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Some Inequalities Concerning Cross-Intersecting Families

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Let a, b and n be integers with $2 \leqslant a \leqslant b$ and $n \geqslant a+b$. Suppose that $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are nontrivial cross-intersecting families. Then $|\mathscr{A}|+|\mathscr{B}| \leqslant 2+\binom{n}{b}-2\binom{n-a}{b}+\binom{n-2a}{b}$. This result is best possible.

1. Introduction

Let $[n] := \{1, 2, \ldots, n\}$ be an n-element set. For an integer k, $0 \leqslant k \leqslant n$, we denote by $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. A family $\mathscr{F} \subset \binom{[n]}{k}$ is called *nontrivial* if $\bigcap_{F \in \mathscr{F}} F = \emptyset$. Two families, $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$, are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. A family $\mathscr{F} \subset \binom{[n]}{k}$ is called *intersecting* if $\mathscr{A}$ and $\mathscr{A}$ are cross-intersecting.

Let us recall the following two fundamental results.

Theorem A (Erdős, Ko and Rado [1]). *Let k and n be integers with $n \geqslant 2k$. If $\mathscr{F} \subset \binom{[n]}{k}$ is intersecting, then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$.*

Theorem B (Hilton and Milner [6]). *Let k and n be integers with $n \geqslant 2k$. If $\mathscr{F} \subset \binom{[n]}{k}$ is nontrivial intersecting, then $|\mathscr{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

In [4], Füredi proposed the following conjectures.

Conjecture 1. *Let a, b and n be integers with $n > a+b$. Suppose that $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are cross-intersecting families. Then $|\mathscr{A}||\mathscr{B}| \leqslant \binom{n-1}{a-1}\binom{n-1}{b-1}$.*

Conjecture 2. *Let a, b and n be integers with $a \leqslant b$ and $n \geqslant a+b$. Suppose that $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are cross-intersecting families. If $|\mathscr{A}| \geqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ and $\mathscr{A}$ is nontrivial, then $|\mathscr{B}| \leqslant \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + \binom{n-a-1}{b-a}$.*

Conjecture 3. *Let a, b and n be integers with $a \leqslant b$ and $n \geqslant a+b$. Suppose that $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are nontrivial cross-intersecting families. Then*

$$|\mathscr{A}| + |\mathscr{B}| \leqslant \left|\binom{[a+1]}{a}\right| + \left|\left\{B \in \binom{[n]}{b} : |[a+1] \cap B| \geqslant 2\right\}\right|.$$

Conjecture 1 was known to be true if $n \geqslant \max\{2a, 2b\}$ (see [10], [13]). But if $n < \max\{2a, 2b\}$ then the conjecture is not true in general. A simple counterexample is given in Section 2.

In Section 3, we show that Conjecture 2 is a direct consequence of a theorem of Mörs.

Conjecture 3 is false even if we fix $|\mathscr{A}| = a+1$. In this case, the best construction is the following. Let

$$A_i := \{1, \ldots, a-1\} \cup \{a+i\} \quad \text{for } 0 \leqslant i < a,$$

and set

$$\mathscr{A} := \{A_0, \ldots, A_{a-1}\} \cup \{\{a, \ldots, 2a-1\}\},$$
$$\mathscr{B} := \left\{B \in \binom{[n]}{b} : A \cap B \neq \emptyset \text{ for all } A \in \mathscr{A}\right\}.$$

If we do not restrict $|\mathscr{A}|$, the following construction is much better.

Example. Choose disjoint $A_0, A_1 \in \binom{[n]}{a}$, and set $\mathscr{A}_0 := \{A_0, A_1\}$,

$$\mathscr{B}_0 := \left\{B \in \binom{[n]}{b} : B \cap A_0 \neq \emptyset,\ B \cap A_1 \neq \emptyset\right\}.$$

Then $\mathscr{A}_0$ and $\mathscr{B}_0$ are nontrivial cross-intersecting families. ($\mathscr{A}_0$ has size 2.)

Actually, if $b \geqslant a+2$ then we have the following result.

Theorem 1. *Let a, b and n be integers with $2 \leqslant a \leqslant b-2$ and $n \geqslant a+b$. Suppose that two families $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are cross-intersecting, and the family $\mathscr{A}$ is nontrivial. Then, $|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}_0| + |\mathscr{B}_0|$ holds. For $n > a+b$, equality holds if and only if $\mathscr{A} \cong \mathscr{A}_0$ and $\mathscr{B} \cong \mathscr{B}_0$.*

Note that in the above theorem it is not assumed that $\mathscr{B}$ is nontrivial. We prove Theorem 1 in Section 5. If $|\mathscr{A}|$ is relatively small then the same inequality holds for the cases $b = a$ or $b = a+1$ as well.

Theorem 2. *Let a, b and n be integers with $2 \leqslant a \leqslant b$ and $n \geqslant a+b$. Suppose that two families $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$ are cross-intersecting, and the family $\mathscr{A}$ is nontrivial. Then the following statements hold.*

(i) *If* $b = a + 1$ *and* $|\mathscr{A}| \leqslant \binom{n-1}{a-1} + \binom{n-2}{a-1}$, *then* $|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}_0| + |\mathscr{B}_0|$. *For* $n > a + b$, *equality holds if and only if* $\mathscr{A} \cong \mathscr{A}_0$ *and* $\mathscr{B} \cong \mathscr{B}_0$.

