bar{6}} := \{H \in \mathscr{F}(1345)\colon 6 \notin F\}$. Then, we have $W(\mathscr{F}_6) \leqslant w^5 v$ and $W(\mathscr{F}_{\bar{6}}) \leqslant w^4 v^2 \alpha$. Thus, we have

$$W(\tilde{\mathscr{F}}(1345)) \leqslant w^4 v^2 (w + v\alpha).$$

We can apply the same thing to $\tilde{\mathscr{F}}(2345)$, because $\{2,3,4,5\} \cup *(8) \notin \mathscr{F}$ follows from the shiftedness of $\mathscr{F}$. Thus,

$$\sum_{|S|=4} \leqslant 2w^4 v^2 (w + v\alpha) + 3w^4 v =: W_{52}.$$

(The former corresponds to 1345, 2345, and the latter corresponds to 1234, 1235, 1245.) Therefore, we have

$$W(\mathscr{F}) \leqslant W_5 + W_{52} + w^5 < w^2.$$

Case 5.3: $\{2,3\} \cup *(4) \in \mathscr{F}$ and $\{1,3,4,5\} \cup *(8) \in \mathscr{F}$. Since $\{2,3\} \cup *(4) \succ [2,7] \cup *(10)$ and

$$(\{1,3,4,5\} \cup *(8)) \cap ([2,7] \cup *(10)) \cap (\{1,2,3,6,7,8\} \cup *(9)) = \{3\},$$

we have $\{1,2,3,6,7,8\} \cup *(9) \notin \mathscr{F}$. This implies that $f(S) \leqslant \alpha^6$ if $|S| = 3$. If $|S| = 2$ and $S \neq \{1,2\}$, then we have $f(S) \leqslant w^2\alpha^4$ by Lemma 5. Thus, we have

$$W(\mathscr{F}) \leqslant w^2\alpha^{10} \cdot v^5 + 5w^2\alpha^7 \cdot wv^4 + (9w^2\alpha^4 + \alpha^6)w^2v^3 + 10\alpha^6 \cdot w^3v^2 + 5w^4v + w^5 < w^2.$$

Case 6: $\{1,3\} \cup *(4) \notin \mathscr{F}$. By using Proposition 1, we may assume that $R := \{1,2,3\} \cup *(6) \notin \mathscr{F}$. This implies that

$$f(123),\ f(124),\ f(125) \leqslant \alpha^3.$$

Since $\{1,2,6,7,8\} \cup *(9) \succ R$, we have $\{1,2,6,7,8\} \cup *(9) \notin \mathscr{F}$, and thus, $f(12) \leqslant \alpha^6$. Therefore, (using Lemma 5) we have

$$W(\mathscr{F}) \leqslant w^2\alpha^{10} \cdot v^5 + 5w^2\alpha^7 \cdot wv^4 + (9w^2\alpha^4 + \alpha^6)w^2v^3 + (7w^2\alpha + 3\alpha^3)w^3v^2 + (4+\alpha)w^4v + w^5 < w^2.$$

This completes the proof of Proposition 2 and Theorem 2 at the same time.

4. PROOF OF THEOREM 3

Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family. This family is clearly 2-wise 2-intersecting, too. Therefore, by the Erdős–Ko–Rado theorem (cf. [1, 2, 5, 19]) it follows that $|\mathscr{F}| \leqslant \binom{n-2}{k-2}$ if $n \geqslant 3(k-1)$. So we may assume that $n < 3k$.

