

Exponents of Uniform L -Systems

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We have determined all exponents of (n, k, L) -systems of $k \leq 12$ except for essentially two cases, which are related to the Steiner systems $S(11, 5, 4)$ and $S(12, 6, 5)$. This requires several new constructions. Also some refinements of the previous methods are necessary to get suitable upper bounds. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let $L \subset [0, k-1]$. A family \mathcal{F} is called an (n, k, L) -system (or (k, L) -system, L -system for short) if $\mathcal{F} \subset \binom{[n]}{k}$ and $|F \cap F'| \in L$ holds for all distinct $F, F' \in \mathcal{F}$. Define

$$m(n, k, L) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } (n, k, L)\text{-system}\}.$$

If there exist constants c, c', α depending only on k and L , and satisfying

$$cn^\alpha < m(n, k, L) < c'n^\alpha,$$

then we define $\alpha(k, L) := \alpha$ which is called the exponent of (k, L) -system.

Conjecture 1. For all k and L , $\alpha(k, L)$ exists.

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THEOREM 1 [7]. *For every rational number $q \geq 1$ there are infinitely many choices of k and L such that $\alpha(k, L) = q$.*

In [5], Frankl determined almost all exponents for $k \leq 7$. In this paper, we will determine all exponents for $k \leq 10$. For $k = 11, 12$, we cannot determine $\alpha(11, \{0, 1, 2, 3, 5\})$ and $\alpha(12, \{0, 1, 2, 3, 4, 6\})$. These two cases are related to the Steiner systems $S(11, 5, 4)$ and $S(12, 6, 5)$. Except essentially these two cases, we will determine all exponents for $k \leq 12$.

Let us summarize the contents of this paper. In this section, we introduce the notion of an intersection structure, and the rank of an intersection structure. In Section 2, we list known reductions and constructions related to $\alpha(k, L)$. In Section 3, we discuss the cases in which we cannot determine $\alpha(k, L)$ using exclusively results in Section 2, and we present new constructions and methods which give bounds for the exponents. Intersection structures will play an important role in Section 3. The table of exponents is presented in the last section.

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and an edge $F \in \mathcal{F}$ define

$$\mathcal{I}(F, \mathcal{F}) := \{F \cap F' : F' \in \mathcal{F} - \{F\}\}.$$

Füredi proved the following fundamental result.

THEOREM 2 [10]. *Given $k, t \geq 2$, there exists a constant $c = c(k, t) > 0$ such that every family $\mathcal{F} \subset \binom{[n]}{k}$ contains a k -partite subfamily $\mathcal{F}^* \subset \mathcal{F}$ with k -partition $[n] = X_1 \cup \dots \cup X_k$ satisfying (1)–(3). For $F \subset [n]$ define the projection $\pi(F)$ of F by $\pi(F) := \{i : F \cap X_i \neq \emptyset\}$. For a family \mathcal{A} set $\pi(\mathcal{A}) := \{\pi(A) : A \in \mathcal{A}\}$.*

- (1) $|\mathcal{F}^*| > c |\mathcal{F}|$.
- (2) For any two edges $F_1, F_2 \in \mathcal{F}^*$, $\pi(\mathcal{I}(F_1, \mathcal{F}^*)) = \pi(\mathcal{I}(F_2, \mathcal{F}^*))$.
- (3) $\mathcal{I}(F, \mathcal{F}^*)$ is closed under intersection, i.e., $A, B \in \mathcal{I}(F, \mathcal{F}^*)$ implies $A \cap B \in \mathcal{I}(F, \mathcal{F}^*)$.

We call $\pi(\mathcal{I}(F, \mathcal{F}^*))$ the intersection structure (IS) of \mathcal{F} and denote it by $\text{IS}(\mathcal{F})$ (it may depend on the particular choice of \mathcal{F}^* but it does not influence our investigations in the sequel). If \mathcal{F} is an L -system then $\text{IS}(\mathcal{F})$ is (non-uniform) L -system closed under intersection, too. For an IS $\mathcal{I} \subset 2^{[k]}$, we define its rank by

$$\text{rank}(\mathcal{I}) := \min \left\{ t : \Delta_t(\mathcal{I}) \neq \binom{[k]}{t} \right\},$$

where Δ_t denotes the t -th shadow, i.e., $\Delta_t(\mathcal{I}) := \{J \in \binom{[k]}{t} : J \subset I \text{ for some } I \in \mathcal{I}\}$. Note that $\text{rank}(\mathcal{I})$ is always an integer.