(ii) *If* $b = a$ *and* $|\mathscr{A}| \leqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ *then* $|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}_0| + |\mathscr{B}_0|$. *For* $n > a + b$ *and* $a \geqslant 3$, *equality holds if and only if* $\mathscr{A} \cong \mathscr{A}_0$ *and* $\mathscr{B} \cong \mathscr{B}_0$.

Using Theorems 1 and 2, we have the following.

Theorem 3. *Let* a, b *and* n *be integers with* $2 \leqslant a \leqslant b$ *and* $n \geqslant a+b$. *Suppose that* $\mathscr{A} \subset \binom{[n]}{a}$ *and* $\mathscr{B} \subset \binom{[n]}{b}$ *are nontrivial cross-intersecting families. Then* $|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}_0| + |\mathscr{B}_0|$. *For* $n > a + b$ *and* $b \geqslant 3$, *equality holds if and only if* $\mathscr{A} \cong \mathscr{A}_0$ *and* $\mathscr{B} \cong \mathscr{B}_0$.

Since Theorems 1, 2, 3 are trivial if $n = a + b$, throughout this paper we consider the case $n > a + b$.

2. Counterexample to Conjecture 1

Define

$$\mathscr{A} := \left\{A \in \binom{[n]}{a} : \{1,2\} \cap A \neq \emptyset\right\},$$
$$\mathscr{B} := \left\{B \in \binom{[n]}{b} : \{1,2\} \subset B\right\}.$$

These two families are cross-intersecting, and

$$|\mathscr{A}| = \binom{n-1}{a-1} + \binom{n-2}{a-1}, \qquad |\mathscr{B}| = \binom{n-1}{b-1} - \binom{n-2}{b-1}.$$

Set $\delta := |\mathscr{A}||\mathscr{B}| - \binom{n-1}{a-1}\binom{n-1}{b-1}$. Then $\delta > 0$ is equivalent to

$$\frac{(n-1)(b-a)}{(n-b)(n-a)} > 1. \tag{2.1}$$

Let $n = (2 - \alpha)b$, $a = (1 - \beta)b$, where

$$0 < \alpha < \beta < 1. \tag{2.2}$$

Then $n > a + b$ holds and condition (2.1) is equivalent to

$$(1 - 1/b)\beta > (1 - \alpha)^2. \tag{2.3}$$

If we choose α, β and b so that (2.2) and (2.3) hold, then $n > a + b$, but $\delta > 0$. For example, choose an integer $c > 5$ and set $n = 17c$, $a = 5c$ and $b = 10c$: then the pair of $\mathscr{A}$ and $\mathscr{B}$ is a counterexample to Conjecture 1.

3. The Mörs theorem

Let $\mathscr{F} \subset \binom{[n]}{k}$ and $0 < l < k$. The lth shadow $\Delta_l(\mathscr{F})$ of $\mathscr{F}$ is defined by

$$\Delta_l(\mathscr{F}) := \{G : |G| = l,\ G \subset F \text{ for some } F \in \mathscr{F}\}.$$

Let us define the colex order on $\binom{[n]}{k}$ by

$$A < B \quad \text{if and only if} \quad \max\{A - B\} < \max\{B - A\}.$$

Define Colex(k, j) to be the first j sets in $\binom{\mathbb{N}}{k}$ with respect to the colex order. Let us define $\|\mathscr{F}\| := \bigcup_{F \in \mathscr{F}} F$. For given integers n, k, i, l, what is the minimum of $|\Delta_l(\mathscr{F})|$ if $\mathscr{F} \subset \binom{[n]}{k}$, $\|\mathscr{F}\| = n$ and $|\mathscr{F}| = i$? The Mörs theorem (Theorem C below) gives the complete answer to this question.

Let n, k, i be integers with $n/k \leqslant i \leqslant \binom{n}{k}$. Let us construct a family $\mathscr{F}_0 \subset \binom{[n]}{k}$ with $\|\mathscr{F}_0\| = n$, $|\mathscr{F}| = i$ as follows. Define $g := \max\{j : n - \|\mathrm{Colex}(k, j)\| \leqslant (i - j)k\}$, $h := \|\mathrm{Colex}(k, g)\|$. For $1 \leqslant j < i - g$, define $F_j := \{(j-1)k + h + 1, \dots, jk + h\}$. Further, define $G := \{(i - g - 1)k + h + 1, \dots, n, 1, 2, \dots, k - (n - (i - g - 1)k - h)\}$. Finally, define

$$\mathscr{F}_0 := \mathrm{Colex}(k, g) \cup \{F_1, \dots, F_{i-g-1}, G\}.$$

Theorem C (Mörs [12]). *Let n, k, i, l be integers with $1 \leqslant l < k \leqslant n$, $n/k \leqslant i \leqslant \binom{n}{k}$. Suppose that $\mathscr{F} \subset \binom{[n]}{k}$, $\|\mathscr{F}\| = n$, $|\mathscr{F}| = i$. Then $|\Delta_l(\mathscr{F})| \geqslant |\Delta_l(\mathscr{F}_0)|$.*

If $n \leqslant 2k$, the situation is much simpler. In this case, the optimal family $\mathscr{F}_0$ is given by

$$\mathscr{F}_0 := \mathrm{Colex}(k, i-1) \cup \{h+1, \dots, n, 1, 2, \dots, k + h - n\}.$$