Let $\delta > 0$ be given. We shall prove $|\mathscr{F}| < (1+\delta)\binom{n-2}{k-2}$ for sufficiently large n. Set $w := 0.5017$ and $v := 1 - w$. By Theorem 2, we must have $W_w(\mathscr{G}) \leqslant w^2$ for any 3-wise 2-intersecting family $\mathscr{G} \subset 2^{[n]}$. Choose $\varepsilon > 0$ sufficiently small so that

$$(1+\delta/2)(1-\varepsilon)^4 > 1, \tag{4}$$

$$0.501 < (1-\varepsilon)w. \tag{5}$$

Define an open interval $I := ((1-\varepsilon)wn, (1+\varepsilon)wn)$. Choose $n_0 = n_0(\delta, \varepsilon)$ sufficiently large so that

$$\sum_{i \in I} \binom{n}{i} w^i v^{n-i} > 1 - \varepsilon \qquad \text{for all } n > n_0, \tag{6}$$

$$\varepsilon > 2/((1-\varepsilon)wn) \qquad \text{for all } n > n_0. \tag{7}$$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $1/3 < k/n < 0.501$. Suppose that $|\mathcal{F}| = (1+\delta)\binom{n-2}{k-2}$. We shall derive a contradiction by constructing a 3-wise 2-intersecting family $\mathcal{G} \subset 2^{[n]}$ with $W_w(\mathcal{G}) > w^2$. Set $\mathcal{F}^c := \{[n] - F : F \in \mathcal{F}\}$. Define

$$\mathcal{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathcal{F}^c))^c \left(\subset \bigcup_{i=k}^{n} \binom{[n]}{i} \right).$$

Then,

$$W_w(\mathcal{G}) = \sum_{\ell=0}^{n-k} |\Delta_\ell(\mathcal{F}^c)| w^{n-\ell} v^\ell$$
$$= \sum_{i=k}^{n} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

Since $k < 0.501n < (1-\varepsilon)wn$ by (5) and $I \subset [k, n]$, we have

$$W_w(\mathcal{G}) \geqslant \sum_{i \in I} |\Delta_{n-i}(\mathcal{F}^c)| w^i v^{n-i}.$$

LEMMA 6. *$|\Delta_{n-i}(\mathcal{F}^c)| \geqslant (1+\frac{\delta}{2})\binom{n-2}{n-i}$ for $i \in I$.*

Proof. Let x ($x \leqslant n-3$) be a real satisfying $\binom{x}{n-k-1} = \delta\binom{n-2}{n-k}$. Then, $|\mathcal{F}^c| = |\mathcal{F}| = (1+\delta)\binom{n-2}{k-2} = \binom{n-2}{n-k} + \binom{x}{n-k-1}$. By the Kruskal–Katona theorem, we have $|\Delta_{n-i}(\mathcal{F}^c)| \geqslant \binom{n-2}{n-i} + \binom{x}{n-i-1}$. To prove $\binom{x}{n-i-1} \geqslant \frac{\delta}{2}\binom{n-2}{n-i}$, it is sufficient to show

$$\frac{\binom{x}{n-i-1}}{\binom{x}{n-k-1}} \geqslant \frac{\frac{\delta}{2}\binom{n-2}{n-i}}{\delta\binom{n-2}{n-k}},$$

or equivalently,

$$\frac{(i-2)\cdots(k-1)}{(x-n+i+1)\cdots(x-n+k+2)} \geqslant \frac{n-k}{2(n-i)}.$$

Let us check that LHS $> 1 >$ RHS. Since LHS $\geqslant (\frac{i-2}{x-n+i+1})^{i-k}$ and $x \leqslant n-3$, we have LHS > 1. On the other hand, $1 >$ RHS is equivalent to $(n+k)/2 \geqslant i$. Using $n < 3k$ and (5), we certainly have $(n+k)/2 \geqslant (n + n/3)/2 = 2n/3 \geqslant (1+\varepsilon)wn \geqslant i$. This completes the proof of the lemma. ∎

By the lemma, we have

$$W_w(\mathscr{G}) \geqslant \sum_{i\in I} \left(1+\frac{\delta}{2}\right)\binom{n-2}{n-i} w^i v^{n-i}$$
$$= \sum_{i\in I} \left(1+\frac{\delta}{2}\right)\frac{i}{n}\frac{i-1}{n-1}\binom{n}{i} w^i v^{n-i}.$$

