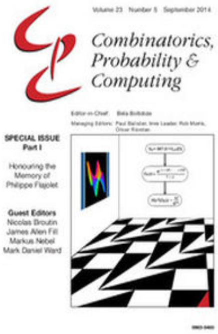

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Uniform Intersecting Families with Covering Number Restrictions

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It is known that any k-uniform family with covering number t has at most k^t t-covers. In this paper, we deal with intersecting families and give better upper bounds for the number of t-covers. Let $p_t(k)$ be the maximum number of t-covers in any k-uniform intersecting families with covering number t. We prove that, for a fixed t,

$$p_t(k) \leqslant k^t - \frac{1}{\sqrt{2}} \left\lfloor \frac{t-1}{2} \right\rfloor^{\frac{3}{2}} k^{t-1} + O(k^{t-2}).$$

In the cases of $t = 4$ and 5, we also prove that the coefficient of k^{t-1} in $p_t(k)$ is exactly $\binom{t}{2}$.

1. Introduction

Let X be a finite set. The family of all k-element subsets of X is denoted by $\binom{X}{k}$. A family $\mathscr{F} \subset \binom{X}{k}$ is called k-uniform. The vertex set of $\mathscr{F}$, denoted by $V(\mathscr{F})$, is defined to be $\bigcup_{F \in \mathscr{F}} F$, which is a subset of X in general. An element of $\mathscr{F}$ is called an *edge* of $\mathscr{F}$. A family $\mathscr{F} \subset \binom{X}{k}$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every $F, G \in \mathscr{F}$. A set $C \subset X$ is called a *cover* of $\mathscr{F}$ if it intersects every edge of $\mathscr{F}$, *i.e.*, $C \cap F \neq \emptyset$ holds for all $F \in \mathscr{F}$. A cover C is also called a t-cover if $|C| = t$. The *covering number* $\tau(\mathscr{F})$ of $\mathscr{F}$ is the minimum cardinality of any cover of $\mathscr{F}$.

For a family $\mathscr{F} \subset \binom{X}{k}$ and an integer $t \geqslant 1$, define

$$\mathscr{C}_t(\mathscr{F}) = \left\{ C \in \binom{X}{t} : C \cap F \neq \emptyset \text{ for all } F \in \mathscr{F} \right\}.$$

Note that $\mathscr{C}_t(\mathscr{F}) = \emptyset$ for $t < \tau(\mathscr{F})$. Define

$$p_t(k) = \max\left\{|\mathscr{C}_t(\mathscr{F})| : \mathscr{F} \subset \binom{X}{k} \text{ is intersecting and } \tau(\mathscr{F}) = t\right\},$$

where 'max' is also taken over all X. Gyárfás [6] proved that $|\mathscr{C}_t(\mathscr{F})| \leqslant k^t$. What happens if we know that $\mathscr{F}$ is intersecting? In Gyárfás's inequality, equality is attained only if $\mathscr{F}$ consists of t pairwise disjoint sets, so, in particular, for $t \geqslant 2$, only if $\mathscr{F}$ is nonintersecting. The aim of the present paper is to attain better bounds for $p_t(k)$.

It is shown in [2] (see also [3] and [5]) that the maximum size of k-uniform intersecting families with covering number t is $(p_{t-1}(k) + o(1))\binom{n}{k-t}$ as the number of vertices n tends to infinity. So, it is greatly important to determine the value $p_t(k)$. See [1], [2], [3] and [7] for results on the maximum size of k-uniform intersecting families with covering number restrictions.

It is easy to see that $p_1(k) = k$. For $t = 2$ and 3, the value $p_t(k)$ is determined in [2], [3] and [4].

Theorem A [2]. *For $k \geqslant 2$, $p_2(k) = k^2 - k + 1$.*

Theorem B [3, 4]. *For $k = 3$ and $k \geqslant 9$, $p_3(k) = k^3 - 3k^2 + 6k - 4$.*

The following conjecture appears in [4].

Conjecture 1 [4]. For a fixed t, $p_t(k) = k^t - \binom{t}{2}k^{t-1} + O(k^{t-2})$.

The coefficient of k^{t-1} in this conjecture is best possible if it is true.

