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Covers in Uniform Intersecting Families and a Counterexample to a Conjecture of Lovász

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We discuss the maximum size of uniform intersecting families with covering number at least τ. Among others, we construct a large k-uniform intersecting family with covering number k, which provides a counterexample to a conjecture of Lovász. The construction for odd k can be visualized on an annulus, while for even k on a Möbius band.

1. INTRODUCTION

Let X be a finite set. $\binom{X}{k}$ denotes the family of all k-element subsets of X. We always assume that $|X|$ is sufficiently large with respect to k. A family $\mathscr{F} \subset \binom{X}{k}$ is called k-uniform. The vertex set of $\mathscr{F}$ is X and denoted by $V(\mathscr{F})$. An element of $\mathscr{F}$ is called an edge of $\mathscr{F}$. $\mathscr{F} \subset \binom{X}{k}$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every $F, G \in \mathscr{F}$. A set $C \subset X$ is called a *cover* of $\mathscr{F}$ if it intersects every edge of $\mathscr{F}$, i.e., $C \cap F \neq \emptyset$ holds for all $F \in \mathscr{F}$. A cover C is also called t-cover if $|C| = t$. The *covering number* $\tau(\mathscr{F})$ of $\mathscr{F}$ is the minimum cardinality of the covers of $\mathscr{F}$. The degree of a vertex x is defined by $\deg(x) := \#\{F \in \mathscr{F} : x \in F\}$.

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For a family $\mathscr{A} \subset 2^X$ and vertices $x, y \in X$, we define $\mathscr{A}(x) := \{A - \{x\}: x \in A \in \mathscr{A}\}$, $\mathscr{A}(\bar{x}) := \{A : x \notin A \in \mathscr{A}\}$, $\mathscr{A}(\bar{x}\bar{y}) := \{A : x, y \notin A \in \mathscr{A}\}$, etc., and for $Y \subset X$, $\mathscr{A}(Y) := \{A - Y: Y \subset A \in \mathscr{A}\}$, $\mathscr{A}(\bar{Y}) := \{A \in \mathscr{A} : Y \cap A = \emptyset\}$.

For a family $\mathscr{F} \subset \binom{X}{k}$ and an integer $t \geqslant 1$, define $\mathscr{C}_t(\mathscr{F}) = \{C \in \binom{X}{t} : C \cap F \neq \emptyset$ holds for all $F \in \mathscr{F}\}$. Note that $\mathscr{C}_t(\mathscr{F}) = \emptyset$ for $t < \tau(\mathscr{F})$. Define

$$p_t(k) = \max \left\{ |\mathscr{C}_t(\mathscr{F})| : \mathscr{F} \subset \binom{X}{k} \text{ is intersecting and } \tau(\mathscr{F}) \geqslant t \right\}.$$

Let us first list some useful facts concerning $p_t(k)$. Choosing $|\mathscr{F}| = 1$, one has $p_1(k) = k$.

(1) $p_{t+1}(k) \leqslant k p_t(k)$.

Proof. Take $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ intersecting, $\tau(\mathscr{F}) = t+1$ and $|\mathscr{C}_{t+1}(\mathscr{F})| = p_{t+1}(k)$. Define $\mathscr{C} = \mathscr{C}_{t+1}(\mathscr{F})$. Let $F \in \mathscr{F}$ be an arbitrary member of $\mathscr{F}$. By definition, $F \cap C \neq \emptyset$ holds for every $C \in \mathscr{C}$. Thus $|\mathscr{C}| \leqslant \sum_{x \in F} |\mathscr{C}(x)|$ holds. Therefore, in order to establish (1) it is sufficient to prove $|\mathscr{C}(x)| \leqslant p_t(k)$ for all $x \in F$. Consider $\mathscr{F}(\bar{x})$. It is intersecting and $t \leqslant \tau(\mathscr{F}(\bar{x})) \leqslant \tau(\mathscr{F}) = t+1$. Moreover, $\mathscr{C}(x) \subset \mathscr{C}_t(\mathscr{F}(\bar{x}))$ is immediate from the definitions. Thus $|\mathscr{C}(x)| = 0$ holds if $\tau(\mathscr{F}(\bar{x})) = t+1$ and $|\mathscr{C}(x)| \leqslant p_t(k)$, otherwise. ∎

(2) *For* $\mathscr{F} \subset \binom{X}{k}$, *intersecting*, $\tau(\mathscr{F}) = t$ *and an arbitrary set* $A \in \binom{X}{a}$ *with* $a < t$, *one has* $|\mathscr{C}_t(\mathscr{F})(A)| \leqslant p_{t-a}(k)$.