Note that

$$\frac{i}{n}\frac{i-1}{n-1} \geqslant \left(\frac{i-1}{n}\right)^2 \geqslant \frac{((1-\varepsilon)wn-1)^2}{n^2} \geqslant (1-\varepsilon)^2 w^2 - \frac{2(1-\varepsilon)w}{n}$$
$$= (1-\varepsilon)^2 w^2\left(1-\frac{2}{(1-\varepsilon)wn}\right) > (1-\varepsilon)^3 w^2 \qquad \text{(by (7))}.$$

Therefore,

$$W_w(\mathscr{G}) \geqslant \left(1+\frac{\delta}{2}\right)(1-\varepsilon)^3 w^2 \sum_{i\in I}\binom{n}{i} w^i v^{n-i}$$
$$> \left(1+\frac{\delta}{2}\right)(1-\varepsilon)^4 w^2 \qquad \text{(by (6))}$$
$$> w^2 \qquad \text{(by (4))},$$

which is a contradiction. This completes the proof of Theorem 3.

5. PROOF OF THEOREM 4

For a family $\mathscr{F} \subset 2^{[n]}$, set $\mathscr{F}_i := \mathscr{F} \cap \binom{[n]}{i}$. First, we prove the following version of the Erdős–Ko–Rado theorem (see [3] for 3-wise 1-intersecting families).

PROPOSITION 3. *Let* $\mathscr{F} \subset 2^{[n]}$ *be a* 3-*wise* 2-*intersecting Sperner family with* $k/n < 0.501$. *Then*, $\sum_{i=1}^{k} |\mathscr{F}_i| \binom{n-2}{i-2}^{-1} \leqslant 1 + o(1)$.

Proof. Let $\delta > 0$ be given. We prove $\sum_{i=1}^{k} |\mathscr{F}_i| \binom{n-2}{i-2}^{-1} \leqslant 1+\delta$ for $n > n_0(\delta)$ by induction on the number of non-zero $|\mathscr{F}_i|$'s.

If this number is one, then the inequality follows from Theorem 3. If it is not the case, then let p be the smallest and r the second-smallest index for which $|\mathscr{F}_i| \neq 0$. Set $\mathscr{F}_p^c := \{[n]-F\colon F \in \mathscr{F}_p\}$. Then, $\mathscr{F}_p^c \subset \binom{[n]}{n-p}$ is (2-wise) $(n-2p+2)$-intersecting. By the Katona's shadow theorem for intersecting family (see Section 2.3), we have

$$\frac{|\Delta_{n-r}(\mathscr{F}_p^c)|}{|\mathscr{F}_p^c|} \geqslant \frac{\binom{2(n-p)-(n-2p+2)}{n-r}}{\binom{2(n-p)-(n-2p+2)}{n-p}} = \frac{\binom{n-2}{r-2}}{\binom{n-2}{p-2}}.$$

Set $\mathscr{G} := \{G \in \binom{[n]}{r}: G \supset \exists F \in \mathscr{F}_p\}$. Since $\mathscr{G} = (\Delta_{n-r}(\mathscr{F}_p^c))^c$, we have $|\mathscr{G}| \binom{n-2}{r-2}^{-1} \geqslant |\mathscr{F}_p| \binom{n-2}{p-2}^{-1}$. Note that $\mathscr{H} := (\mathscr{F} - \mathscr{F}_p) \cup \mathscr{G}$ is also 3-wise 2-intersecting Sperner family, and the number of non-zero $|\mathscr{H}_i|$'s is one less. Therefore, by the induction hypothesis we have

$$\sum_{i=1}^{k} \frac{|\mathscr{F}_i|}{\binom{n-2}{i-2}} \leqslant \sum_{i=1}^{k} \frac{|\mathscr{H}_i|}{\binom{n-2}{i-2}} \leqslant 1 + \delta,$$

which completes the proof of the proposition. ∎

Let us now prove Theorem 4. Let $\delta > 0$ be given. Suppose that $\mathscr{F} \subset 2^{[n]}$ is a 3-wise 2-intersecting Sperner family. We show $|\mathscr{F}| < (1+\delta)\binom{n-2}{\lceil (n-2)/2 \rceil}$ for $n > n_0(\delta)$. Set $k := \lfloor 0.501n \rfloor$. By Proposition 3, we have

$$1 + \frac{\delta}{2} > \sum_{i=1}^{k} \frac{|\mathscr{F}_i|}{\binom{n-2}{i-2}} \geqslant \sum_{i=1}^{k} \frac{|\mathscr{F}_i|}{\binom{n-2}{\lceil (n-2)/2 \rceil}}.$$