Example 1. Let T be any tournament with vertex set $\{1, 2, \ldots, t\}$, and let α_i be the outdegree of the vertex i of T. Choosing t sets of vertices $X_1, X_2, \ldots, X_t$ such that $|X_i| = k - \alpha_i$ for $(1 \leqslant i \leqslant t)$, we define a family $\mathscr{F}_i$ for each i $(1 \leqslant i \leqslant t)$ as follows:

$$\mathscr{F}_i = \{X_i \cup A \ : \ |A| = \alpha_i,\ |A \cap X_j| = 1 \text{ if and only if } i \text{ dominates } j\}.$$

Then, $\mathscr{F} = \bigcup_{i=1}^t \mathscr{F}_i$ is a k-uniform intersecting family and $\tau(\mathscr{F}) = t$ if $k \geqslant t$. Now, we can get a t-cover of $\mathscr{F}$ by choosing any one vertex from each X_i $(1 \leqslant i \leqslant t)$. Hence

$$\begin{aligned}|\mathscr{C}_t(\mathscr{F})| &\geqslant \prod_{i=1}^t |X_i| = \prod_{i=1}^t (k - \alpha_i) = k^t - \left(\sum_{i=1}^t \alpha_i\right) k^{t-1} + O(k^{t-2}) \\ &= k^t - \binom{t}{2} k^{t-1} + O(k^{t-2}).\end{aligned}$$ □

In view of this example, we make the following conjecture.

Conjecture 2. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathscr{F}) = t$. Let $X_1, X_2, \ldots, X_t$ be pairwise disjoint subsets of X, and suppose that $\mathscr{F}$ is partitioned into t classes of edges $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_t$, and that, for each i, every edge $F \in \mathscr{F}_i$ contains X_i. Then, $\sum_{i=1}^t (k - |X_i|) \geqslant \binom{t}{2}$.

Obviously, Conjecture 1 implies Conjecture 2. One of the main results in this paper is the other implication. In fact, we prove the following theorem, in which the function $b(t)$ is defined to be the minimum value of $\sum_{i=1}^{t}(k-|X_i|)$ among the families satisfying the assumption of Conjecture 2. Note that $b(t) \leqslant \binom{t}{2}$, by Example 1.

Theorem 1.1. $p_t(k) = k^t - b(t)k^{t-1} + O(k^{t-2})$.

We prove Theorem 1.1 in Section 2.

For a general t, we prove the following theorem in Section 3.

Theorem 1.2. $b(t) \geqslant \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}$.

Corollary 1.1. $p_t(k) \leqslant k^t - \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}k^{t-1} + O(k^{t-2})$.

Moreover, in Section 4, we determine the exact value for $b(4)$ and $b(5)$, showing that Conjecture 2, and hence Conjecture 1, is true for $t \leqslant 5$.

In the subsequent argument, we use the following propositions without explicit reference.

Proposition 1.1. [6] $p_t(k) \leqslant k^t$.

For a family $\mathscr{A} \subset 2^X$ and vertices $x, y \in X$, we define

$$\begin{aligned}
\mathscr{A}(x) &= \{A \in \mathscr{A} \ : \ x \in A\},\\
\mathscr{A}(\overline{x}) &= \{A \in \mathscr{A} \ : \ x \notin A\},\\
\mathscr{A}(xy) &= \{A \in \mathscr{A} \ : \ x \in A,\ y \in A\},\\
\mathscr{A}(x\overline{y}) &= \{A \in \mathscr{A} \ : \ x \in A,\ y \notin A\},\ \textit{etc.},
\end{aligned}$$

and for $Y \subset X$,

$$\begin{aligned}
\mathscr{A}(Y) &= \{A \in \mathscr{A} \ : \ Y \subset A\},\\
\mathscr{A}(\overline{Y}) &= \{A \in \mathscr{A} \ : \ Y \cap A = \emptyset\}.
\end{aligned}$$

Proposition 1.2 [4]. *Suppose that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{F}) = t$. Let $\mathscr{C} = \mathscr{C}_t(\mathscr{F})$. Then, for any subset A of X with $|A| < t$, we have $|\mathscr{C}(A)| \leqslant p_{t-|A|}(k)$.*

2. Proof of Theorem 1.1

Throughout this section, we assume that t is a fixed positive integer, k is large compared to t, and that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{F}) = t$ such that $|\mathscr{C}_t(\mathscr{F})| \geqslant k^t - \binom{t}{2}k^{t-1}$. We simply write $\mathscr{C}$ for $\mathscr{C}_t(\mathscr{F})$.

For $A \in \mathscr{F}$ and $x \in A$, define

$$\begin{aligned}
\gamma_i(x, A) &= |\{C \in \mathscr{C}(x) \ : \ |C \cap A| = i\}|,\\
c(x, A) &= \sum_{i=1}^{t} \frac{1}{i}\gamma_i(x, A).
\end{aligned}$$

We call $c(x,A)$ the *contribution* of $x \in A$ for $|\mathscr{C}|$, because it is easy to see that $|\mathscr{C}| = \sum_{x\in A} c(x,A)$. Moreover, by definition, we have $|\mathscr{C}(x)| = \sum_{i=1}^{t} \gamma_i(x,A)$.