Proof. This follows from $\mathscr{C}_t(\mathscr{F})(A) = \mathscr{C}_{t-a}(\mathscr{F}(\bar{A}))$. ∎

The following was proved implicitly in [3]. For a simple proof, see [4].

(3) $p_2(k) = k^2 - k + 1$.

Using a construction described in the next section, it is not difficult to check that $p_3(k) \geqslant (k-1)^3 + 3(k-1)$ holds for all $k \geqslant 3$. Actually, this inequality is proved to be an equality if $k \geqslant 9$ in [4]. (The proof is not simple.) Later we prove $p_3(3) = 14$. The case $4 \leqslant k \leqslant 8$ remains open.

The following is proved in [5].

(4) *For* $k \geqslant k_0$, $p_4(k) = k^4 - 6k^3 + O(k^2)$, $p_5(k) = k^5 - 10k^4 + O(k^3)$.

Let us define

$$r(k) := \max\{|\mathscr{F}| : \mathscr{F} \text{ is } k\text{-uniform and intersecting with } \tau(\mathscr{F}) = k\}.$$

For example, $r(2) = 3$ and the only extremal configuration is a triangle. Note that, $\mathscr{C}_k(\mathscr{F}) \supset \mathscr{F}$ for every intersecting k-uniform hypergraph, and equality must hold if $|\mathscr{F}| = r(k)$ holds (together with $\tau(\mathscr{F}) = k$). Recall also, that $r(k) \leqslant k^k$ was proved by Erdős and Lovász [2]. Clearly, $p_k(k) \geqslant r(k)$. This inequality is likely to be strict for all $k \geqslant 3$. E.g. for $k = 3$ consider the family $\mathscr{F} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}, \{2, 4, 5\}, \{4, 6, 1\}, \{6, 2, 3\}\}$. Then $\mathscr{F} \subset \binom{[6]}{3}$ and $\tau(\mathscr{F}) = 3$ imply $|\mathscr{C}_3(\mathscr{F})| = \binom{6}{3} - |\mathscr{F}| = 14$ ($G \notin \mathscr{C}_3(\mathscr{F})$ iff G is the complement of some $F \in \mathscr{F}$). On the other hand, $r(3) = 10$ is known. (See Appendix.)

(5) *Suppose that* $\mathscr{F} \subset \binom{X}{3}$ *is an intersecting family with* $\tau(\mathscr{F}) = 3$. *Then there exists* $x \in X$ *such that* $\deg(x) \geqslant 3$, *and* $|\mathscr{F}| \geqslant 6$.

Proof. We can choose $F, F' \in \mathscr{F}$ such that $F = \{1, 2, 3\}$, $F' = \{1, 4, 5\}$. There exists $G \in \mathscr{F}$ such that $G \cap \{2, 4\} = \emptyset$. If $1 \in G$, then $\deg(1) \geqslant 3$. Otherwise we may assume $G = \{3, 5, 6\}$. We can choose $G' \in \mathscr{F}$ such that $G' \cap \{3, 4\} = \emptyset$. Since $F' \cap G' \neq \emptyset$, we have $G' \cap \{1, 5\} \neq \emptyset$. This implies $\deg(1) \geqslant 3$ or $\deg(5) \geqslant 3$.

Next we prove $|\mathscr{F}| \geqslant 6$. Assume on the contrary that $|\mathscr{F}| \leqslant 5$. We choose $x \in X$ such that $\deg(x) \geqslant 3$. Thus the number of edges which do not contain x is at most 2. Let F and F' be such edges. Choose $y \in F \cap F'$. Then $\{x, y\}$ is a cover of $\mathscr{F}$, which contradicts $\tau(\mathscr{F}) = 3$. ∎

(6) $p_3(3) = 14$.