The following statement is an immediate consequence of Theorem 2. (cf. [12])

THEOREM 3 [12]. *If \mathcal{F} is an (n, k, L) -system then*

$$|\overline{\mathcal{F}}| \leq n^\alpha,$$

where $\alpha := \text{rank}(\text{IS}(\overline{\mathcal{F}}))$.

Proof. Take some $A \in \binom{[k]}{\alpha}$ such that $A \notin \Delta_\alpha(\mathcal{J})$. Then for every set B satisfying $\pi(B) = A$ there is at most one member F of the family \mathcal{F}^* such that $B \subset F$ holds. ■

Define

$$\text{rank}(k, L) := \max\{\text{rank}(\mathcal{J}) : \mathcal{J} \subset 2^{[k]} \text{ is a closed } L\text{-system}\}.$$

Using this notation, the above inequality is restated as follows.

Reduction 1 [12].

$$\alpha(k, L) \leq \text{rank}(k, L). \tag{1}$$

Conjecture 2 [12]. $\text{rank}(k, L) - 1 < \alpha(k, L)$.

The above conjecture is true if $\alpha = 2$ (see [10]). Initially we started our research on this topic in order to disprove this conjecture, but so far our results verified it for $k \leq 10$ and also for almost all cases if $k = 11$ or 12 .

In the sequel, we often write $\alpha(9, 025)$ or $\text{rank}(9, 025)$ instead of $\alpha(9, \{0, 2, 5\})$ or $\text{rank}(9, \{0, 2, 5\})$.

2. REDUCTIONS AND CONSTRUCTIONS

Let $L = \{l_0, l_1, \dots, l_s\}$, $0 \leq l_0 < l_1 < \dots < l_s < k$. Define $L - l := \{l_0 - l, l_1 - l, \dots, l_s - l\} \cap [0, \infty]$, $L/d := \{l_0/d, l_1/d, \dots, l_s/d\}$, and $A + B := \{a + b : a \in A, b \in B\}$.

Reduction 2 [5]

$$\alpha(k, L) = \alpha(k - l_0, L - l_0). \tag{2}$$

From now on, we may assume that $l_0 = 0$.

Reduction 3 [10].

$$\alpha(k, L) = \alpha(k/d, L/d) \quad \text{if } d = \gcd\{l_0, \dots, l_s\}. \tag{3}$$

It is not hard to see that

Reduction 4.

$$\alpha(k, [0, t-1]) = t. \quad (4)$$

Very little is known about the exact value of $m(n, k, L)$ in general. Let us mention that Rödl proved the following much stronger statement.

THEOREM 4 [13].

$$m(n, k, [0, t-1]) = (1 - o(1)) \frac{\binom{n}{t}}{\binom{k}{t}}.$$

Frankl and Füredi extended (4) in the following way:

Reduction 5 [8]. Let $L = [0, t_1 - 1] \cup [k - t_2, k - 1]$, $t_1 + t_2 < k$. Then

$$\alpha(k, L) = \max\{t_1, t_2\}. \quad (5)$$

2.1. Lower Bounds

Reduction 6. For $L' \subset L$

$$\alpha(k, L) \geq \alpha(k, L'). \quad (6)$$

Reduction 7. If $k = k_1 + k_2$ and $L \cup \{k\} \supset (L_1 \cup \{k_1\}) + (L_2 \cup \{k_2\})$, then

$$\alpha(k, L) \geq \alpha(k_1, L_1) + \alpha(k_2, L_2). \quad (7)$$

EXAMPLE 1. $\alpha(9, 012456) \geq \alpha(4, 01) + \alpha(5, 01) = 2 + 2 = 4$.

Reduction 8 [1].

$$\alpha(k, L) \geq 2 \quad \text{if} \quad k = \sum_{l_i \in L} a_i l_i, \quad (8)$$

where a_i is a non-negative integer.

EXAMPLE 2. $\alpha(9, 0258) \geq 2$. (Indeed $9 = 0 \cdot 0 + 2 \cdot 2 + 1 \cdot 5 + 0 \cdot 8$.)

2.2. Upper Bounds

Reduction 9 [3].

$$\alpha(k, L) \leq |L|. \quad (9)$$

Reduction 10 [3].

$$\alpha(k, L) \leq |L| - 1 \tag{10}$$

unless both

$$(l_1 - l_0) |(l_2 - l_1)| \cdots |(l_s - l_{s-1})| (k - l_s), \tag{11}$$

and

$$\frac{l_2 - l_1}{l_1 - l_0} \leq \frac{l_3 - l_2}{l_2 - l_1} \leq \cdots \leq \frac{l_s - l_{s-1}}{l_{s-1} - l_{s-2}} \leq \frac{k - l_s}{l_s - l_{s-1}} \tag{12}$$

hold.