Lemma 2.1. *For any pair of edges A and B in $\mathscr{F}$, either $|A\cap B| < t^2$ or $|A \cap B| > k - t^2$ holds.*

Proof. Define $a = |A\cap B|$. We assume that $t^2 \leqslant a \leqslant k - t^2$, and estimate the contribution of each vertex $x \in A$ for $|\mathscr{C}|$.

If $x \in A - B$, then every t-cover $C \in \mathscr{C}$ with $C \cap A = \{x\}$ must contain some vertex $y \in B - A$. So, for fixed $y \in B - A$, we have $|\mathscr{C}(xy)| \leqslant p_{t-2}(k) \leqslant k^{t-2}$. Hence,

$$\gamma_1(x,A) \leqslant |B-A|\, k^{t-2} = (k-a)k^{t-2}.$$

Thus,

$$\begin{aligned} c(x,A) &\leqslant \gamma_1(x,A) + \frac{1}{2}(|\mathscr{C}(x)| - \gamma_1(x,A)) \\ &= \frac{1}{2}(\gamma_1(x,A) + |\mathscr{C}(x)|) \\ &\leqslant \frac{1}{2}((k-a)k^{t-2} + k^{t-1}) \\ &= k^{t-1} - \frac{a}{2}k^{t-2}. \end{aligned}$$

If $x \in A\cap B$, then we have $c(x,A) \leqslant |\mathscr{C}(x)| \leqslant p_{t-1}(k) \leqslant k^{t-1}$. By summing up all contributions of $x \in A$, we get

$$\begin{aligned} |\mathscr{C}| = \sum_{x\in A} c(x,A) &\leqslant (k-a)\left(k^{t-1} - \frac{a}{2}k^{t-2}\right) + ak^{t-1} \\ &= k^t - \frac{a}{2}k^{t-1} + \frac{a^2}{2}k^{t-2}. \end{aligned}$$

Since $t^2 \leqslant a \leqslant k - t^2$, the RHS of the above inequality attains its maximum when $a = t^2$. So, $|\mathscr{C}| \leqslant k^t - \frac{t^2}{2}k^{t-1} + \frac{t^4}{2}k^{t-2}$, which contradicts the assumption that $|\mathscr{C}| \geqslant k^t - \binom{t}{2}k^{t-1}$, for k sufficiently large. □

The result of Lemma 2.1 implies that the set of edges in $\mathscr{F}$ is partitioned into the equivalence classes $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_r$, where $|A\cap B| > k - t^2$ if and only if A and B are in the same class $\mathscr{F}_i$.

Lemma 2.2. *For each i ($1 \leqslant i \leqslant r$), we have $|\bigcap_{F\in\mathscr{F}_i} F| > k - t^2$.*

Proof. Fix i and $A \in \mathscr{F}_i$. Let $X_i = \bigcap_{F\in\mathscr{F}_i} F$ and $a = |X_i|$. We assume that $a \leqslant k - t^2$. If $x \in A - X_i$, then there exists an edge $B \in \mathscr{F}_i$ such that $x \notin B$. Note that $|A \cap B| > k - t^2$ and hence $|B - A| < t^2$. By the same argument used in Lemma 2.1, we have $\gamma_1(x,A) \leqslant$

$|B - A|\, k^{t-2} < t^2 k^{t-2}$. Therefore,

$$\begin{aligned} c(x, A) &\leqslant \frac{1}{2}(\gamma_1(x, A) + |\mathscr{C}(x)|) \\ &< \frac{1}{2}(t^2 k^{t-2} + k^{t-1}). \end{aligned}$$

If $x \in X_i$, then $c(x, A) \leqslant |\mathscr{C}(x)| \leqslant k^{t-1}$. Thus,

$$\begin{aligned} |\mathscr{C}| = \sum_{x \in A} c(x, A) &< (k - a)\frac{1}{2}(t^2 k^{t-2} + k^{t-1}) + ak^{t-1} \\ &= \frac{1}{2}(k^t + t^2 k^{t-1} + a(k^{t-1} - t^2 k^{t-2})) \\ &\leqslant \frac{1}{2}(k^t + t^2 k^{t-1} + (k - t^2)(k^{t-1} - t^2 k^{t-2})) \\ &= k^t - \frac{t^2}{2}k^{t-1} + \frac{t^4}{2}k^{t-2}. \end{aligned}$$

This is a contradiction. □

By Lemma 2.2, $\tau(\mathscr{F}_i) = 1$ holds for each i $(1 \leqslant i \leqslant r)$. And so we have that $r \geqslant t$ must hold, since $\tau(\mathscr{F}) = t$.