Proof. First we consider the case that there exist $F, F' \in \mathscr{F}$ such that $|F \cap F'| = 2$. Let $F = \{1, 2, 3\}$, $F' = \{1, 2, 4\}$, and $\mathscr{C} = \mathscr{C}_3(\mathscr{F})$. By (2) and (3), $|\mathscr{C}(1)| \leqslant 7$ and $|\mathscr{C}(2)| \leqslant 7$. Thus, since $F, F' \in \mathscr{C}(1) \cap \mathscr{C}(2)$, we have $|\mathscr{C}(1) \cup \mathscr{C}(2)| \leqslant 7 + 7 - 2 = 12$. Suppose $|\mathscr{C}| \geqslant 15$. Then $|\mathscr{C}(\bar{1}\bar{2})| \geqslant 3$. Every member of $\mathscr{C}(\bar{1}\bar{2})$ must meet F at $\{3\}$ and F' at $\{4\}$, and hence $\{3, 4, 5\}$, $\{3, 4, 6\}$, $\{3, 4, 7\} \in \mathscr{C}$. Since $\mathscr{F}(\bar{3}\bar{4}) \neq \emptyset$, we must have $\{5, 6, 7\} \in \mathscr{F}(\bar{3}\bar{4})$. But $F \cap \{5, 6, 7\} = \emptyset$, a contradiction.

Now we assume that $|F \cap F'| = 1$ holds for all distinct edges $F, F' \in \mathscr{F}$. Let $\mathscr{C} = \mathscr{C}_3(\mathscr{F})$. We may assume that $\deg(1) \geqslant 3$ (by (5)) and $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\} \in \mathscr{F}$. Note that if $F \in \mathscr{F}(\bar{1})$ then $F \in \binom{\{2,3\}}{1} \cup \binom{\{4,5\}}{1} \cup \binom{\{6,7\}}{1}$. Consequently, there exist no other edges containing 1, i.e., $\deg(1) = 3$. Hence by (5), we have $\mathscr{F}(\bar{1}) \geqslant 3$. Thus, we have $|\mathscr{C}(\bar{1})| \leqslant 2^3 - |\mathscr{F}(\bar{1})| \leqslant 5$. Therefore, $|\mathscr{C}| = |\mathscr{C}(1)| + |\mathscr{C}(\bar{1})| \leqslant 7 + 5 = 12$. ∎

It is not difficult to check $p_{t+1}(k+1) \geqslant (k+1)\, p_t(k)$ holds for $t < k$ and $p_{k+1}(k+1) \geqslant (k+1)\, p_k(k) + 1$ for $t = k$. Similarly $r(k+1) \geqslant (k+1)\, r(k) + 1$. This together with $r(2) = 3$, we obtain

(7) $r(k) \geqslant \lfloor k!(e-1) \rfloor$.

Actually, (7) was proved by Erdős and Lovász [2].

(8) *Let* $k > k_0(\tau)$, $|X| > n_0(k)$. *Suppose that* $\mathscr{F} \subset \binom{X}{k}$ *is an intersecting family with covering number* τ. *Then*, $|\mathscr{F}| \leqslant p_{\tau-1}(k)\binom{|X|-\tau}{k-\tau} + O(|X|^{k-\tau-1})$ *holds.*

The above claim is proved in [4] for $\tau = 4$. One can prove the general case in the same way.

2. A Counterexample to a Conjecture of Lovász

Erdős and Lovász [2] proved that the maximum size of k-uniform intersecting families with covering number k is at least $\lfloor k!(e-1) \rfloor$ and at most k^k. Lovász [10] conjectured that $\lfloor k!(e-1) \rfloor$ is the exact bound. This conjecture is true for $k = 2, 3$. However, for the case $k \geqslant 4$, this conjecture turns out to be false. In this section, we will construct k-uniform intersecting family with covering number k whose size is greater than $((k+1)/2)^{k-1}$.

The constructions are rather complicated, therefore we first give an outline of them. There is a particular element x_0 which will have the unique highest degree in general. We construct an intersecting family $\mathscr{G} \subset \binom{X-\{x_0\}}{k}$ with $\tau(\mathscr{G}) = \tau - 1$. ($\tau = k$ in the Erdős–Lovász case, and $\tau \leqslant k$ in general.) Next we define $\mathscr{B} := \{\{x_0\} \cup C : C \in \bigcup_{t=\tau-1}^{k-1} \mathscr{C}_t(\mathscr{G})\}$. Finally, the family $\mathscr{F}_0 = \mathscr{F}_0(k, \tau)$ is defined as $\mathscr{F}_0 := \mathscr{G} \cup \{F \in \binom{X}{k}, \exists B \in \mathscr{B}, B \subset F\}$. Now we give the two examples, according to the parity of τ.