Reduction 11 [4].

$$\begin{aligned} \alpha(k, L) &\leq \max\{\alpha(k, L - \{l\}), \alpha(l, L \cap [0, l-1]) + \alpha(k-l, L-l)\} \\ &\text{for all } 0 < l \in L. \end{aligned} \tag{13}$$

Reduction 12 [5].

$$\begin{aligned} \alpha(k, L) &\leq \max\{\alpha(k, L - \{l\}), (k-l) + \alpha(l, L \cap (L - (k-l)))\} \\ &\text{for all } 0 < l \in L. \end{aligned} \tag{14}$$

Reduction 13 [9]. Suppose q is a power of the prime p . Let μ_0, \dots, μ_s be distinct residues modulo q . Suppose $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $k \equiv \mu_0 \pmod{q}$, and if $F, F' \in \mathcal{F}$, $F \neq F'$ then $|F \cap F'| \equiv \mu_i \pmod{q}$ for some $1 \leq i \leq s$. If there exists an integer-valued polynomial $g(x)$ of degree d such that $p \nmid g(k)$ but $p \mid g(x)$ for $x \equiv \mu_i \pmod{q}$, $i = 1, \dots, s$, then

$$|\mathcal{F}| \leq \binom{n}{d}. \tag{15}$$

EXAMPLE 3. $\alpha(10, 0347) \leq 2$. Take $p = 2$, $q = 4$, $g(x) = \binom{x+1}{2}$.

2.3. Application

THEOREM 5. *Suppose $L \subset [0, k-3]$ and define $L' := L \cup \{k-1\}$. Then $\alpha(k, L') = \max\{\alpha(k, L), 1 + \alpha(k-1, L \cap (L-1))\}$.*

Proof. By Reduction 12, we have

$$\alpha(k, L') \leq \max\{\alpha(k, L), 1 + \alpha(k-1, L' \cap (L'-1))\}.$$

If $\alpha(k, L) \geq 1 + \alpha(k-1, L' \cap (L' - 1))$ then using Reduction 6, we have

$$\alpha(k, L) \leq \alpha(k, L') \leq \alpha(k, L).$$

Now suppose $\alpha(k, L) < 1 + \alpha(k-1, L' \cap (L' - 1))$. Using Reduction 7, we have

$$\alpha(k, L') \leq \alpha(1, \{0\}) + \alpha(k-1, L' \cap (L' - 1)) \leq \alpha(k, L').$$

By the definition of L' , we have $L' \cap (L' - 1) = L \cap (L - 1)$. Thus, it follows that $\alpha(k, L') = 1 + \alpha(k-1, L \cap (L - 1))$ in this case. ■

Conjecture 3. Suppose $L \subset [0, k-3]$ and define $L' := L \cup \{k-1\}$. Then $\alpha(k, L') = \alpha(k, L)$ holds.

In view of our results this conjecture is true for $k \leq 10$ and also in almost all cases for $k = 11, 12$ as well (cf. the tables at the end of the paper).

2.4. Constructions Using Finite Geometries

Construction 1. Let \mathcal{F} be the set of m -dimensional affine subspaces of a d -dimensional vector space over $GF(q)$, and let $n = q^d$. Then \mathcal{F} is an $(n, q^m, \{0, 1, q, q^2, \dots, q^{m-1}\})$ -system of size $\Theta(n^{m+1})$.

EXAMPLE 4. $\alpha(8, 0124) = 4, \alpha(9, 013) = 3$.

Construction 2. Let \mathcal{F} be the set of m -dimensional projective subspaces of a d -dimensional vector space over $GF(q)$, and let $n = q^d$. Then \mathcal{F} is an $(n, (q^{m+1} - 1)/(q - 1), \{0, 1, q + 1, q^2 + q + 1, \dots, (q^m - 1)/(q - 1)\})$ -system of size $\Theta(n^{m+1})$.

EXAMPLE 5. $\alpha(7, 013) = 3$. (Set $q = m = 2$.)

Using quadratic forms in projective spaces the following result was obtained in [4].

THEOREM 6. $\alpha(10, 0124) = 4, \alpha(11, 012357) = 5$.