Let us show how Conjecture 2 follows from Theorem C (see also [5]). Note that

$$\begin{aligned} |\mathscr{A}| &\geqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1 \\ &= \binom{n-2}{n-a} + \binom{n-3}{n-a-1} + \cdots + \binom{n-a-1}{n-2a+1} + \binom{n-a-1}{n-a-1}. \end{aligned}$$

By the Mörs theorem, we have

$$\begin{aligned} |\mathscr{B}| &\leqslant \binom{n}{b} - \binom{n-2}{b} - \binom{n-3}{b-1} - \cdots - \binom{n-a-1}{b-a+1} - \binom{n-a-1}{b-1} \\ &= \binom{n-1}{b-1} + \binom{n-a-1}{b-a} - \binom{n-a-1}{b-1}. \end{aligned}$$

4. Tools for proofs

In this section, we list several inequalities concerning binomial coefficients (see [2], [3], [10], [11]). These inequalities will be used in later sections.

Lemma 1. *Let $b \geqslant a$, $a \geqslant e+3$ and $n \geqslant a+b$. Then inequality $P(j, n)$ holds for $0 \leqslant e \leqslant a-3$ and $0 \leqslant j \leqslant e+1$,*

$$P(j, n): \quad \binom{n-a+e}{b-1-j} - \binom{n-2a+e}{b-1-j} > \binom{n-a+e}{e+1-j}.$$

Proof. We prove $P(j, n)$ by double induction on j and n. Fix a, b and e.

If $j = e+1$, then the desired inequality is

$$P(e+1,n): \quad \binom{n-a+e}{b-2-e} - \binom{n-2a+e}{b-2-e} > 1.$$

Since $b \geqslant a \geqslant e+3$, we have $b-2-e \geqslant 1$. Thus $P(e+1,n)$ holds for all $n \geqslant a+b$.

Now fix $0 < j \leqslant e$ and assume that $P(j,n)$ holds for all $n \geqslant a+b$. We prove

$$P(j-1,n): \quad \binom{n-a+e}{b-j} - \binom{n-2a+e}{b-j} > \binom{n-a+e}{e+2-j}$$

using induction on n.

First we check the case $n = a+b$, that is,

$$P(j-1,a+b): \quad \binom{b+e}{b-j} - \binom{b-a+e}{b-j} > \binom{b+e}{e+2-j}.$$

The above inequality is trivial if $b-a+e \leqslant b-j$. So assume $a < e+j$. By the induction hypothesis $P(j,a+b)$, it follows that

$$\binom{b+e}{b-1-j} - \binom{b-a+e}{b-1-j} > \binom{b+e}{e+1-j} = \frac{e+2-j}{b-1+j}\binom{b+e}{e+2-j}.$$

Thus, to prove $P(j-1,a+b)$, it suffices to show

$$\binom{b+e}{b-j} - \binom{b-a+e}{b-j} > \frac{b-1+j}{e+2-j}\left(\binom{b+e}{b-1-j} - \binom{b-a+e}{b-1-j}\right),$$

or, equivalently,

$$\binom{b+e}{b-j}\left(1 - \frac{b-1+j}{e+2-j}\cdot\frac{b-j}{e+1+j}\right) > \binom{b-a+e}{b-j}\left(1 - \frac{b-1+j}{e+2-j}\cdot\frac{b-j}{e+1+j-a}\right).$$

The above inequality clearly holds.

Next we fix n and assume $P(j-1,n)$. We prove $P(j-1,n+1)$. Using the induction hypotheses $P(j-1,n)$ and $P(j,n)$, we have

$$\begin{aligned}
&\binom{n+1-a+e}{b-j} - \binom{n+1-2a+e}{b-j} \\
&\quad = \left\{\binom{n-a+e}{b-j} - \binom{n-2a+e}{b-j}\right\} + \left\{\binom{n-a+e}{b-j-1} - \binom{n-2a+e}{b-j-1}\right\} \\
&\quad > \binom{n-a+e}{e+2-j} + \binom{n-a+e}{e+1-j} = \binom{n-a+e+1}{e+2-j}
\end{aligned}$$

This proves $P(j-1,n+1)$, and by induction $P(j-1,n)$ holds for all $n \geqslant a+b$. □

Lemma 2. *Let n and a be integers with $n > 2a$, $a > 1$. Define $f(n,a) := \binom{n-1}{a} + \binom{n-2a}{a} - 2\binom{n-a-1}{a} - \binom{n-1}{a-1}$. Then we have $f(n,a) > 0$.*

Proof. We prove $f(n,a) > 0$ by double induction on n and a. It is easily checked that $f(n,1) = 0$ and $f(2a,a) = 0$. Fix n and a, and assume $f(n,a) \geqslant 0$ and $f(n,a-1) \geqslant 0$. Using these assumptions, let us prove $f(n,a+1)>0$. Since

$$\begin{aligned} f(n+1,a) &\\ = &\left\{\binom{n-1}{a}+\binom{n-2a}{a}-2\binom{n-a-1}{a}-\binom{n-1}{a-1}\right\}+\\ &\left\{\binom{n-1}{a-1}+\binom{n-2a}{a-1}-2\binom{n-a-1}{a-1}-\binom{n-1}{a-2}\right\}\\ = & f(n,a)+f(n,a-1)+2\binom{n-a-1}{a-2}-2\binom{n-2a}{a-2}\left(1+\frac{a-2}{2(n-3a+3)}\right),\end{aligned}$$

it suffices to show that

$$\frac{(n-a-1)\cdots(n-2a+2)}{(n-2a)\cdots(n-3a+3)}\geqslant 1+\frac{a-2}{2(n-3a+3)}.$$

