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An Erdős–Ko–Rado Theorem for Direct Products

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Let n_i, k_i be positive integers, $i = 1, \ldots, d$, satisfying $n_i \geqslant 2k_i$. Let $X_1, \ldots, X_d$ be pairwise disjoint sets with $|X_i| = n_i$. Let $\mathscr{H}$ be the family of those $(k_1 + \cdots + k_d)$-element sets which have exactly k_i elements in X_i, $i = 1, \ldots, d$. It is shown that if $\mathscr{F} \subset \mathscr{H}$ is an intersecting family then $|\mathscr{F}|/|\mathscr{H}| \leqslant \max_i k_i/n_i$, and this is best possible. The proof is algebraic, although in the $d = 2$ case a combinatorial argument is presented as well.

1. INTRODUCTION

Let X be an n-element set, and for $k < n$ let $\binom{X}{k}$ denote the family of all k-subsets of X. A family $\mathscr{F} \subset \binom{X}{k}$ is called *t-intersecting* if for all $F, F' \in \mathscr{F}$ one has $|F \cap F'| \geqslant t$. Also, intersecting stands for 1-intersecting.

One of the oldest and most important results in extremal set theory is the following.

ERDŐS–KO–RADO THEOREM (cf. [2, 3, 8]). *Suppose that* $n \geqslant (k - t + 1)(t + 1)$ *and let* $\mathscr{F} \subset \binom{X}{k}$ *be a t-intersecting family. Then*

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t} \tag{1}$$

holds.

Over the years, many extensions and sharpenings of this result have proved: see [1] for survey. The following problem arose in connection with a recent result of Sali [7].

Suppose that $n = n_1 + \cdots + n_d$, $k = k_1 + \cdots + k_d$, and $X = X_1 \cup \cdots \cup X_d$, with $|X_i| = n_i$. Define

$$\mathscr{H} = \left\{ F \in \binom{X}{k} : |F \cap X_i| = k_i \text{ for } i = 1, \ldots, d \right\}.$$

What is the maximal size of an intersecting subfamily of $\mathscr{H}$?

For an arbitrary element $x \in X$ the family $\mathscr{H}_x = \{H \in \mathscr{H} : x \in H\}$ is obviously intersecting. Moreover, if $x \in X_i$ then $|\mathscr{H}_x|/|\mathscr{H}| = k_i/n_i$.

THEOREM 1. *Suppose that* $\mathscr{F} \subset \mathscr{H}$ *is intersecting and* $k_d/n_d \leqslant \cdots \leqslant k_d/n_d \leqslant 1/2$. *Then* $|\mathscr{F}|/|\mathscr{H}| \leqslant k_1/n_1$ *holds.*

The proof of this result, which is presented in the next section, is based on the eigenvalue argument of Lovasz [5]. In Section 3, a combinatorial argument is provided for the case $d = 2$. An application of this result is given in [4]. It is based on the cyclic permutation method of Katona [6].

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2. Proof of Theorem 1

Let M_i denote the symmetric 0–1 matrix the rows and columns of which are indexed by the k_i-subsets of X_i, and the entry being 1 iff the corresponding two sets are disjoint. It is well-known that the eigenvalues of M_i are

$$(-1)^j\binom{n_i-k_i-j}{k_i-j}$$

with corresponding multiplicities

$$\binom{n_i}{j}-\binom{n_i}{j-j}, \qquad j=0,\ldots,k_i.$$

Let $M=M_1\oplus\cdots\oplus M_d$ be the tensor product of the matrices M_i. Then if a row (column) of M is indexed by $(F_1,\ldots,F_d)((G_1,\ldots,G_d))$, respectively, then the corresponding entry is 1 or 0 according to whether or not the intersection of $F_1\cup\cdots\cup F_d$ and $G_1\cup\cdots\cup G_d$ is empty. That is, M has its rows and columns indexed by the members of $\mathscr{H}$, and the corresponding entry is 1 or 0 according to whether or not the intersection of the sets is empty. Since the eigenvalues of M are the products of those of M_i, the largest eigenvalue of M is

$$\lambda=\binom{n_i-k_1}{k_i}\times\cdots\times\binom{n_d-k_d}{k_d},$$

it corresponds to the all-one vector. The smallest eigenvalue is

$$\mu=-\binom{n_1-k_1-1}{k_1-1}\binom{n_2-k_2}{k_2}\times\cdots\times\binom{n_d-k_d}{k_d}.$$

Let I (J) denote the identity (all-one) matrices of order $|\mathscr{H}|$, respectively. Then

$$N=M-\mu I-\frac{\lambda-\mu}{|\mathscr{H}|}J$$

is positive semidefinite.