3. EXPONENTS BEYOND PREVIOUS REDUCTIONS AND CONSTRUCTIONS

Let \mathcal{F} be an (n, k, L) -system and set $\mathcal{I} := \text{IS}(\mathcal{F})$. Define the generator set \mathcal{I}^* of \mathcal{I} by

$$\mathcal{I}^* := \{I \in \mathcal{I} : \exists I' \in \mathcal{I} \text{ such that } I \subset I', I \neq I'\}.$$

3.1. $\alpha(12, 0145) = 2.5$

By computer search, we found that $\text{rank}(12, 0145) = 3$ and there are two ISs \mathcal{I}_1 and \mathcal{I}_2 of rank 3, with corresponding generator sets

$$\mathcal{I}_1^* = \{xY, yZ, zX\},$$

$$\mathcal{I}_2^* = \{X, xY, xZ, yZ\},$$

where $X \cup Y \cup Z = [12]$, $|X| = |Y| = |Z| = 4$, $xY := \{\{x\} \cup Y : x \in X\}$ etc.

THEOREM 7. *Let $a \geq 3$ and \mathcal{F} be a $(3an, 3a, \{0, 1, a, a+1\})$ -system. Let $X \cup Y \cup Z = [3a]$, $|X| = |Y| = |Z| = a$. Suppose that $\text{IS}(\mathcal{F})$ is generated by*

$$\mathcal{I}_1^* = \{xY, yZ, zX\},$$

or

$$\mathcal{I}_2^* = \{X, xY, xZ, yZ\},$$

where $xY := \{\{x\} \cup Y : x \in X\}$ etc. Then

$$|\mathcal{F}| = O(n^{2.5}).$$

Proof of Theorem 7. By using Theorem 2, we may assume that \mathcal{F} is $3a$ -partite with partition $[3an] = V_1 \cup \dots \cup V_{3a}$, $|V_1| = \dots = |V_{3a}| = n$. For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$ and $\mathcal{F}|_A := \{F \cap \tilde{A} : F \in \mathcal{F}\}$. For $G \subset \tilde{Y}$, define

$$N(G) := \left\{ \bigcup F : F \subset \tilde{X}, F \cup G \in \mathcal{F}|_{X \cup Y} \right\}.$$

LEMMA 1. *If $G, G' \subset \tilde{Y}$, $|G \cap G'| = 1$ then $N(G) \cap N(G') = \emptyset$.*

LEMMA 2. *For all $y \in \tilde{Y}$, $\sum_{y \in G} |N(G)| \leq |V_1| = n$.*

LEMMA 3. $\#\{G \subset \tilde{Y} : |N(G)| > \sqrt{n}\} < n^{1.5}$.

Proof. Let $\mathcal{G} := \{G \subset \tilde{Y} : |N(G)| > \sqrt{n}\} = \{G_1, \dots, G_q\}$. Suppose that $q \geq n^{1.5}$. For $y \in \tilde{Y}$ set $\text{deg}(y) := \#\{G \in \mathcal{G} : y \in G\}$. Then it follows that

$$n \geq \sum_{y \in G \in \mathcal{G}} |N(G)| > \sqrt{n} \text{deg}(y)$$

for every fixed $y \in \tilde{Y}$. Thus we have

$$aq = \sum_{y \in \tilde{Y}} \deg(y) < |\tilde{Y}| \sqrt{n} = an^{1.5},$$

that is $q < n^{1.5}$, a contradiction. \blacksquare

Since any two edges in \mathcal{F} intersect in at most $a + 1$ vertices, one has

$$|\mathcal{F}| = \#\{G \cup H \in \mathcal{F} \mid_{Y \cup Z}: G \subset \tilde{Y}, H \subset \tilde{Z}\}.$$

Define

$$\mathcal{F}_0 = \{F \in \mathcal{F} : F \cap \tilde{Y} \in \mathcal{G}\}.$$

By the ISs, any two edges in \mathcal{F} which intersect in a vertices on \tilde{Y} are disjoint on \tilde{Z} . Thus, we have

$$|\mathcal{F}_0| \leq |\mathcal{G}| |V_{3a}| < n^{2.5}.$$

Let $\mathcal{F}_1 := \mathcal{F} - \mathcal{F}_0$. Then for all $G \in \mathcal{F}_1 \mid_Y$ we have $|N(G)| \leq \sqrt{n}$. Fix $\mathcal{F}_1 \mid_X$ is an $(a, 01)$ -system, it follows that

$$\#\{F \subset \tilde{X} : F \cup G \in \mathcal{F} \mid_{X \cup Y}\} \leq m(|N(G)|, a, 01) \leq |N(G)|^2.$$