On the other hand, by the LYM inequality, we have

$$1 \geqslant \sum_{i=k+1}^{n} \frac{|\mathscr{F}_i|}{\binom{n}{i}} \geqslant \sum_{i=k+1}^{n} \frac{|\mathscr{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathscr{F}| \leqslant \left(1 + \frac{\delta}{2}\right) \binom{n-2}{\lceil (n-2)/2 \rceil} + \binom{n}{\lfloor 0.501n \rfloor + 1} < (1+\delta) \binom{n-2}{\lceil (n-2)/2 \rceil}$$

for sufficiently large n.

6. PROOF OF THEOREM 5

An r-wise t-intersecting family $\mathscr{F} \subset 2^{[n]}$ is called non-trivial if $|\bigcap_{F \in \mathscr{F}} F| < t$. Define

$$g_{w,r,t}(n) := \max\{W_w(\mathscr{F}) : \mathscr{F} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise} t\text{-intersecting}\}.$$

Proposition 4. $g_{w,4,2}(n) \leqslant 0.999w^2$ *if* $w \leqslant 0.5015$.

Proof. Let $\mathscr{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting family. In the proof of Theorem 2, we checked the inequality $W_w(\mathscr{F}) \leqslant w^2$. In exactly the same way, we can check

$$W_w(\mathscr{F}) < 0.999w^2 \qquad \text{for } w \leqslant 0.5015$$

in all cases but Case 2.

Now let $\mathscr{F} \subset 2^{[n]}$ be a non-trivial 4-wise 2-intersecting family. We follow the proof of Theorem 2 and all we have to deal with is only Case 2. Suppose that there exist F_1, F_2, F_2 such that $F_1 \cap F_2 \cap F_3 = \{1,2\}$. Then every $F \in \mathscr{F}$ must contain $\{1,2\}$, which is not possible because $\mathscr{F}$ is non-trivial. Thus, we may assume that $\{F \backslash \{1,2\} : \{1,2\} \subset F \in \mathscr{F}\}$ is 3-wise 1-intersecting. Then, by Theorem 1, we have

$$W_w(\mathscr{F}_{12}) \leqslant w^3 \qquad \text{for } w \leqslant 2/3.$$

For $\mathscr{F}_{1\bar{2}}, \mathscr{F}_{\bar{1}2}, \mathscr{F}_{\bar{1}\bar{2}}$, we use the same estimation in Case 2 of proof of Theorem 2, but this time we redefine $\alpha \in (0,1)$ as the unique root (in the interval) of the equation $(1-w)x^4 - x + w = 0$. (cf. $\alpha \approx 0.543689$ if $w = 1/2$.) Then, one can check in inequalities (2) and (3) that

$$W_w(\mathscr{F}) < 0.93w^2 \qquad \text{for } w \leqslant 2/3.$$

This completes the proof. (Note that one can construct (see [9]) a non-trivial 4-wise 2-intersecting family $\mathscr{F} \subset 2^{[n]}$ with $\lim_{n\to\infty} W_w(\mathscr{F}) = w^2$ if $w > 2/3$.) ∎

Proposition 5. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 4-wise 2-intersecting family with $k/n < 0.501$, $n > n_0$. Then, $|\mathscr{F}| \leqslant \binom{n-2}{k-2}$. Moreover, if $\mathscr{F}$ is non-trivial, then $|\mathscr{F}| < 0.9999\binom{n-2}{k-2}$.*

Proof. The proof is similar to the proof of Theorem 3, and we give a sketch here. Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 4-wise 2-intersecting family. If $\mathscr{F}$ fixes 2-element set, then $|\mathscr{F}| \leqslant \binom{n-2}{k-2}$. So we may assume that $\mathscr{F}$ is non-trivial. Suppose that $|\mathscr{F}| \geqslant 0.9999\binom{n-2}{k-2}$, and set $w := 0.501$, $v := 1 - w$. We shall derive a contradiction by constructing a non-trivial 4-wise 2-intersecting family $\mathscr{G} \subset 2^{[n]}$ with $W_w(\mathscr{G}) > 0.999w^2$.