Lemma 2.3. $r = t$.

Proof. Suppose that $r \geqslant t + 1$. Choose one edge F_i from each $\mathscr{F}_i$, $1 \leqslant i \leqslant t + 1$, and define $\mathscr{H} = \{F_1, F_2, \ldots, F_{t+1}\}$. The degree of a vertex x in $\mathscr{H}$ is the number of edges in $\mathscr{H}$ containing x. Let Y be the set of those vertices whose degree in $\mathscr{H}$ is at least two. Note that $|F_i \cap F_j| < t^2$ if $i \neq j$, and hence $|Y| < \binom{t+1}{2}t^2$. On the other hand, every t-cover of $\mathscr{F}$ must contain some vertex in Y. Thus,

$$\begin{aligned} |\mathscr{C}| &\leqslant \sum_{y \in Y} |\mathscr{C}(y)| \leqslant |Y|\, p_{t-1}(k) \\ &< \binom{t+1}{2} t^2 k^{t-1}. \end{aligned}$$

This is a contradiction. □

For each i $(1 \leqslant i \leqslant t)$, define $X_i = \bigcap_{F \in \mathscr{F}_i} F$ and $\alpha_i = k - |X_i|$. By Lemma 2.2, we have $\alpha_i < t^2$.

The vertex sets $X_1, X_2, \ldots, X_t$ are pairwise disjoint, for otherwise $\mathscr{F}$ can be covered by at most $t - 1$ vertices.

Lemma 2.4. $|\mathscr{C}| = k^t - \left(\sum_{i=1}^{t} \alpha_i\right) k^{t-1} + O(k^{t-2})$.

Proof. Define

$$\mathscr{C}' = \left\{ C \in \binom{X}{t} \;:\; |C \cap X_i| = 1 \text{ for all } i,\ 1 \leqslant i \leqslant t \right\}.$$

Obviously, $\mathscr{C}' \subset \mathscr{C} = \mathscr{C}_t(\mathscr{F})$, and

$$|\mathscr{C}'| = \prod_{i=1}^{t} |X_i| = \prod_{i=1}^{t}(k - \alpha_i) = k^t - \left(\sum_{i=1}^{t} \alpha_i\right) k^{t-1} + O(k^{t-2}).$$

Hence, in order to prove the lemma, it suffices to show that $|\mathscr{C} - \mathscr{C}'| = O(k^{t-2})$.

For each i $(1 \leqslant i \leqslant t)$, let $\mathscr{C}_i$ be the set of t-covers C of $\mathscr{F}$ such that $C \cap X_i = \emptyset$. Fix i and $A \in \mathscr{F}_i$. Since every t-cover $C \in \mathscr{C}_i$ contains some vertex in $A - X_i$, there exists a vertex $x \in A - X_i$ such that $|\mathscr{C}_i(x)| \geqslant \frac{1}{\alpha_i}|\mathscr{C}_i|$. Now, there exists an edge $B \in \mathscr{F}_i$ such that $x \notin B$. Since every cover $C \in \mathscr{C}_i(x)$ must contain some vertex in $B - X_i$, there exists a vertex $y \in B - X_i$ such that $|\mathscr{C}_i(xy)| \geqslant \frac{1}{\alpha_i}|\mathscr{C}_i(x)| \geqslant \frac{1}{\alpha_i^2}|\mathscr{C}_i|$.

On the other hand, $|\mathscr{C}_i(xy)| \leqslant |\mathscr{C}(xy)| \leqslant p_{t-2}(k) \leqslant k^{t-2}$. The last two inequalities imply $|\mathscr{C}_i| \leqslant \alpha_i^2 k^{t-2} < t^4 k^{t-2}$. Thus,

$$|\mathscr{C} - \mathscr{C}'| \leqslant \sum_{i=1}^{t} |\mathscr{C}_i| < t^5 k^{t-2} = O(k^{t-2}).$$

This completes the proof of Lemma 2.4. □

Now we can easily prove Theorem 1.1. Suppose that k is sufficiently large with respect to t. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathscr{F}) = t$ such that $|\mathscr{C}_t(\mathscr{F})| = p_t(k)$. Because we know that $b(t) \leqslant \binom{t}{2}$ (see Example 1), we have

$$|\mathscr{C}_t(\mathscr{F})| \geqslant k^t - b(t)k^{t-1} \geqslant k^t - \binom{t}{2}k^{t-1}.$$

Then, by Lemma 2.4,

$$|\mathscr{C}_t(\mathscr{F})| \leqslant k^t - b(t)k^{t-1} + O(k^{t-2}).$$

This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We assume that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{F}) = t$. Let $X_1, X_2, \ldots, X_t$ be pairwise disjoint subset of X. Suppose that $\mathscr{F}$ is partitioned into t classes of edges $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_t$, and that, for each i, every edge $F \in \mathscr{F}_i$ contains X_i,

Let $|X_i| = k - \alpha_i$ for $1 \leqslant i \leqslant t$.