Thus, we have

$$\begin{aligned} |\mathcal{F}_1| &= \sum_{G \in \mathcal{F}_1 \mid_Y} \#\{F \subset \tilde{X} : F \cup G \in \mathcal{F} \mid_{X \cup Y}\} \\ &\leq \sum_{G \in \mathcal{F}_1 \mid_Y} |N(G)|^2 \\ &\leq \sqrt{n} \sum_{G \in \mathcal{F}_1 \mid_Y} |N(G)| \\ &= \sqrt{n} \sum_{y \in \tilde{Y}} \left(\sum_{y \in G} |N(G)| \right) \\ &\leq \sqrt{n} |\tilde{Y}| n = an^{2.5}. \end{aligned}$$

Consequently we have

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \leq (1 + a)n^{2.5},$$

which completes the proof of Theorem 7. \blacksquare

THEOREM 8. *If $a \geq 4$ then the IS of $(3a, \{0, 1, a, a + 1\})$ -system of rank 3 is \mathcal{I}_1 or \mathcal{I}_2 .*

Actually, this was our conjecture, but Füredi gave us a proof. The following proof is based on his idea.

Proof of Theorem 8. For $a=4, 5$, we can verify this fact directly by computer search. Let us assume $a \geq 6$. Let \mathcal{I} be an IS of $(3a, \{0, 1, a, a+1\})$ -system of rank 3.

LEMMA 4. *If $I_1, I_2 \in \mathcal{I}$, $|I_1| = |I_2| = a$, then $I_1 \cap I_2 = \emptyset$.*

Proof. Assume, on the contrary, that $I_1 \cap I_2 = \{w\}$. Let $I_1 = \{u_1, \dots, u_{a-1}, w\}$ and $I_2 = \{v_1, \dots, v_{a-1}, w\}$. For every $i, 1 \leq i < a$, we can find $J_i \in \mathcal{I}$ such that $\{u_i, v_i\} \subset J_i$, $a \leq |J_i| \leq a+1$. If $|I_1 \cap J_i| = a$ then $J_i = I_1 \cup \{v_i\}$ and $|I_2 \cap J_i| = 2$. This is a contradiction. Thus $|I_1 \cap J_i| = |I_2 \cap J_i| = 1$ must hold. Define $J'_i := J_i - \{u_i, v_i\} \subset [3a] - (I_1 \cup I_2)$. Then $a-2 \leq |J'_i| \leq a-1$. Thus, $|J'_1 \cup J'_2 \cup J'_3| \geq 3(a-2) - 3 = 3a-9$. On the other hand, $|\cup J'_i| \leq 3a - (2a-1) = a+1$. Therefore, we have $a+1 \geq 3a-9$, i.e., $a \leq 5$, a contradiction. ■

Define $\mathcal{I}' := \{I \in \mathcal{I} : |I| = a\}$, $\mathcal{J} := \{J \in \mathcal{I} : |J| = a+1\}$. By the above lemma, edges in \mathcal{I}' are disjoint.

Case 1. $\mathcal{I}' = \emptyset$. In this case, two edges in \mathcal{I} meet at most one vertex. Since $\text{rank}(\mathcal{I}) = 3$, every pair (two element set) in $[3a]$ is covered (contained) by an edge in \mathcal{I} . Thus, we need at least $\binom{3a}{2} / \binom{a+1}{2} \geq 4$ edges in \mathcal{I} . This implies $|\cup_{J \in \mathcal{I}} J| \geq 4(a+1) - \binom{4}{2} > 3a$, a contradiction.

Case 2. $\mathcal{I}' = \{I_1\}$. Define $\mathcal{J}' := \{J \in \mathcal{I} : I_1 \not\subset J\}$. To cover all pairs in $[3a] - I_1$, we need at least $\binom{2a}{2} / \binom{a+1}{2} \geq 3$ edges in \mathcal{J}' . Then $|\cup_{J \in \mathcal{J}'} J - I_1| \geq 3a - 3 > 2a$, a contradiction.

Case 3. $\mathcal{I}' = \{I_1, I_2\}$. Choose $u, v \in [3a] - (I_1 \cup I_2)$, and $J_1 \in \mathcal{I}$ containing both u and v . We may assume $|I_1 \cap J_1| = 1$, $|I_2 \cap J_1| \leq 1$. Choose $w \in I_1 - J_1$ and $J_2 \in \mathcal{I}$ containing both u and w . Then $|J_1 \cap J_2| = |I_1 \cap J_2| = 1$, $|I_2 \cap J_2| \leq 1$. Thus, $|I_1 \cup I_2 \cup J_1 \cup J_2| \geq a + a + (a+1) + (a+1) - 5 > 3a$, a contradiction.