Example 1 (The Case $\tau = 2s + 2$). Let $h = k - s$. First we define an infinite k-uniform family $\mathscr{G}^* = \mathscr{G}^*(h)$ as follows. Let

$$V(\mathscr{G}^*) := \{(2i, 2j) : i \in \mathbf{Z},\ 0 \leqslant j < h\} \cup \{(2i+1, 2j+1) : i \in \mathbf{Z},\ 0 \leqslant j < h\}.$$

We define a broom structure $\mathscr{G}_i$ as follows. A broom $\mathscr{G}_i$ has a broomstick

$$S_i := \{(i, j) : (i, j) \in V(\mathscr{G}^*)\},\ (|S_i| = h)$$

and tails

$$\mathscr{T}_i := \{\{(i, j_0), (i+1, j_1), (i+2, j_2), \ldots, (i+s, j_s)\} : j_{t+1} - j_t \in \{1, -1\} \text{ for } 0 \leqslant \forall t < s\}$$

where

$$j_0 := \begin{cases} h & \text{if } \ h+i \text{ is even} \\ h-1 & \text{if } \ h+i \text{ is odd.} \end{cases}$$

Set $\mathcal{G}_i := \{S_i \cup T : T \in \mathcal{T}_i\}$. Note that $\mathcal{G}_i$ is a k-uniform family with size $|\mathcal{T}_i| = 2^s$. Now define $\mathcal{G}^* := \bigcup_{i \in \mathbf{Z}} \mathcal{G}_i$.

Next we define an equivalence relation $R(s)$ on $V(\mathcal{G}^*)$ induced by

$$(i, j) \equiv (i + 2s + 1,\ 2h - 1 - j) \text{ for all } i \in \mathbf{Z} \text{ and } 0 \leqslant j \leqslant 2h - 1.$$

Note that this equivalence transforms the infinite tape into a Möbius band. Finally, we define $\mathcal{G}$ as a quotient family of $\mathcal{G}^*$ by $R(s)$, that is, $\mathcal{G} := \mathcal{G}^*/R(s)$. Note that $|V(\mathcal{G})| = (2s+1)h$.

Example 2 (The Case $\tau = 2s+1$). Let $h = k - s$, and

$$\begin{aligned} V(\mathcal{G}) := {} & \{(2i, 2j) : i \in \mathbf{Z}_{2s},\ 0 \leqslant i < s,\ 0 \leqslant j \leqslant h\} \\ & \cup \{(2i+1, 2j+1) : i \in \mathbf{Z}_{2s},\ 0 \leqslant i < S,\ 0 \leqslant j \leqslant h\} \\ & - \{(2i, 0) : i \in \mathbf{Z}_{2s},\ s \leqslant 2i < 2s\} \\ & - \{(2i+1, 2h+1) : i \in \mathbf{Z}_{2s},\ s \leqslant 2i+1 < 2s\} \end{aligned}$$

Note that $|V(\mathcal{G})| = s(2h+1)$. We define a broom structure $\mathcal{G}_i$ as follows. A broom $\mathcal{G}_i$ has a broomstick

$$S_i := \{(i, j) : (i, j) \in V(\mathcal{G})\},$$

$$(|S_0| = \cdots = |S_{s-1}| = h+1,\ |S_s| = \cdots = |S_{2s-1}| = h)$$

and tails

$$\mathcal{T}_i := \{\{(i, j_0), (i+1, j_1), (i+2, j_2), ..., (i+u, j_u)\} : j_{t+1} - j_t \in \{1, -1\} \text{ for } 0 \leqslant \forall t < u\}$$

where

$$u := \begin{cases} s-1 & \text{if } \ i \in \{0, 1, ..., s-1\} \pmod{2s} \\ s & \text{if } \ i \in \{s, s+1, ..., 2s-1\} \pmod{2s}, \end{cases}$$

and

$$j_0 := \begin{cases} h & \text{if } \ h+i \text{ is even} \\ h+1 & \text{if } \ h+i \text{ is odd.} \end{cases}$$

Set $\mathcal{G}_i := \{S_i \cup T : T \in \mathcal{T}_i\}$, and define $\mathcal{G} := \bigcup_{0 \leqslant i < 2s} \mathcal{G}_i$.

Remark 1. In both examples, any edge of type $\{x_0, x_1, ..., x_{\tau-2}\}$ $(x_j \in S_j$ for all $0 \leqslant j \leqslant \tau - 2)$ is a cover of $\mathcal{G}$. This implies that $|\mathcal{C}_{\tau-1}(\mathcal{G})| \geqslant \prod_{i=0}^{\tau-2} |S_i|$.

Now we check that the above constructions satisfy the required conditions. It is easy to see that the family $\mathscr{G}$ is intersecting. But $\tau(\mathscr{G}) = \tau - 1$ is not trivial. We only prove the case $\tau = 2s+2$, because the proof for the case $\tau = 2s+1$ is very similar.