Let $\mathscr{F}\subset\mathscr{H}$ be an intersecting family and let $v=v(\mathscr{F})$ be its characteristic vector: it is a 0–1 vector of length $|\mathscr{H}|$, with entries indexed by the members of $\mathscr{H}$, in the same order as the rows and columns of M.

Since N is positive semidefinite, we have the following inequality:

$$0\leqslant vNv^t=vMv^t-\mu vIv^t-\frac{\lambda-\mu}{|\mathscr{H}|}vJv^t=-\mu\,|\mathscr{F}|-\frac{\lambda-\mu}{|\mathscr{H}|}|\mathscr{F}|^2. \tag{2}$$

Consequently,

$$|\mathscr{F}|/|\mathscr{H}|\leqslant-\mu/(\lambda-\mu)=\binom{n_1-k_1-1}{k_1-1}\Big/\left(\binom{n_1-k_1}{k_1}+\binom{n_1-k_1-1}{k_1-1}\right)=k_1/n_1.$$

Using the proof of (1) given by Wilson [8], one obtains the following, more general, result in the same way.

Theorem 2. *Let n_i, k_i and t_i be positive integers satisfying $n_i\geqslant(k_i-t_i+1)(t_i+1)$, $i=1,\ldots,d$. Suppose that $\mathscr{F}\subset\mathscr{H}$ has the property that, for every F, $G\in\mathscr{F}$, there exists*

$1 \leqslant i \leqslant d$ *such that* $|F \cap G \cap X_i| \geqslant t_i$ *holds. Then*

$$|\mathscr{F}|/|\mathscr{H}| \leqslant \max_i \binom{n_i - t_i}{k_i - t_i} \Big/ \binom{n_i}{k_i},$$

and this is best possible.

3. The Combinatorial Proof of Theorem 1 in the Case $d = 2$

Let $x_1, \ldots, x_m$ be a cyclic permutation, $1 < r < m$ an integer. Consider a collection $\mathscr{R}$ of cyclic intervals of length r. Let $\mathscr{R}'$ be the collection of those intervals of length $r - 1$ which are contained in some member of $\mathscr{R}$. It is easy to see that $|\mathscr{R}'| \geqslant \min\{m, |\mathscr{R}| + 1\}$ holds. Applying this argument repeatedly implies that, for every $1 \leqslant i < r$, the number of cyclic intervals of length i which are contained in some member of $\mathscr{R}$ is at least $\min\{m, |\mathscr{R}| + r - i\}$. This simple result can be called the Circular Kruskal–Katona Theorem.

Recall that two families of sets are called cross-intersecting if each member of one intersects each member of the other. We need the following.

Proposition. *Suppose that* $\mathscr{C}$ *and* $\mathscr{D}$ *are a cross-intersecting families of cyclic intervals of respective lengths c and d,* $c + d \leqslant m$. *Then the following hold*:
(*a*) $|\mathscr{C}| + |\mathscr{D}| \leqslant m$
(*b*) $|\mathscr{C}| + |\mathscr{D}| \leqslant c + d$ *if both* $\mathscr{C}$ *and* $\mathscr{D}$ *are non-empty.*

Proof. Set $r = m - c$ and let $\mathscr{R}$ consist of the complements of the members of $\mathscr{C}$. Let $\mathscr{G}$ consist of those cyclic intervals of length d which are contained in some member of $\mathscr{R}$. By the cross-intersecting property $\mathscr{G} \cap \mathscr{D} = \varnothing$, and by the Circular Kruskal–Katona Theorem,

$$|\mathscr{D}| \leqslant m - \min\{m, |\mathscr{C}| + m - c - d\}.$$

Using $|\mathscr{D}| \geqslant 1$,

$$|\mathscr{C}| + |\mathscr{D}| \leqslant c + d$$

follows.