Let us check the above inequality:

$$\begin{aligned}\text{LHS} &= \left(1+\frac{a-1}{n-2a}\right)\cdots\left(1+\frac{a-1}{n-3a+3}\right)\\ &> 1+\frac{a-1}{n-2a}+\cdots+\frac{a-1}{n-3a+3}>1+\frac{a-2}{2(n-3a+3)} = \text{RHS}.\end{aligned}$$

This proves $f(n+1,a)>0$. □

Lemma 3. *Let n and a be integers with $n>2a+1$, $a>0$. Define $f(n,a):=\binom{n}{a+1}-\binom{n}{a}-2\binom{n-a-1}{a+1}+\binom{n-2a}{a+1}-\binom{n-2a-2}{a}$. Then $f(n,a)>0$.*

Proof. We prove $f(n,a)>0$ by double induction on n and a. One can easily check that $f(n,0)=0$ and $f(2a+1,a)=0$. Fix n and a, and assume that $f(n,a)>0$ and $f(n,a-1)>0$. Using these assumptions, let us prove $f(n+1,a)>0$. In fact,

$$\begin{aligned} f(n+1,a) = &\left\{\binom{n}{a+1}-\binom{n}{a}-2\binom{n-a-1}{a+1}+\binom{n-2a}{a+1}-\binom{n-2a-2}{a}\right\}\\ &\left\{\binom{n}{a}-\binom{n}{a-1}-2\binom{n-a-1}{a}+\binom{n-2a}{a}-\binom{n-2a-2}{a-1}\right\}\\ = & f(n,a)+f(n,a-1)+2\left\{\binom{n-a}{a}-\binom{n-a-1}{a}\right\}+\\ &\left\{\binom{n-2a}{a}-\binom{n-2a-2}{a}\right\}+\left\{\binom{n-2a}{a-1}-\binom{n-2a-2}{a-1}\right\}\\ > & 0.\end{aligned}$$

□

For an integer k and a real $x\geqslant k$, define $\binom{x}{k}:=\prod_{i=0}^{k-1}(x-i)/k!$.

Lemma 4. *Let s,t and n be integers with $n>s+t$. Define a real valued function $f(x):=-\binom{x}{s}+\binom{x}{n-t}$. Then the following statements hold.*

(i) *Suppose that $1+\frac{(n-s-t)v}{s(v-n+t+1)}<\binom{v}{s}/\binom{v}{n-t}$. Then $f'(x)<0$ holds for all real numbers $x\leqslant v$.*

(ii) *Let u, v be real numbers with $u < v$, and let $w \in \{u, v\}$. Suppose that $f'(u) < 0$ and $f(w) = \max\{f(u), f(v)\}$. Then $f(w) \geqslant f(x)$ holds for all real numbers x, $u \leqslant x \leqslant v$.*

Proof. **(i)** Since $f'(x) = -\binom{x}{s}\sum_{j=0}^{s-1}\frac{1}{x-j} + \binom{x}{n-t}\sum_{j=0}^{n-t-1}\frac{1}{x-j}$, $f'(x) < 0$ is equivalent to

$$\left(\sum_{j=0}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right) < \binom{x}{s}\Bigg/\binom{n}{n-t} = \frac{(n-t)\cdots(s+1)}{(x-s)\cdots(x-n+t+1)}. \tag{4.1}$$

By simple estimation, we have

$$\text{LHS} = 1 + \left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right) \leqslant 1 + \frac{n-t-s}{x-n+t+1}\cdot\frac{x}{s}.$$

Thus, to prove (4.1), it suffices to show that

$$(x-s)\cdots(x-n+t+1)\left(1 + \frac{n-t-s}{x-n+t+1}\cdot\frac{x}{s}\right) < (n-t)\cdots(s+1). \tag{4.2}$$

Since the LHS of (4.2) is increasing with x, it suffices to show (4.2) for $x = v$, that is,

$$1 + \frac{n-t-s}{v-n+t+1}\cdot\frac{v}{s} < \binom{v}{s}\Bigg/\binom{v}{n-t}.$$

But this was our assumption.

(ii) Suppose on the contrary that $f(w) < f(x)$ holds for some x, $x > u$. Then, we may assume that there exist p, q which satisfy

$$u < p < q \leqslant v,$$
$$f'(p) = f'(q) = 0,$$
$$f(p) < f(w) < f(q).$$

If $f'(x) = 0$, it follows that

$$\binom{x}{s} = \binom{x}{n-t}\left\{1 + \left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right)\right\}.$$

Substituting this into $f(x)$, we define a new function:

$$g(x) := -\binom{x}{n-t}\left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right).$$

Note that $g(x) = f(x)$ holds if $f'(x) = 0$. Thus, $f(w) < g(q)$ must hold. We derive a contradiction by showing that $f(w) \geqslant g(x)$ or, equivalently,

$$\left\{\binom{w}{s} - \binom{w}{n-t}\right\}\sum_{j=0}^{s-1}\frac{1}{x-j} \leqslant \binom{x}{n-t}\sum_{j=s}^{n-t-1}\frac{1}{x-j}$$

holds for all $x \geqslant p$. We may assume that $\binom{w}{s} - \binom{w}{n-t}$ is nonnegative. Then the LHS is decreasing with x. On the other hand, the RHS is increasing with x. Therefore, it suffices to check the inequality for $x = p$, that is, $f(w) \geqslant g(p) = f(p)$. But this was our assumption. $\square$

Lemma 5. *Let a, b and n be integers with $n > a + b$. Define a real valued function $f(y) := -\binom{y}{b-1} + \binom{y}{n-a-1}$. Then, the following hold.*

(i) *If $b \geqslant a + 3$ then $f(y) < f(n-a-1)$ holds for $n - a - 1 < y \leqslant n - 1$.*
(ii) *If $b = a + e$ then $f(y) < f(n-a-1)$ holds for $n - a - 1 < y \leqslant n - 3 + e$, $e = 0, 1, 2$.*

Proof. Set $s := b - 1$ and $t := a + 1$.