Define $s = \lfloor \frac{t-1}{2} \rfloor$. Let $F_1, F_2, \ldots, F_s$ be edges of $\mathscr{F}$ such that F_i and F_j are in the different classes of $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_t$ if $i \neq j$. Define

$$\mathscr{H} = \{F_1, F_2, \ldots, F_s\}.$$

The degree of a vertex x in $\mathscr{H}$ is denoted by $\deg_{\mathscr{H}}(x)$. Let us choose $F_1, F_2, \ldots, F_s$ so that $\sum_{x \in V(\mathscr{H})}(\deg_{\mathscr{H}}(x) - 1)$ is maximal. We may assume that $F_i \in \mathscr{F}_i$ for each i $(1 \leqslant i \leqslant s)$. Let $x_1, x_2, \ldots, x_s$ be the s vertices of $\mathscr{H}$ whose degrees in $\mathscr{H}$ are as large as possible. Define $d = \min_{1 \leqslant i \leqslant s} \deg_{\mathscr{H}}(x_i)$. Now,

$$\deg_{\mathscr{H}}(x_i) \geqslant d \quad \text{for each } i \ (1 \leqslant i \leqslant s), \quad \text{and}$$
$$\deg_{\mathscr{H}}(y) \leqslant d \quad \text{for each } y \in V(\mathscr{H}) - \{x_1, x_2, \ldots, x_s\}.$$

Case 1. $d \geqslant \sqrt{s/2}$.
Since $\deg_{\mathscr{H}}(x_i) \geqslant d$ for each i $(1 \leqslant i \leqslant s)$, we have

$$\sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) \geqslant s(d-1) \geqslant \frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s.$$

On the other hand,

$$\sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) = ks - |V(\mathscr{H})| \leqslant ks - \sum_{i=1}^{s} |X_i| = \sum_{i=1}^{s} \alpha_i.$$

Hence, we have $\sum_{i=1}^{s} \alpha_i \geqslant \frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s$. Moreover, since $\mathscr{F}$ is intersecting, at most one of $\alpha_{s+1}, \ldots, \alpha_t$ is 0. Thus,

$$\begin{aligned} \sum_{i=1}^{t} \alpha_i &\geqslant \sum_{i=1}^{s} \alpha_i + (t - s - 1) \\ &\geqslant \left(\frac{1}{\sqrt{2}} s^{\frac{3}{2}} - s \right) + s = \frac{1}{\sqrt{2}} s^{\frac{3}{2}}. \end{aligned}$$

Case 2. $d < \sqrt{s/2}$.
For each i $(1 \leqslant i \leqslant s)$, choose one vertex $y_i \in X_i$. Since $\tau(\mathscr{F}) = t > 2s$, there exists an edge $G \in \mathscr{F}$ such that $G \cap \{x_1, \ldots, x_s, y_1, \ldots, y_s\} = \emptyset$. We may assume that $G \in \mathscr{F}_{s+1}$. We will find an edge $F_l \in \mathscr{H}$ such that the family $(\mathscr{H} - \{F_l\}) \cup \{G\}$ contradicts the maximality of $\sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1)$.

Let Y be the set of vertices y in $V(\mathscr{H})$ with $\deg_{\mathscr{H}}(y) \geqslant 2$, and define $a_i = |F_i \cap Y|$ for $1 \leqslant i \leqslant s$. Then

$$\sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) = \sum_{y \in Y} (\deg_{\mathscr{H}}(y) - 1) = \sum_{i=1}^{s} a_i - |Y|.$$

Obviously, $|Y| \leqslant \sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1)$ holds, and hence

$$\sum_{i=1}^{s} a_i = \sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) + |Y| \leqslant 2 \sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1).$$

If $\sum_{x \in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) \geqslant s(\sqrt{s/2} - 1)$, then, by the same argument as used in Case 1, we are done. Hence, we may assume that $\sum_{i=1}^{s} a_i < 2s(\sqrt{s/2} - 1)$. Therefore, there exists some l $(1 \leqslant l \leqslant s)$ such that $a_l < 2(\sqrt{s/2} - 1) = \sqrt{2s} - 2$.

