Extremal k-edge-hamiltonian Hypergraphs

Péter Frankl, [a] Gyula Y. Katona [b 1]

[a] *CNRS, 75006 Paris, France*

[b] *Department of Computer and Information Sciences, Technical University of Budapest, 1521, P. O. B.: 91, Hungary*

Abstract

An r-uniform hypergraph is k-edge-hamiltonian iff it still contains a hamiltonian chain after deleting any k edges of the hypergraph. What is the minimum number of edges in such a hypergraph? We give lower and upper bounds for this question for several values of r and k.

Key words: k-edge-hamiltonian, hamiltonian cycle, hypergraph

1 Introduction

Let $\mathcal{H}$ be a r–uniform hypergraph on the vertex set $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ where $n > r$. For simplicity of notation v_{n+x} with $x \geq 0$ denotes the same vertex as v_x (unless stated otherwise). The set of the edges, r–element subsets of V, is denoted by $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, \ldots, E_m\}$. We will write simply V for $V(\mathcal{H})$ and $\mathcal{E}$ for $\mathcal{E}(\mathcal{H})$ if no confusion can arise.

In [1] the authors defined the notion of a hamiltonian-chain.

Definition 1 *A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a* hamiltonian chain *iff for each $1 \leq i \leq n$ $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} =: E_j$ is an edge of $\mathcal{H}$. An ordering $(v_1, v_2, \ldots, v_{l+1})$ of a subset of the vertex set is called an* open chain of length l *between v_1 and v_{l+1} iff for each $1 \leq i \leq l-r+2$ there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. An open chain of length $n-1$ is an* open hamiltonian chain*. A cyclic ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called a* chain of length l *iff for every $1 \leq i \leq l$*

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there exists an edge E_j of $\mathcal{H}$ such that $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} = E_j$. (Now v_{l+x} denotes the same vertex as v_x).

Definition 2 *A hypergraph is* hamiltonian *if it contains a hamiltonian-chain and it is k-*edge-hamiltonian *if by the removal of any k edges a hamiltonian hypergraph is obtained.*

The notion of the degree is also extended, it is defined below in full generality, however, only some special cases will be used.

Definition 3 *The* degree of a fixed l–tuple *of distinct vertices, $\{v_1, v_2, \ldots, v_l\}$, in a r–uniform hypergraph is the number of edges of the hypergraph containing the set $\{v_1, v_2, \ldots, v_l\}$. It is denoted by $d_r(v_1, v_2, \ldots, v_l)$. Furthermore $\delta_r^{(l)}(\mathcal{H})$ denotes the minimum of $d_r(v_1, v_2, \ldots, v_l)$ over all l-tuples of vertices in $\mathcal{H}$. The* neighbourhood *of a vertex v is defined by*

$$N_{\mathcal{H}}(v) := \{E - \{v\} \mid v \in E, E \in \mathcal{E}(\mathcal{H})\}.$$

The main aim of the present article is to investigate minimum size k-edge-hamiltonian hypergraphs. In [2,3] the authors settle this question for graphs.

Theorem 4 [2,3] *The number of edges in a minimum k-edge-hamiltonian graph on $n \geq k+3$ vertices is $\lceil n(k+2)/2 \rceil$.*

Since the degree of any vertex in a r-uniform hamiltonian chain is r, the minimum degree in a k-edge-hamiltonian hypergraph is at least $r+k$, so the number of edges is at least $\lceil n(r+k)/r \rceil$. For $r = 2$ this shows that the constructions in the above theorem are best possible. However, for $r > 2$ this lower bound is not best possible.

2 3-uniform hypergraphs

If a hypergraph contains $k+1$ edge-disjoint hamiltonian chains, then it is clearly k-edge-hamiltonian. This observation leads to the trivial upper bound on the minimum number of edges: $(k+1)n$. If $k = 1$ then the following slightly better upper bound is obtained.

Theorem 5 *There exists a 1-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{11}{6}n + o(n).$$

PROOF. We present only the construction due to space limitation. Let $\mathcal{V}(\mathcal{H}) := \{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ where $p = \lceil n/6 \rceil$ and $q = n - p$. There are two types of edges in $\mathcal{H}$. The first kind of edges form a chain on $\{v_1, \ldots, v_q\}$,

$$\mathcal{E}_1(\mathcal{H}) := \{\{v_i, v_{i+1}, v_{i+2}\} \mid 1 \leq i \leq q\}.$$

The second kind connects the rest of the vertices to this chain:

$$\mathcal{E}_2(\mathcal{H}) := \Big\{\{w_i, v_{5(i-1)+j}, v_{5(i-1)+j+1}\} \Bigm| 1 \leq i \leq p, 1 \leq j \leq 6\Big\}.$$

This means that the neighbouhood of w_i is an ordinary graph, a path of length 6 formed by vertices $v_{5(i-1)+1}, \ldots, v_{5(i-1)+7}$. The neighbourhood of w_{i+1} is also a path of length 6, which begins at $v_{5(i-1)+6}$, so $v_{5(i-1)+6}, v_{5(i-1)+7} \in N(w_i) \cap N(w_{i+1})$ (except maybe for $N(w_1)$ and $N(w_p)$ where the overlap is larger if $6 \nmid n$). Let $\mathcal{E}(\mathcal{H}) := \mathcal{E}_1(\mathcal{H}) \cup \mathcal{E}_2(\mathcal{H})$, then it is clear that $|\mathcal{E}(\mathcal{H})| = q + 6p = n + 5\lceil n/6 \rceil = 11n/6 + o(n)$. (See Fig. 1.)

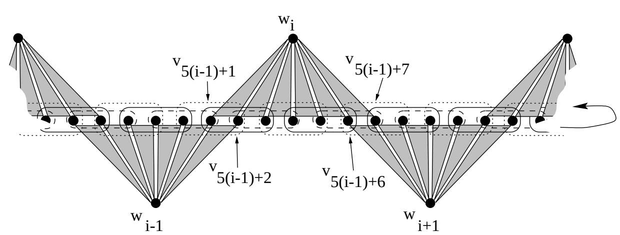

Fig. 1. 3-uniform 1-edge-hamiltonian hypergraph

It is not too difficult to show that this hypergraph is 1-edge-hamiltonian.

Theorem 6 *For any 1-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{14}{9} n$$

holds.

Theorem 7 *There exists a 2-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{13}{5} n + o(n).$$

Conjecture 8 *For any k-edge-hamiltonian 3-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{k + 2\sqrt{k} + o(k)}{3} n.$$

holds.

The above conjecture is proved for $k = 2, 3, 4, 5$.

3 1-edge-hamiltonian hypergraphs

Theorem 9 *There exists a 1-edge-hamiltonian r-uniform hypergraph $\mathcal{H}$ on n vertices with*

$$|\mathcal{E}(\mathcal{H})| = \frac{4r-1}{2r}n + o(n).$$

Theorem 10 *For any 1-edge-hamiltonian 4-uniform hypergraph $\mathcal{H}$ on n vertices*

$$|\mathcal{E}(\mathcal{H})| \geq \frac{3}{2}n$$

holds.

4 An application

Theorem 11 *If a 3-uniform hypergraph $\mathcal{H}$ on n vertices has no hamiltonian chain then*

$$|\mathcal{E}(\mathcal{H})| \leq \binom{n}{3}\left(1 - \frac{12}{11n}\right)$$

holds.

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