(i) Set $v := n - 1$. Then, we have

$$1 + \frac{(n-s-t)v}{s(v-n+t+1)} = \frac{(n-a-1)(n-b+1) - (n-a-b)}{(b-1)(a+1)},$$

$$\binom{v}{s} \Big/ \binom{v}{n-t} = \frac{(n-a-1)\cdots(n-b+1)}{(b-1)\cdots(a+1)} \geqslant \frac{(n-a-1)(n-b-1)}{(b-1)(a+1)}.$$

Using Lemma 4(i), we have $f'(y) < 0$ for $y \leqslant n - 1$.

(ii) Set $v := n - 4 + e$. Using Lemma 4(i), one can check $f'(y) < 0$ for $y \leqslant n - 4 + e$. Next, define $u := n - 4 + e$, $v := n - 3 + e$, $w := u$. Using Lemma 4(ii), one can check $f(y) \leqslant f(n-4+e) = f(n-3+e)$ for $n - 4 + e \leqslant y \leqslant n - 3 + e$. $\square$

5. Proof of Theorem 1

Let $n > a + b$ and consider cross-intersecting families $\mathscr{A} \subset \binom{[n]}{a}$ and $\mathscr{B} \subset \binom{[n]}{b}$. Define

$$P(t) := \max\{|\mathscr{A}| + |\mathscr{B}| : |\mathscr{A}| = t, \mathscr{A} \text{ and } \mathscr{B} \text{ are cross-intersecting and } \mathscr{A} \text{ is nontrivial}\}.$$

Our goal is to show $P(|\mathscr{A}|) \leqslant P(2)$ for $2 \leqslant |\mathscr{A}| \leqslant \binom{n}{a}$.

Define the complement of $\mathscr{A}$ by $\mathscr{A}^c := \{[n] - A : A \in \mathscr{A}\} \subset \binom{[n]}{n-a}$, and recall from Section 3 that the bth shadow of $\mathscr{A}^c$ is

$$\Delta_b(\mathscr{A}^c) := \left\{F \in \binom{[n]}{b} : F \cap A = \emptyset \text{ for some } A \in \mathscr{A}\right\}.$$

Since $\mathscr{A}$ is nontrivial, we have $\bigcup_{F \in \mathscr{A}^c} F = [n]$. The cross-intersecting property implies $\Delta_b(\mathscr{A}^c) \cap \mathscr{B} = \emptyset$.

Case 1. $|\mathscr{A}| \leqslant \binom{n-1}{a-1}$.
In this case, we assume $b \geqslant a + 1$ instead of $b \geqslant a + 2$. (We will use this part of the proof for a proof of Theorems 2 and 3 later.) Suppose that $|\mathscr{A}| = |\mathscr{A}^c| \leqslant \binom{n-1}{a-1}$ is fixed. Then, in order to maximize $|\mathscr{A}| + |\mathscr{B}|$, we have to choose $\mathscr{A}$ so that $|\Delta_b(\mathscr{A}^c)|$ is minimal. (Then

$\mathscr{B} := \binom{[n]}{b} - \Delta_b(\mathscr{A}^c)$ has the maximal size.) By the Mörs theorem, the optimal family is the following. Let $\mathscr{F} \subset \binom{[n]}{n-a}$ be the first $|\mathscr{A}| - 1$ sets with respect to the colex order. Let $\bigcup_{E\in\mathscr{F}} E = \{1, 2, \ldots, x\}$ and define $F := \{1, \ldots, x-a\} \cup \{x+1, \ldots, n\}$. Finally, the optimal family $\mathscr{A}$ is given by $\mathscr{A}^c = \mathscr{F} \cup \{F\}$. Then we have

$$P(|\mathscr{A}|) = P(|\mathscr{F}| + 1) = |\mathscr{F}| + 1 + \binom{n}{b} - |\Delta_b(\mathscr{F} \cup \{F\})|.$$

Lemma 6. *Let $b \geqslant a$. For any integer x, $n - a < x \leqslant n-2$, we have $P(2) > P(\binom{x}{n-a} + 1)$.*

Proof. Let $\mathscr{A}^c = \mathscr{F} \cup \{F\}$ and $|\mathscr{F}| = \binom{x}{n-a}$. In this case, $\mathscr{F} = \binom{[x]}{n-a}$ and $F = \{1, \ldots, x - a\} \cup \{x+1, \ldots, n\}$ hold. Thus, $|\Delta_b(\mathscr{A}^c)| = \binom{x}{b} + \binom{n-a}{b} - \binom{x-a}{b}$. Therefore, we have

$$P\left(\binom{x}{n-a} + 1\right) = \binom{x}{n-a} + 1 + \binom{n}{b} - \binom{x}{b} - \binom{n-a}{b} + \binom{x-a}{b}.$$