Case 4. $|\mathcal{I}'| = 3$. In this case, it is not difficult to check that $\mathcal{I} = \mathcal{I}_1$ or \mathcal{I}_2 and we leave this task to the reader. This completes the proof of Theorem 8. ■

Remark 1. For $a=3$, there are (exactly) three ISs of $(9, 0134)$ -system of rank 3, i.e., $\mathcal{I}_1, \mathcal{I}_2$, and the Steiner system $S(9, 3, 2)$. This Steiner system is a $(9, 013)$ -system of rank 3. It is worth noting that $\alpha(9, 0134) = 3$ and this exponent can be achieved only if the IS is $S(9, 3, 2)$. This means that a family whose IS has the highest rank can not necessarily achieve the (highest) exponent. Actually, an $(n, 9, 0134)$ -system whose IS is \mathcal{I}_1 or \mathcal{I}_2 has only $O(n^{2.5})$ edges.

Construction 3. Let p be a prime. We will construct a $(kp, k, 01)$ -system \mathcal{L} of size p^2 . \mathcal{L} will be k -partite with partition $V_1 \cup \dots \cup V_k$, $V_i = [(i-1)p, ip-1]$. Define

$$L_j^i := (j, j+i, j+2i, \dots, j+(k-1)i)$$

and set $\mathcal{L}^i := \{L_j^i : j \in V_1\}$, and $\mathcal{L} := \bigcup_{0 \leq i < p} \mathcal{L}^i$. (All calculations are performed module p .) Then \mathcal{L}^i is a $(k, \{0\})$ -system and \mathcal{L} is a $(k, 01)$ -system. The sizes are $|\mathcal{L}^i| = p$, $|\mathcal{L}| = p^2$.

Construction 4. Choose a prime p between $n/2k$ and n/k . Then we can construct an $(n, k, 01)$ -system \mathcal{F} of size $\Theta(n^2)$ using Construction 3. Further, we can divide \mathcal{F} as

$$\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{cn}$$

so that every \mathcal{F}_i is an $(n, k, \{0\})$ -system of size $\Theta(n)$.

Construction 5. Let us construct a $(kn, k, 01)$ -system \mathcal{F} of size $\Theta(n^{1.5})$. \mathcal{F} will be k -partite with partition $V_1 \cup \dots \cup V_k$, $|V_1| = \dots = |V_k| = n$. For each i , divide $V_i = \bigcup_{j=1}^{\sqrt{n}} V_i^j$ where $|V_i^1| = \dots = |V_i^{\sqrt{n}}| = \sqrt{n}$. Define $W^j := \bigcup_{i=1}^k V_i^j$ ($|W^j| = k\sqrt{n}$). Let $\mathcal{F}_j \subset \binom{W^j}{k}$ be a $(k\sqrt{n}, k, 01)$ -system of size $\Theta(n)$. Finally define

$$\mathcal{F} := \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{\sqrt{n}}.$$

Then \mathcal{F} is an $(n, k, 01)$ -system of size $\Theta(n^{1.5})$. If $F \in \mathcal{F}_i$, $F' \in \mathcal{F}_j$, $i \neq j$, then clearly F and F' are disjoint.

Construction 6. Let us construct a $(3an, 3a, \{0, 1, a, a+1\})$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^{2.5})$ and $\text{IS}(\mathcal{F})$ is generated by \mathcal{I}_1^* (see Theorem 7). Let $[3an] = V_1 \cup \dots \cup V_{3a}$, $|V_1| = \dots = |V_{3a}| = n$, $[3a] = X \cup Y \cup Z$, $|X| = |Y| = |Z| = a$. For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$.

First we construct an $(an, a, 01)$ -system \mathcal{A} on \tilde{X} using Construction 5. Then

$$\mathcal{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\sqrt{n}}$$

and each \mathcal{A}_i is an $(a\sqrt{n}, a, 01)$ -system of size $\Theta(n)$. Now we divide each \mathcal{A}_i as

$$\mathcal{A}_i = \mathcal{A}_i^1 \cup \dots \cup \mathcal{A}_i^{c\sqrt{n}}$$

using Construction 4. Thus each \mathcal{A}_i^j is an $(a\sqrt{n}, a, \{0\})$ -system of size $\Theta(\sqrt{n})$. Set $\mathcal{A}_i^j = \{A(i, j, 1), \dots, A(i, j, c\sqrt{n})\}$.