Now define $\mathscr{H}' = (\mathscr{H} - \{F_l\}) \cup \{G\}$. Let $Z = V(\mathscr{H} - \{F_l\}) \cap G$. Recall that G contains none of the vertices $x_1, \ldots, x_s$. So the degree of every vertex of Z in $\mathscr{H}$ (and hence in $\mathscr{H} - \{F_l\}$) is at most $d < \sqrt{s/2}$, while G must intersect with $s-1$ edges of $\mathscr{H} - \{F_l\}$.

Therefore, we have $|Z| \geqslant \frac{s-1}{d} > \sqrt{2s} - \sqrt{2/s}$. Thus,

$$\begin{aligned}\sum_{x\in V(\mathscr{H}')} (\deg_{\mathscr{H}'}(x) - 1) &= \sum_{x\in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) - a_l + |Z| \\ &> \sum_{x\in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1) - (\sqrt{2s} - 2) + (\sqrt{2s} - \sqrt{2/s}) \\ &\geqslant \sum_{x\in V(\mathscr{H})} (\deg_{\mathscr{H}}(x) - 1).\end{aligned}$$

This contradicts the maximality of $\sum_{x\in V(\mathscr{H})}(\deg_{\mathscr{H}}(x) - 1)$. □

4. $p_4(k)$ and $p_5(k)$

In this section, we show that Conjecture 2, and hence Conjecture 1, is true for $t = 4$ and $t = 5$.

Theorem 4.1. $p_4(k) = k^4 - 6k^3 + O(k^2)$.

Theorem 4.2. $p_5(k) = k^5 - 10k^4 + O(k^3)$.

Proof of Theorem 4.1. We will use Theorem 1.1. Let $\mathscr{F} \subset \binom{X}{2}$ be an intersecting family with $\tau(\mathscr{F}) = 4$. Let X_1, X_2, X_3 and X_4 be pairwise disjoint subsets of X. Suppose that $\mathscr{F}$ is partitioned into four classes $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ and $\mathscr{F}_4$ such that, for each i ($1 \leqslant i \leqslant 4$), every edge $F \in \mathscr{F}_i$ contains X_i. We may assume that $|X_1| \geqslant |X_2| \geqslant |X_3| \geqslant |X_4|$. We want to show that $\sum_{i=1}^4 (k - |X_i|) \geqslant 6$.

We use the following notation. For $I \subset \{1, 2, 3, 4\}$, define $\mathscr{F}_I = \bigcup_{i\in I} \mathscr{F}_i$ and $X_I = \bigcup_{i\in I} X_i$. If $I = \{i, j, \ldots\}$, then we write $\mathscr{F}_{ij\cdots}$ and $X_{ij\cdots}$ instead of $\mathscr{F}_{\{i,j,\ldots\}}$ and $X_{\{i,j,\ldots\}}$, respectively. Note that $\tau(\mathscr{F}_I) = |I|$, for otherwise, *i.e.*, if $\tau(\mathscr{F}_I) < |I|$, then $\mathscr{F}$ can be covered by at most three vertices.

Case 1. $|X_1| = k$.
If $|X_2| \leqslant k - 2$, then $\sum_{i=1}^4 (k - |X_i|) \geqslant 6$, and we are done. So we may assume that $|X_2| = k - 1$. In this case, for any $F \in \mathscr{F}_{12}$, we have $F \subset X_{12}$, *i.e.*, $F \cap X_{34} = \emptyset$. Since $\tau(\mathscr{F}_{12}) = 2$, every edge $G \in \mathscr{F}_{34}$ contains at least two vertices of X_{12}, in order to intersect with all edges in $\mathscr{F}_{12}$. Hence we have $|X_3| \leqslant k - 2$. We may assume that $|X_3| = k - 2$. Then $V(\mathscr{F}_{123}) = X_{123}$. In particular, for every edge $F \in \mathscr{F}_{123}$, we have $F \cap X_4 = \emptyset$. Since $\tau(\mathscr{F}_{123}) = 3$, every edge $G \in \mathscr{F}_4$ must contain at least three vertices of X_{123}. Hence $|X_4| \leqslant k - 3$. Thus $\sum_{i=1}^4 (k - |X_i|) \geqslant 6$ has been proved.

Case 2. $|X_1| \leqslant k - 1$.
We may assume that $|X_1| = |X_2| = |X_3| = k - 1$ and that $|X_4| = k - 1$ or $k - 2$. Let $H \in \mathscr{F}_4$. Since $|H - X_4| \leqslant 2$ and $\tau(\mathscr{F}_{123}) = 3$, $H - X_4$ does not cover $\mathscr{F}_{123}$. This implies that there exists an edge $F \in \mathscr{F}_{123}$ such that $F \cap H \subset X_4$. We may assume that $F \in \mathscr{F}_1$. In particular, $F \subset X_{14}$. Then every edge $G \in \mathscr{F}_i$ ($i = 2, 3$) consists of X_i and some vertex in $F \subset X_{14}$. In this situation, it is easy to see that either some edges $G \in \mathscr{F}_2$ and $G' \in \mathscr{F}_3$ do not intersect, or $\tau(\mathscr{F}_{12})$ or $\tau(\mathscr{F}_{13})$ is one, a contradiction. □

Our proof of Theorem 4.2 is lengthy and tedious, so we give only a part of the proof.