Let $f(x) := \binom{x}{n-a} - \binom{x}{b} + \binom{x-a}{b}$. We want to show $f(x) < f(n-a)$ for $n - a < x \leqslant n-2$. Let us define $g(x) := f(x) - f(x+1)$. It suffices to show $g(n-a+e) > 0$ for $0 \leqslant e \leqslant a-3$. This follows from Lemma 1 by setting $j = 0$. □

Lemma 7. *Let $b \geqslant a$. For any integer x, $n - a < x \leqslant n-2$, we have $P(\binom{x}{n-a} + 1) \geqslant P(\binom{x}{n-a} + 2)$. Equality holds if and only if $x = n-2$ and $a = b = 2$.*

Proof. We calculated $P(\binom{x}{n-a} + 1)$ in the proof of Lemma 6. Now we consider the case $|\mathscr{A}^c| = \binom{x}{n-a} + 2 = \binom{x}{n-a} + \binom{n-a-1}{n-a-1} + 1$. This time, we have $\mathscr{F} = \binom{[x]}{n-a} \cup \{1, \ldots, n-a-1, x+1\}$ and $F = \{1, \ldots, x-a+1\} \cup \{x+2, \ldots, n\}$. Thus,

$$P\left(\binom{x}{n-a} + 2\right) = \binom{x}{n-a} + 2 + \binom{n}{b} - \binom{x}{b} - \binom{n-a-1}{b-1} - \binom{n-a}{b} + \binom{x-a+1}{b}.$$

Therefore,

$$P\left(\binom{x}{n-a} + 1\right) - P\left(\binom{x}{n-a} + 2\right) = \binom{n-a-1}{b-1} - \binom{x-a}{b-1} - 1 \geqslant 0.$$

□

Lemma 8. *Let $b \geqslant a + 1$, and let x be an integer with $n - a \leqslant x \leqslant n - 2$. If $\binom{x}{n-a} + 2 \leqslant |\mathscr{A}| \leqslant \binom{x+1}{n-a}$ then $P(\binom{x}{n-a} + 2) \geqslant P(|\mathscr{A}|)$. Equality holds if and only if $|\mathscr{A}| = \binom{x}{n-a} + 2$.*

Proof. Choose a real y, $n - a - 1 \leqslant y < x$, so that $|\mathscr{A}| = \binom{x}{n-a} + \binom{y}{n-a-1} + 1$. In this case, it follows that $\mathscr{A}^c = \mathscr{F} \cup \{F\}$,

$$\mathscr{F} \subset \binom{[x]}{n-a} \cup \left\{G \cup \{x+1\} : G \in \binom{[\lceil y \rceil]}{n-a-1}\right\},$$
$$F = \{1, \ldots, x-a+1\} \cup \{x+2, \ldots, n\}.$$

Using the Kruskal–Katona theorem ([7], [8], [9]), we have

$$\begin{aligned} P(|\mathscr{A}|) &= |\mathscr{A}| + \binom{n}{b} - |\Delta_b(\mathscr{A}^c)| \\ &\leqslant \binom{x}{n-a} + \binom{y}{n-a-1} + 1 + \\ &\qquad \binom{n}{b} - \binom{x}{b} - \binom{y}{b-1} - \binom{n-a}{b} + \binom{x-a+1}{b}. \end{aligned}$$

Now define a real valued function $f(y) := -\binom{y}{b-1} + \binom{y}{n-a-1}$ for $n-a-1 \leqslant y < n-2$. By Lemma 5, we have $f(y) \leqslant f(n-a-1)$, that is, $P(|\mathscr{A}|) \leqslant P(\binom{x}{n-a} + 2)$. Equality holds if and only if $y = n-a-1$, that is, $|\mathscr{A}| = \binom{x}{n-a} + 2$. □

By Lemmas 6, 7 and 8, we have $P(2) \geqslant P(|\mathscr{A}|)$ for $2 < |\mathscr{A}| \leqslant \binom{n-1}{a-1}$. Equality holds only if $a = b = 2$. Since we have assumed $b \geqslant a+1$, we obtain $P(2) > P(|\mathscr{A}|)$.

Case 2. $|\mathscr{A}| > \binom{n-1}{a-1}$.
By the Erdős–Ko–Rado theorem ([1]), $\mathscr{A}$ is nontrivial no matter how we choose $\mathscr{A}$. Suppose that $|\mathscr{A}| = |\mathscr{A}^c| > \binom{n-1}{a-1}$ is fixed. Then, to maximize $|\mathscr{A}| + |\mathscr{B}|$, we have to choose $\mathscr{A}$ so that $|\Delta_b(\mathscr{A}^c)|$ is minimal. By the Kruskal–Katona theorem, we may assume that $\mathscr{A}^c$ is the first $|\mathscr{A}|$ sets with respect to the colex order. Choose a real y, $n-a-1 \leqslant y \leqslant n-1$, so that $|\mathscr{A}^c| = \binom{n-1}{n-a} + \binom{y}{n-a-1}$. Then we have

$$\begin{aligned} P(|\mathscr{A}|) &= |\mathscr{A}| + \binom{n}{b} - |\Delta_b(\mathscr{A}^c)| \\ &\leqslant \binom{n-1}{n-a} + \binom{y}{n-a-1} + \binom{n}{b} - \binom{n-1}{b} - \binom{y}{b-1}. \end{aligned}$$

Let us define a real valued function $f(y) := -\binom{y}{b-1} + \binom{y}{n-a-1}$ for $n-a-1 \leqslant y \leqslant n-1$. Then, by our assumption $b \geqslant a+2$ and Lemma 5, we have $f(y) \leqslant f(n-a-1)$. Thus,

$$P(|\mathscr{A}|) \leqslant P\left(\binom{n-1}{a-1} + 1\right) = P\left(\binom{n-1}{a-1}\right) + 1 - \binom{n-a-1}{b-1} < P(2).$$

This completes the proof of Theorem 1. □

6. Proof of Theorem 2

The proof is similar to the proof of Theorem 1. We leave some of the computations in the proof of Theorem 2 to the reader. We use the same definitions and notation as in the proof of Theorem 1.