Let us consider properties of covers of $\mathscr{T}_0$. Define $I_t := \bigcup_{T \in \mathscr{T}_0} (S_t \cap T)$, $J_t := \bigcup_{l=0}^{t} I_l$, and fix a cover $C \in \mathscr{C}(\mathscr{T}_0)$. A vertex $y_i \in S_i$ is called suspicious (under C) if there exists $T = \{y_0, y_1, ..., y_s\} \in \mathscr{T}_0$ ($y_j \in S_j$ for all $0 \leqslant j \leqslant s$) such that $\{y_0, y_1, ..., y_i\} \cap C = \emptyset$. Let $L = L(C)$ be the set of all suspicious vertices.

Let us start with a trivial but useful fact.

Claim 1. If $C \cap I_{i+1} = \emptyset$ then $|L \cap I_{i+1}| \geqslant |L \cap I_i| + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on I_i.

The following fact is easily proved by induction on i.

Claim 2. Let $a = |C \cap I_i|$. Suppose that $|C \cap J_l| \leqslant l$ for all $0 \leqslant l < i$. Then $|L \cap I_i| \geqslant i - a + 1$ and equality holds only if $L \cap I_i$ consists of consecutive vertices on I_i.

The following is a direct consequence of the above fact.

PROPOSITION 1. *Suppose that $|C \cap J_l| \leqslant l$ for all $0 \leqslant l < i$ and $L \cap I_i = \emptyset$. Then $|C \cap J_i| \geqslant i+1$ and equality holds only if $C \cap I_i$ consists of consecutive vertices on I_i.*

PROPOSITION 2. $\tau(\mathscr{G}) = 2s+1$.

Proof. Let C be any cover for $\mathscr{G}$. For each $0 \leqslant i \leqslant 2s$, we define the interval $W_i = [i, i+r] \pmod{2s+1}$ so that r is the minimum non-negative integer satisfying $|C \cap (S_i \cup S_{i+1} \cup \cdots \cup S_{i+r})| \geqslant r+1$. In fact, such an integer r exists by Proposition 1. The following claim can be shown easily.

Claim 3. If W_i and W_j have non-empty intersection, then $W_i \subset W_j$ or $W_j \subset W_i$ holds.

Using this, we can choose disjoint intervals from $W_0, W_1, ..., W_{2s}$ whose union is exactly $[0, 2s]$. And so, $|C| \geqslant 2s+1$. This completes the proof of $\tau(\mathscr{G}) = 2s+1$. ∎

Now we know that $\mathscr{F}_0 := \mathscr{G} \cup \{F \in \binom{X}{k}, \exists B \in \mathscr{B},\ B \subset F\}$ is intersecting, and $\tau - 1 \leqslant \tau(\mathscr{F}_0) \leqslant \tau$. We can check that $\tau(\mathscr{F}_0) = \tau$ using the following easy fact.

PROPOSITION 3. *Let $\mathscr{G} \subset \binom{X - \{x_0\}}{k}$ be an intersecting family with $\tau(\mathscr{G}) = \tau - 1$. Define $\mathscr{B} := \{\{x_0\} \cup C : C \in \bigcup_{t=\tau-1}^{k-1} \mathscr{C}_t(\mathscr{G})\}$, $\mathscr{F} := \mathscr{G} \cup \{F \in \binom{X}{k}$,*

$\exists B \in \mathscr{B}, B \subset F\}$. *Then* $\tau(\mathscr{F}) = \tau$ *if and only if for all* $C \in \mathscr{C}_{\tau-1}(\mathscr{G})$ *there exists* $C' \in \mathscr{C}_{\tau-1}(\mathscr{G})$ *such that* $C \cap C' = \varnothing$.

Lovász conjectured that $r(k) = \lfloor k!(e-1) \rfloor < e^2((k+1)/e)^{k+1}$. Our construction beats this conjecture as follows. Let $\mathscr{G}$ be a k-uniform intersecting family defined in Example 1 or Example 2. Then $\tau(\mathscr{G}) = k$. By Remark 1, we have the following lower bound.

THEOREM 1.

$$r(k) > |\mathscr{C}_{k-1}(\mathscr{G})| > \begin{cases} \left(\dfrac{k}{2}+1\right)^{k-1} & \text{if } k \text{ is even,} \\ \left(\dfrac{k+3}{2}\right)^{(k-1)/2} \left(\dfrac{k+1}{2}\right)^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Thus, our construction is exponentially larger than Erdős–Lovász construction.