Proof of Theorem 4.2. As assumed in the proof of Theorem 4.1, let $\mathscr{F} \subset \binom{X}{2}$ be an intersecting family with $\tau(\mathscr{F}) = 5$. Let X_1, X_2, X_3, X_4 and X_5 be pairwise disjoint subsets of X. Suppose that $\mathscr{F}$ is partitioned into five classes $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ and $\mathscr{F}_5$ such that, for each i ($1 \leqslant i \leqslant 5$), every edge $F \in \mathscr{F}_i$ contains X_i.

We use the same notation used in the proof of Theorem 4.1. Also, we use the following facts.

(1) For $I \subset \{1,2,3,4,5\}$, we have $\tau(\mathscr{F}_I) = |I|$.

(2) For $F \in \mathscr{F}_i$ and $G \in \mathscr{F}_j$ ($i \neq j$), if $F \cap (G - X_j) = \emptyset$, then $F \cap X_j \neq \emptyset$.

(3) Let $I \subset \{1,2,3,4,5\}$. Suppose that $V(\mathscr{F}_I) \cap X_j = \emptyset$. Then, for every $F \in \mathscr{F}_j$, $F - X_j$ covers $\mathscr{F}_I$. In particular, $|F - X_j| = k - |X_j| \geqslant |I|$.

We may assume that $|X_1| \geqslant |X_2| \geqslant |X_3| \geqslant |X_4| \geqslant |X_5|$. Now, we want to show that $\sum_{i=1}^{5}(k - |X_i|) \geqslant 10$. So, we may also assume that $|X_1| \geqslant k - 1$. We distinguish the following five cases.

Case 1. $|X_1| = k$ and $|X_2| = k - 1$.

Case 2. $|X_1| = k$ and $|X_2| \leqslant k - 2$.

Case 3. $|X_1| = |X_2| = |X_3| = k - 1$.

Case 4. $|X_1| = |X_2| = k - 1$ and $|X_3| \leqslant k - 2$.

Case 5. $|X_1| = k - 1$ and $|X_2| \leqslant k - 2$.

Here, we consider only the last case (Case 5), which is in a sense the most complicated case. The other cases are similar but easier.

Now, we may assume that $|X_1| = k - 1$ and $|X_2| = |X_3| = |X_4| = |X_5| = k - 2$.

Subcase 5.1. There exists an edge $A_1 \in \mathscr{F}_1$ such that $A_1 \cap X_{2345} \neq \emptyset$.
We may assume that $A_1 = X_1 \cup \{e_1\}$ with $e_1 \in X_5$. Let E_0 be an edge in $\mathscr{F}_5$. Note that $|(E_0 - X_5) \cup \{e_1\}| = 3$. So $(E_0 - X_5) \cup \{e_1\}$ does not cover $\mathscr{F}_{2345}$. We may assume that there exists an edge $B_1 \in \mathscr{F}_2$ such that $B_1 \cap ((E_0 - X_5) \cup \{e_1\}) = \emptyset$. This edge B_1 must intersect with A_1 and E_0. Hence B_1 must contain a vertex a_1 of X_1 and a vertex e_2 of X_5 ($e_2 \neq e_1$), *i.e.*, $B_1 = X_2 \cup \{a_1, e_2\}$.

Consider the set $\{a_1, e_1, e_2\}$, which does not cover $\mathscr{F}_{1345}$. We may assume that there exists an edge $C_1 \in \mathscr{F}_3$ such that $C_1 \cap \{a_1, e_1, e_2\} = \emptyset$. Since C_1 must intersect with A_1 and B_1, we can write $C_1 = X_3 \cup \{a_2, b_1\}$ with $a_2 \in X_1 - \{a_1\}$ and $b_1 \in X_2$.

Now, the set $\{a_2, b_1, e_1\}$ does not cover $\mathscr{F}_{1245}$. So there exists an edge $D_1 \in \mathscr{F}_4$ such that $D_1 \cap \{a_2, b_1, e_1\} = \emptyset$. Since D_1 must intersect with A_1 and C_1, D_1 must contain some vertex in X_1 and some vertex in X_3. Also, D_1 intersects with B_1, and hence D_1 must contain a_1. Let $D_1 = X_4 \cup \{a_1, c_1\}$ with $c_1 \in X_3$.