***Proof of Theorem 2* (i)**
Case 1. $|\mathscr{A}| \leqslant \binom{n-1}{a-1}$.
The proof of this case is exactly same as the proof of Theorem 1.

Case 2. $\binom{n-1}{a-1} < |\mathscr{A}| \leqslant \binom{n-1}{a-1} + \binom{n-2}{a-1}$.
Choose a real y, $n-a-1 \leqslant y \leqslant n-2$, so that $|\mathscr{A}^c| = \binom{n-1}{n-a} + \binom{y}{n-a-1}$. Then we have

$$\begin{aligned} P(|\mathscr{A}|) &= |\mathscr{A}| + \binom{n}{b} - |\Delta_b(\mathscr{A}^c)| \\ &\leqslant \binom{n-1}{n-a} + \binom{y}{n-a-1} + \binom{n}{b} - \binom{n-1}{b} - \binom{y}{b-1}. \end{aligned}$$

Let us define a real valued function $f(y) := -\binom{y}{b-1} + \binom{y}{n-a-1}$ for $n-a-1 \leqslant y \leqslant n-2$. Then, by Lemma 5, we have $f(y) \leqslant f(n-a-1)$. Thus,

$$P(|\mathscr{A}|) \leqslant P\left(\binom{n-1}{a-1} + 1\right) = P\left(\binom{n-1}{a-1}\right) + 1 - \binom{n-a-1}{b-1} < P(2). \qquad \square$$

***Proof of Theorem 2* (ii)**
Let us settle the case $a = b = 2$ first. In this case, it is not difficult to check that $|\mathscr{A}| + |\mathscr{B}| \leqslant 6 = |\mathscr{A}_0| + |\mathscr{B}_0|$ by hand. Equality holds if and only if $\{\mathscr{A}, \mathscr{B}\} \cong \{\mathscr{A}_0, \mathscr{B}_0\}$ or $\mathscr{A} = \mathscr{B} = \{12, 13, 23\}$ or $\{\mathscr{A}, \mathscr{B}\} \cong \{\{12, 23, 34\}, \{13, 23, 24\}\}$.

From now on, we assume $a = b \geqslant 3$.

Case 1. $|\mathscr{A}| \leqslant \binom{n-2}{a-2} + \binom{n-3}{a-2}$.
We follow the proof of Theorem 1. This time, Lemmas 6 and 7 are still valid. Instead of Lemma 8, we use the following.

Lemma 9. *Let x be any integer with $n-a \leqslant x \leqslant n-3$. If $\binom{x}{n-a} + 2 \leqslant |\mathscr{A}| \leqslant \binom{x+1}{n-a}$ then $P(\binom{x}{n-a} + 2) \geqslant P(|\mathscr{A}|)$* $\square$

We can prove the above lemma in exactly the same way as in the proof of Lemma 8. Now using Lemmas 6, 7, 9, it follows that $P(2) < P(|\mathscr{A}|)$ for $2 < |\mathscr{A}| \leqslant \binom{n-2}{a-2} + \binom{n-3}{a-2}$.

Case 2. $\binom{n-2}{a-2} + \binom{n-3}{a-2} < |\mathscr{A}| \leqslant \binom{n-2}{a-2} + \cdots + \binom{n-a-1}{a-2} + 1 = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$.
For an integer x, $2 \leqslant x \leqslant a+1$, let us define

$$\begin{aligned} g(x) &:= \binom{n-2}{a-2} + \cdots \binom{n-x}{a-2} + 1, \\ h(x) &:= \binom{n}{a} - \binom{n-2}{n-a-2} - \cdots - \binom{n-x}{n-a-2} - \binom{n-a-1}{a-1}. \end{aligned}$$

Note that if $|\mathscr{A}| = g(x)$ then, by the Kruskal–Katona theorem, we have $|\mathscr{B}| \leqslant h(x)$. Note also that $h(a+1) = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1 = g(a+1)$. Thus, if $|\mathscr{A}| \geqslant g(a+1)$ then $|\mathscr{B}| \leqslant g(a+1)$.