In the same way, we define $(an, a, 01)$ -systems \mathcal{B} and \mathcal{C} with vertex set \tilde{Y} and \tilde{Z} respectively. Finally define

$$\mathcal{F} := \{A(i, j, l) \cup B(j, h, m) \cup C(h, i, l+m) : 1 \leq i, j, h, l, m \leq c \sqrt{n}\}.$$

Construction 7. Let us construct a $(3an, 3a, \{0, 1, a, a+1\})$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^{2.5})$ and $\text{IS}(\mathcal{F})$ is generated by \mathcal{I}_2^* (see Theorem 7). Let $[3an] = V_1 \cup \dots \cup V_{3a}$, $|V_1| = \dots = |V_{3a}| = n$, $[3a] = X \cup Y \cup Z$, $|X| = |Y| = |Z| = a$. For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$.

Define $\mathcal{A} = \bigcup_{i=1}^{\sqrt{n}} \mathcal{A}_i$ on \tilde{X} as in Construction 6. Set $\mathcal{A}_i = \{A(i, 1), \dots, A(i, cn)\}$. Next construct an $(an, a, 01)$ -system \mathcal{B} on \tilde{Y} using Construction 4. Thus $\mathcal{B} = \bigcup_{i=1}^{cn} \mathcal{B}_i$ where \mathcal{B}_i is an $(an, a, \{0\})$ -system. Set $\mathcal{B}_i = \{B(i, 1), \dots, B(i, cn)\}$. On \tilde{Z} , we construct an $(an, n, \{0\})$ -system $\mathcal{C} = \{C_1, \dots, C_n\}$. Finally define

$$\mathcal{F} := \{A(i, l) \cup B(i, m) \cup C_{l+m} : 1 \leq i \leq \sqrt{n}, 1 \leq l, m \leq cn\}.$$

Theorems 7, 8, and Constructions 6, 7 imply the following.

THEOREM 9. *For $a \geq 4$, one has $\alpha(3a, \{0, 1, a, a+1\}) = 2.5$.*

3.2. $\alpha(12, 013467) = 3.5$

By computer search, we found that $\text{rank}(12, 013467) = 4$ and there is a unique IS \mathcal{I}_3 of rank 4. The generator set is

$$\mathcal{I}_3^* = \{XW, xYW, xZW, yZW, xYZ, XY, XZ\},$$

where $X \cup Y \cup Z \cup W = [12]$, $|X| = |Y| = |Z| = |W| = 3$, $xYZ := \{\{x\} \cup Y \cup Z : x \in X\}$ etc.

THEOREM 10. *Let $a \geq 3$ and \mathcal{F} be $(4an, 4a, \{0, 1, a, a+1, 2a, 2a+1\})$ -system. Let $X \cup Y \cup Z \cup W = [4a]$, $|X| = |Y| = |Z| = |W| = a$. Suppose that $\text{IS}(\mathcal{F})$ is generated by*

$$\mathcal{I}_3^* = \{XW, xYW, xZW, yZW, xYZ, XY, XZ\}.$$

Then,

$$|\mathcal{F}| = O(n^{3.5}).$$

Proof. We may assume that \mathcal{F} is a $4a$ -partite with partition $[4an] = V_1 \cup \dots \cup V_{4a}$, $|V_1| = \dots = |V_{4a}| = n$. For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$ and $\mathcal{F}|_{\tilde{A}} := \{F \cap \tilde{A} : F \in \mathcal{F}\}$. For $H \subset \tilde{W}$ define $\mathcal{F}(H) := \{F - H : H \subset F \in \mathcal{F}\}$. Clearly, $\mathcal{F} = \bigcup_{H \in \mathcal{F}|_{\tilde{W}}} \mathcal{F}(H)$. Looking at the IS, we

find that any two edges in $\mathcal{F} \upharpoonright_W$ are disjoint. Thus $|\mathcal{F} \upharpoonright_W| \leq n$. Note also $\text{IS}(\mathcal{F}(H)) = \mathcal{I}_2$ (see Theorem 7), i.e., the IS of $\mathcal{F}(H)$ is generated by

$$\{X, xY, xZ, yZ\}.$$

Using Theorem 7, we have $|\mathcal{F}(H)| \leq (1+a)n^{2.5}$ for all $H \subset \tilde{W}$. Therefore,

$$|\mathcal{F}| = \left| \bigcup_{H \in \mathcal{F} \upharpoonright_W} \mathcal{F}(H) \right| \leq |\mathcal{F} \upharpoonright_W| |\mathcal{F}(H)| \leq (1+a)n^{3.5}. \quad \blacksquare$$

Construction 8. Let us construct a $(4an, 4a, \{0, 1, a, a+1, 2a, 2a+1\})$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^{3.5})$ and $\text{IS}(\mathcal{F})$ is generated by \mathcal{I}_3^* (see Theorem 10). Let $[4an] = V_1 \cup \dots \cup V_{4a}$, $|V_1| = \dots = |V_{4a}| = n$, $[4a] = X \cup Y \cup Z \cup W$, $|X| = |Y| = |Z| = |W| = a$. For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$.