3. Open Problems

Problem 1. Determine the maximum size of 4-uniform intersecting families with covering number four. Does $r(4) = 42$ hold?

Problem 2. Determine $p_3(k)$ for $4 \leqslant k \leqslant 8$. Does $p_3(k) = k^3 - 3k^2 + 6k - 4$ hold in these cases?

Conjecture 1. Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family with covering number τ. If $k > k_0(\tau)$, $|X| > n_0(k)$, then we have $|\mathscr{F}| \leqslant (k^{\tau-1} - \binom{\tau-1}{2} k^{\tau-2} + c(k, \tau)) \binom{|X|-\tau}{k-\tau} + O(|X|^{k-\tau-1})$, where $c(k, \tau)$ is a polynomial of k and τ, and the degree of k is at most $\tau - 3$.

Using (8), the above conjecture would follow from the following conjecture by setting $\tau = t + 1$.

Conjecture 2. Let $k \geqslant k_0(t)$. Then $p_t(k) = k^t - \binom{t}{2} k^{t-1} + O(k^{t-2})$ holds. ∎

This conjecture is true for $t \leqslant 5$ [5]. It seems that the coefficient of k^{t-2} in the above conjecture is $(t/4)\lfloor (t+1)(t^2 - 4t + 7)/2 \rfloor$.

For the case $\tau = k$, we conjecture the following.

Conjecture 3. For some absolute constant $\frac{1}{2} \leqslant \mu < 1$, $r(k) < (\mu k)^k$ holds. ∎

We close this section with a bold conjecture.

Conjecture 4. Let $k \geqslant \tau \geqslant 4$ and $n > n_0(k)$. Let $\mathscr{F}_0$ be the family defined in Example 1 or Example 2. Suppose that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with covering number τ, then $|\mathscr{F}| \leqslant |\mathscr{F}_0|$ holds. Equality holds if and only if $\mathscr{F}$ is isomorphic to $\mathscr{F}_0$. ∎

This conjecture is true if "$k \geqslant 4$ and $\tau = 2$ [9]," or "$k \geqslant 4$ and $\tau = 3$ [3]," or "$k \geqslant 9$ and $\tau = 4$ [4]." (Inequality holds even if "$k = 3$ and $\tau = 2$," or "$k = 3$ and $\tau = 3$," but the uniqueness of the extremal configuration does not hold in these cases.) Of course, this conjecture is much stronger than Conjecture 1. Note that for $k = \tau$ this conjecture would give the solution to the problem of Erdős–Lovász, and in particular, it would show that the answer to Problem 1 is 42.

4. Appendix

4.1. *Numerical Data*

The following is a table of the size of k-uniform intersecting families with covering number k, i.e., known lower bounds for $r(k)$.

k	Erdős–Lovász construction	Example 1, Example 2
2	3	3
3	10	10
4	41	42
5	206	228
6	1, 237	1, 639
7	8, 660	13, 264
8	69, 281	128, 469
9	623, 530	1, 327, 677
10	6, 235, 301	15, 962, 373
11	68, 588, 312	202, 391, 317
12	823, 059, 745	2, 942, 955, 330
13	10, 699, 776, 686	44, 744, 668, 113
14	149, 796, 873, 605	770, 458, 315, 037
15	2, 246, 953, 104, 076	13, 752, 147, 069, 844
16	35, 951, 249, 665, 217	274, 736, 003, 372, 155

4.2. $k = \tau = 3$

The maximum size of 3-uniform intersecting families with covering number 3 is 10, i.e., $r(3) = 10$. There are 7 configurations which attain the maximum. The following is the list of these extremal configurations.

(#1)	(#2)	(#3)	(#4)
123	123	123	123
12 4	12 4	12 4	12 4
12 5	12 5	12 5	12 5
345	345	345	345
1 34	1 3 6	1 3 6	1 34
1 3 5	1 4 6	1 4 6	1 4 6
1 45	1 56	1 56	1 56
234	23 6	1 34	23 5
23 5	2 4 6	23 6	23 6
2 45	2 56	2 4 6	2 45

(#5)	(#6)	(#7)
123	123	123
12 4	12 4	12 4
12 5	12 5	12 5
345	345	345
1 34	1 34	1 34
1 3 5	1 3 6	1 3 6
1 56	1 56	1 4 7
23 5	23 5	234
2 45	23 6	23 7
23 6	2 4 6	2 4 6

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