Lemma 10. *For any integer x, $2 \leqslant x \leqslant a+1$, we have $P(2) > P(g(x))$.*

Proof. Using the result of Case 1, we have $P(2) > P(g(2))$. Since

$$\begin{aligned} P(g(x)) - P(g(x-1)) &= g(x) + h(x) - g(x-1) - h(x-1) \\ &= \binom{n-x}{a-2} - \binom{n-x}{a-x+2}, \end{aligned}$$

we have

$$P(g(2)) \geqslant P(g(3)) = P(g(4)) \leqslant P(g(5)) \leqslant \cdots \leqslant P(g(a+1)).$$

Thus, it suffices to show $P(2) > P(g(a+1))$. Note that

$$g(a+1) = h(a+1) = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1,$$

and

$$P(a+1) = 2g(a+1) = 2\binom{n-1}{a-1} - 2\binom{n-a-1}{a-1} + 2.$$

Therefore, the desired inequality $P(2) > P(g(a+1))$ is equivalent to

$$\binom{n-1}{a} + \binom{n-2a}{a} - 2\binom{n-a-1}{a} - \binom{n-1}{a-1} > 0.$$

The above inequality follows from Lemma 2. □

Lemma 11. *For any integer x, $2 \leqslant x \leqslant a$, we have $P(g(x)) > P(g(x)+1)$.*

Proof. If $|\mathscr{A}| = g(x) + 1 = g(x) + \binom{n-x-a+1}{n-x-a+1}$, then by the Mörs theorem, we have $|\mathscr{B}| \leqslant h(x) - \binom{n-x-a+1}{a-x+1}$. Thus, $P(g(x)) > P(g(x)+1)$ is equivalent to $\binom{n-x-a+1}{a-x+1} > 1$. This follows from our assumption $n > 2a$. □

Lemma 12. *Let x be an integer with $2 \leqslant x \leqslant a$. If $g(x)+1 \leqslant |\mathscr{A}| \leqslant g(x+1)$ then $P(|\mathscr{A}|) \leqslant \max\{P(g(x)+1), P(g(x+1))\}$.*

Proof. Choose a real y, $n-x-a+1 \leqslant y \leqslant n-x-1$, so that $|\mathscr{A}| = g(x) + \binom{y}{n-x-a+1}$. (Note that if $y = n-x-1$ then $|\mathscr{A}| = g(x+1)$.) Using the Kruskal–Katona theorem, we have $|\mathscr{B}| \leqslant h(x) - \binom{y}{a-x+1}$. Now define a real valued function $f(y) := -\binom{y}{a-x-1} + \binom{y}{n-x-a+1}$ for $n-x-a+1 \leqslant n-x-1$. Our goal is to show $f(y) \leqslant \max\{f(n-x-a+1), f(n-x-1)\}$.

First we settle the case $x = a$. In this case, we have $f(y) = -\binom{y}{1} + \binom{y}{n-2a+1}$. Since $n > 2a$, $f(y)$ is an increasing function. Thus, $f(y) \leqslant f(n-a-1)$ holds.

From now on, we assume $x < a$. Set $s := a-x+1$, $t := x+a-1$, and $v := n-2x$. Using Lemma 4(i), one can check that $f'(y) < 0$ holds for $y \leqslant n-2x$. Thus, we have $f'(n-x-a+1) < 0$. Therefore, $f(y) \leqslant \max\{f(n-x-a+1), f(n-x-1)\}$ follows from Lemma 4(ii). □

By Lemmas 10, 11, 12, we have

$$P(|\mathscr{A}|) \leqslant \max\{P(g(2)), P(g(a+1))\} < P(2). \qquad \square$$

7. Proof of Theorem 3

Recall that $P(|\mathscr{A}|) = \max\{|\mathscr{A}| + |\mathscr{B}|\}$ (see Section 5). If $b \geqslant a+2$, then the theorem follows from Theorem 1.

Case 1. $b = a+1$.
If $|\mathscr{A}| \leqslant \binom{n-1}{a-1} + \binom{n-2}{a-1}$ then the desired inequality ($P(2) > P(|\mathscr{A}|)$ for $2 < |\mathscr{A}| \leqslant \binom{n-1}{a-1} + \binom{n-2}{a-1}$) follows from Theorem 2. So we may assume $|\mathscr{A}^c| > \binom{n-1}{n-a} + \binom{n-2}{n-a-1}$. Then, by the Kruskal–Katona theorem, we have

$$|\mathscr{B}| \leqslant \binom{n}{b} - |\Delta_b(\mathscr{A}^c)| \leqslant \binom{n}{a+1} - \binom{n-1}{a+1} - \binom{n-2}{a} = \binom{n-2}{n-(a+1)}.$$

Define

$$Q(t) := \max\{|\mathscr{A}| + |\mathscr{B}| : |\mathscr{B}| = t,\ \mathscr{A} \text{ and } \mathscr{B} \text{ are cross-intersecting and } \mathscr{B} \text{ is nontrivial}\}.$$

Let $|\mathscr{B}^c| = \binom{y}{n-(a+1)} + 1$ for $n-a-1 \leqslant y < n-2$. Then we have $Q(|\mathscr{B}|) \leqslant f(y) + (\text{constant})$, where $f(y) := -\binom{y}{a} + \binom{y}{n-a-1}$. Using Lemma 5, one can check that $f(y) < f(n-a-1)$ for $n-a-1 < y < n-2$, that is, $Q(2) > Q(|\mathscr{B}|)$ for $2 < |\mathscr{B}| \leqslant \binom{n-2}{n-(a+1)}$. Using Lemma 3, we have $P(2) > Q(2)$. This completes the proof of this case.

Case 2. $b = a$.
Without loss of generality, we may assume that $|\mathscr{A}| \leqslant |\mathscr{B}|$. If $|\mathscr{A}| \geqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ then $|\mathscr{B}| \leqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ (see the computation in the proof of Theorem 2(ii), Case 2). Thus we may assume that $|\mathscr{A}| \leqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$. Then the result follows from Theorem 2. □

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