Choose $\varepsilon > 0$ sufficiently small so that

$$0.9998(1-\varepsilon)^4 > 0.999, \tag{8}$$

$$0.501 < (1-\varepsilon)w. \tag{9}$$

Define an open interval $I := ((1-\varepsilon)wn, (1+\varepsilon)wn)$. Choose $n_0 = n_0(\delta, \varepsilon)$

sufficiently large so that

$$\sum_{i\in I}\binom{n}{i}w^i v^{n-i} > 1-\varepsilon \qquad \text{for all } n > n_0, \tag{10}$$

$$\varepsilon > 2/((1-\varepsilon)wn) \qquad \text{for all } n > n_0. \tag{11}$$

Set $\mathscr{F}^{\mathrm{c}} := \{[n]-F\colon F \in \mathscr{F}\}$ and define

$$\mathscr{G} := \bigcup_{\ell=0}^{n-k} (\Delta_\ell(\mathscr{F}^{\mathrm{c}}))^{\mathrm{c}} \left(\subset \bigcup_{i=k}^{n} \binom{[n]}{i} \right).$$

Then $\mathscr{G}$ is a non-trivial 4-wise 2-intersecting family, and since $k < 0.501n < (1-\varepsilon)wn$ by (9), we have

$$W_w(\mathscr{G}) \geqslant \sum_{i\in I} |\Delta_{n-i}(\mathscr{F}^{\mathrm{c}})| w^i v^{n-i}.$$

Lemma 7. *$|\Delta_{n-i}(\mathscr{F}^{\mathrm{c}})| \geqslant 0.9998\binom{n-2}{n-i}$ for $i \in I$.*

Proof. Let x $(x < n-2)$ be a real satisfying $|\mathscr{F}| \geqslant 0.9999\binom{n-2}{k-2} = \binom{x}{n-k}$. Then, by the Kruskal–Katona theorem, we have $|\Delta_{n-i}(\mathscr{F}^{\mathrm{c}})| \geqslant \binom{x}{n-i}$. To prove $\binom{x}{n-i} \geqslant 0.9998\binom{n-2}{n-i}$, it is sufficient to show

$$\frac{\binom{x}{n-i}}{\binom{x}{n-k}} \geqslant \frac{0.9998\binom{n-2}{n-i}}{0.9999\binom{n-2}{n-k}},$$

or equivalently,

$$\frac{(i-2)\cdots(k-1)}{(x-n+i)\cdots(x-n+k+1)} \geqslant \frac{0.9998}{0.9999}.$$

This is true, because LHS $\geqslant \left(\frac{i-2}{x-n+i}\right)^{i-k} > 1 >$ RHS. This completes the proof of the lemma. ∎

Therefore,

$$\begin{aligned}
W_w(\mathscr{G}) &\geqslant 0.9998 \sum_{i \in I} \binom{n-2}{n-i} w^i v^{n-i} \qquad \text{(by the lemma)} \\
&= 0.9998 \sum_{i \in I} \frac{i}{n} \cdot \frac{i-1}{n-1} \binom{n}{i} w^i v^{n-i} \\
&\geqslant 0.9998 (1-\varepsilon)^3 w^2 \sum_{i \in I} \binom{n}{i} w^i v^{n-i} \qquad \text{(by (11))} \\
&> 0.9998 (1-\varepsilon)^4 w^2 \qquad \text{(by (10))} \\
&> 0.999 w^2 \qquad \text{(by (8))},
\end{aligned}$$

which is a contradiction. This completes the proof of Proposition 5.