Also, the set $\{a_1, c_1, e_1\}$ does not cover $\mathscr{F}_{1235}$. So, there exists an edge $B_2 \in \mathscr{F}_2$ such that $B_2 \cap \{a_1, c_1, e_1\} = \emptyset$. Since B_2 must intersect with A_1 and D_1, we can put $B_2 = X_2 \cup \{a, d_1\}$ where $a \in X_1 - \{a_1\}$ and $d_1 \in X_4$.

Finally, the set $\{a_2, b_1, c_1, d_1\}$ does not cover $\mathscr{F}$. So, there exists an edge $E_1 \in \mathscr{F}_5$ such that $E_1 \cap \{a_2, b_1, c_1, d_1\} = \emptyset$. Since E_1 must intersect with C_1, we can put $E_1 = X_5 \cup \{c_2, x\}$ with $c_2 \in X_3 - \{c_1\}$ and $x \notin \{a_2, b_1, c_1, d_1\}$. Now, $E_1 \cap D_1 \neq \emptyset$ and $E_1 \cap B_2 \neq \emptyset$, while $(D_1 \cup B_2) \cap (X_5 \cup \{c_2\}) = \emptyset$. Hence, $x \in D_1 \cap B_2 = \{d_1\}$. This is a contradiction.

Subcase 5.2. $V(\mathscr{F}_1) \cap X_{2345} = \emptyset$.
In this case, every edge in $\mathscr{F}_{2345}$ contains some vertex of $V(\mathscr{F}_1)$. Let $A_0 \in \mathscr{F}_1$ and $B_0 \in \mathscr{F}_2$. Since $|A_0 - X_1| = 1$ and $|B_0 - X_2| = 2$, $(A_0 - X_1) \cup (B_0 - X_2)$ does not cover $\mathscr{F}_{1345}$. So, we may assume that there exists an edge $C_1 = X_3 \cup \{a_1, b_1\} \in \mathscr{F}_3$ with $a_1 \in X_1$ and $b_1 \in X_2$. Next, the set $(A_0 - X_1) \cup \{a_1, b_1\}$ does not cover $\mathscr{F}_{1245}$. So, we may assume that there exists an edge $D_1 = X_4 \cup \{a_2, c_1\} \in \mathscr{F}_4$ with $a_2 \in X_1 - \{a_1\}$ and $c_1 \in X_3$.

Now, the set $\{a_1, b_1, c_1\}$ does not cover $\mathscr{F}_{1235}$. So, we may assume that there exists an edge $E_1 = X_5 \cup \{c_2, x\} \in \mathscr{F}_5$ with $c_2 \in X_3 - \{c_1\}$ and $x \notin \{a_1, b_1, c_1\}$. But E_1 must intersect with A_0 and D_1, while $(A_0 \cup D_1) \cap (X_5 \cup \{c_2\}) = \emptyset$. Hence $x \in A_0 \cap D_1 = \{a_2\}$, *i.e.*, $x = a_2$. In particular, every edge $E \in \mathscr{F}_5(\overline{a_1 b_1 c_1})$ contains a_2.

Now, the set $\{a_1, a_2, b_1, c_1\}$ does not cover $\mathscr{F}$, but covers $\mathscr{F}_{1235}$. So, there exists an edge $D_2 \in \mathscr{F}_4$ such that $D_2 \cap \{a_1, a_2, b_1, c_1\} = \emptyset$. Since D_2 must intersect with A_0, C_1 and E_1, D_2 contains a vertex of A_0 and the vertex c_2. Let $D_2 = X_4 \cup \{a', c_2\}$, where $a' \in A_0 - \{a_1, a_2\}$ and $c_2 \in X_3 - \{c_1\}$. The argument implies that every edge $D \in \mathscr{F}_4(\overline{a_1 a_2 b_1})$ contains c_2.

Next, consider the set $\{a_1, a_2, b_1, c_2\}$, which does not cover $\mathscr{F}$. This set covers $\mathscr{F}_{123}$, and also, by the result in the last paragraph, covers $\mathscr{F}_4$. So, there exists an edge $E_2 \in \mathscr{F}_5$ such that $E_2 \cap \{a_1, a_2, b_1, c_2\} = \emptyset$. This edge E_2 must contain the vertices a' and c_1.

Now, we can easily see that every edge $F \in \mathscr{F}_{45}(\overline{a_1 b_1})$ must contain one of the vertices c_1 and c_2. This implies that $\mathscr{F}$ is covered by $\{a_1, b_1, c_1, c_2\}$, a contradiction. □

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