Now (a) follows from (b) unless $\mathscr{C}$ and $\mathscr{D}$ is empty, in which case it is trivial.

Next we turn to the proof of Theorem 1, $d = 2$.

Let $x_1, \ldots, x_{n_i}$ be a cyclic ordering of the elements of X_i, $i = 1, 2$. Let $\mathscr{A}_i$ be the collection of the n_i cyclic intervals of length k_i. Define $\mathscr{A} = \{A_1 \cup A_2 : A_i \in \mathscr{A}_i\}$.

Lemma. $\|\mathscr{A} \cap \mathscr{F}| \leqslant k_1 n_2$.

Proof. Define $\mathscr{B} = \{B \in \mathscr{A}_1 : \exists A_2 \in \mathscr{A}_2, (A_1 \cup A_2) \in \mathscr{F}\}$. We distinguish two cases, as follows.
(i) $|\mathscr{B}| \leqslant 2k_i$. Choose some family $\mathscr{B}'$, such that $\mathscr{B} \subset \mathscr{B}' \subset \mathscr{A}_1$, $|\mathscr{B}'| = 2k_1$. Let $\mathscr{B}'$ consist of the sets $B_1, B_2, \ldots, B_{2k_1}$ in this order. Let b_i denote the number of those $A \in \mathscr{A}_2$ for which $(b_i \cup A) \in \mathscr{F}$ holds. Since $B_i \cap B_{i+k_1} = \varnothing$, $b_i + b_{i+k} \leqslant n_2$ follows for $1 \leqslant \mathrm{i} \leqslant \mathrm{k}_1$ from the proposition. Consequently,

$$|\mathscr{A} \cap \mathscr{F}| = b_1 + b_2 + \cdots + b_{2k_1} \leqslant k_1 n_2$$

holds.
(ii) $|\mathscr{B}| > 2k_1$. For $A \in \mathscr{A}_1$, let $\mathscr{D}(A)$ be the collection of those $D \in \mathscr{A}_2$ for which

$(A \cup D) \in \mathscr{F}$ holds. Note that, if $A \cap A' = \varnothing$, then $\mathscr{D}(A)$ and $\mathscr{D}(A')$ are cross-intersecting. Let $d_1, d_2, \ldots, d_{n_1}$ be the numbers $|\mathscr{D}(A)|$, $A \in \mathscr{A}_1$, in decreasing order.

CLAIM. $d_j + d_{n_1+1-j} \leqslant 2k_2$, $1 \leqslant j \leqslant n_1$.

To prove the claim it is sufficient to consider the case $2j \leqslant n_1$. First observe that every cyclic interval of length k_i intersects exactly $2k_i - 1$ cyclic intervals of length k_i. Thus $d_1 \geqslant 2k_2$ would imply $d_{2k_2} = 0$, contrary to our assumptions. Consequently, $d_1 < 2k_2$. Thus the claim is true if $d_{n_1+1-j} = 0$. Therefore we suppose that $d_{n_1+1-j} \geqslant 1$.

Let A_i be the cyclic interval corresponding to the number d_i. In view of the proposition $\{A_1, \ldots, A_j\}$ and $\{A_1, \ldots, A_{n_1+1-j}\}$ cannot be cross-intersecting. That is, there exist $1 \leqslant u \leqslant j$ and $1 \leqslant v \leqslant n_1 + 1 - j$ such that $A_u \cap A_v = \varnothing$—a contradiction. By the proposition, we have $d_j + d_{n_1+1-j} \leqslant d_u + d_v \leqslant 2k_2$, as desired.

Summing up the inequality in the claim for $1 \leqslant j \leqslant n_1$ gives $2\,|\mathscr{A} \cap \mathscr{F}| = 2(d_1 + \cdots + d_{n_1}) \leqslant 2k_2 n_1 \leqslant 2k_1 n_2$, concluding the proof of the lemma.

The lemma says that for each pair of cyclic permutations out of the possible $n_1 n_2$ sets, at most $k_1 n_2$ are in $\mathscr{F}$: that is, a proportion of at most k_1/n_1. Therefore, by averaging (see, e.g., [6]), $|\mathscr{F}|/|\mathscr{H}| \leqslant k_1/n_1$ follows.

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