For a family $\mathscr{F} \subset 2^{[n]}$, set $\mathscr{F}_i := \mathscr{F} \cap \binom{[n]}{i}$. One can prove the next proposition in the same way we proved Theorem 4. (The only difference is to use Proposition 5 instead of Theorem 3.)

Proposition 6. *Let $\mathscr{F} \subset 2^{[n]}$ be a 4-wise 2-intersecting Sperner family with $k/n < 0.501$, $n > n_0$. Then $\sum_{i=1}^{k} |\mathscr{F}_i| \binom{n-2}{i-2}^{-1} \leqslant 1$. Moreover, if $\mathscr{F}$ is non-trivial, then $\sum_{i=1}^{k} |\mathscr{F}_i| \binom{n-2}{i-2}^{-1} < 0.9999$.*

Let us now prove Theorem 5. Let $\mathscr{F} \subset 2^{[n]}$ be a 4-wise 2-intersecting Sperner family. First suppose that $\mathscr{F}$ fixes 2-element set, say $\{1,2\}$. Then $\mathscr{G} := \{F \setminus \{1,2\} : F \in \mathscr{F}\} \subset 2^{[3,n]}$ is a Sperner family. Thus, we have

$$|\mathscr{F}| = |\mathscr{G}| \leqslant \binom{n-2}{\lceil (n-2)/2 \rceil}.$$

Next suppose that $\mathscr{F}$ is non-trivial. Set $k := \lfloor 0.501n \rfloor$. By Proposition 6, we have

$$0.9999 > \sum_{i=1}^{k} \frac{|\mathscr{F}_i|}{\binom{n-2}{i-2}} \geqslant \sum_{i=1}^{k} \frac{|\mathscr{F}_i|}{\binom{n-2}{\lceil (n-2)/2 \rceil}}.$$

On the other hand, by the LYM inequality, we have

$$1 \geqslant \sum_{i=k+1}^{n} \frac{|\mathscr{F}_i|}{\binom{n}{i}} \geqslant \sum_{i=k+1}^{n} \frac{|\mathscr{F}_i|}{\binom{n}{k+1}}.$$

Therefore, we have

$$|\mathscr{F}| \leqslant 0.9999 \binom{n-2}{\lceil (n-2)/2 \rceil} + \binom{n}{\lfloor 0.501n \rfloor + 1} < \binom{n-2}{\lceil (n-2)/2 \rceil}$$

for sufficiently large n. This completes the proof of Theorem 5.

As for 3-wise case, compared to Proposition 6, we have a following difficulty.

EXAMPLE 1. Let $w = \frac{1}{2} + \varepsilon$, $k = \lfloor wn \rfloor$, and set $A = [3, k+2]$. Define a non-trivial 3-wise 2-intersecting family $\mathscr{F}_n \subset \binom{[n]}{k}$ as follows:

$$\mathscr{F}_n := \left\{ \{1,2\} \cup G : |G \cap A| \geqslant \frac{k+2}{2}, G \in \binom{[3,n]}{k-2} \right\} \cup \{A\}.$$

Then one has $\lim_{n\to\infty} |\mathscr{F}_n| / \binom{n-2}{k-2} = 1$.

If we take all superset of $F \in \mathscr{F}_n$, that is, $\mathscr{G}_n := \{G \subset [n]\colon\, G \supset \exists_F \in \mathscr{F}_n\}$, then this family is clearly non-trivial 3-wise 2-intersecting. One can check that $\lim_{n\to\infty} W_w(\mathscr{G}_n) = w^2$ for fixed $w = \frac{1}{2} + \varepsilon$. Thus, Proposition 4 fails for 3-wise 2-intersecting family. However, we may still expect to refine Theorem 3 as follows:

Conjecture 2. Let $\mathscr{F} \subset \binom{[n]}{k}$ be a 3-wise 2-intersecting family with $k/n < 0.501$, $n > n_0$. Then $|\mathscr{F}| \leqslant \binom{n-2}{k-2}$.

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