As in Construction 7, we define $(an, a, 01)$ -systems $\mathcal{A} = \bigcup_{i=1}^{\sqrt{n}} \{A(i, 1), \dots, A(i, cn)\}$ on \tilde{X} and $\mathcal{B} = \bigcup_{i=1}^{cn} \{B(i, 1), \dots, B(i, cn)\}$ on \tilde{Y} . Next we construct $(an, n, \{0\})$ -systems $\mathcal{C} = \{C_1, \dots, C_n\}$ on \tilde{Z} and $\mathcal{D} = \{D_1, \dots, D_n\}$ on \tilde{W} . Finally define

$$\mathcal{F} := \{A(i, h) \cup B(i, j) \cup C_l \cup D_m : 1 \leq i \leq \sqrt{n}, h + j + l + m \equiv 0 \pmod{n}\}.$$

3.3. $\alpha(12, 01459) = 3$

By computer search, we found that $\text{rank}(12, 01459) = 3$ and there are three ISs $\mathcal{I}_1, \mathcal{I}_2$ (defined in Theorem 7) and \mathcal{I}_4 of rank 3. \mathcal{I}_4 is generated by

$$\mathcal{I}_4^* = \{XYz_0, Xz_1, Xz_2, Xz_3, Yz_1, Yz_2, Yz_3, Z\},$$

where $X \cup Y \cup Z = [12]$, $|X| = |Y| = |Z| = 4$, $Z = \{z_0, z_1, z_2, z_3\}$, $XYz_0 := \{X \cup Y \cup \{z_0\}\}$ etc.

Construction 9. Let us construct a $(3an, 3a, \{0, 1, a, a+1, 2a+1\})$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^3)$ and $\text{IS}(\mathcal{F})$ is generated by

$$\mathcal{I}_4^* = \{XYz_0, Xz_1, \dots, Xz_{a-1}, Yz_1, \dots, Yz_{a-1}, Z\},$$

where $[3a] = X \cup Y \cup Z$, $|X| = |Y| = |Z| = a$, $Z = \{z_0, \dots, z_{a-1}\}$.

\mathcal{F} will be a $3a$ -partite with partition $\bigcup_{i \in X \cup Y \cup Z} V_i$ ($|V_i| = n$). For $A \subset [3a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$. On \tilde{X} we construct an $(an, a, \{0\})$ -system $\mathcal{A} = \{A_1, \dots, A_n\}$, and on \tilde{Y} we construct an $(an, a, \{0\})$ -system $\mathcal{B} = \{B_1, \dots, B_n\}$. Set $V_{z_0} = \{p_1, \dots, p_n\} = \mathbf{Z}/n\mathbf{Z}$. On $\bigcup_{i \in Z - \{z_0\}} V_i$ we construct an $((a-1)n, a-1, 01)$ -system $\mathcal{C} = \bigcup_{i=1}^{cn} \mathcal{C}_i$ using Construction 4. Set $\mathcal{C}_i = \{C(i, 1), \dots, C(i, cn)\}$. Finally define

$$\mathcal{F} := \{A_i \cup B_j \cup \{p_{i+j}\} \cup C(i+j, h) : 1 \leq i, j, h \leq cn\}.$$

3.4. $\alpha(12, \{0, 1, 3, 4, 6, 7, 10\}) = 4$

By computer search, we found that $\text{rank}(12, \{0, 1, 3, 4, 6, 7, 10\}) = 4$ and there are two ISs \mathcal{I}_3 (defined in Theorem 10) and \mathcal{I}_5 of rank 4. \mathcal{I}_5 is generated by

$$\mathcal{I}_5^* = \{XYZw_0, XYw_1, XYw_2, XZw_1, XZw_2, YZw_1, \\ YZw_2, XW, YW, ZW\},$$

where $X \cup Y \cup Z \cup W = [12]$, $|X| = |Y| = |Z| = |W| = 3$, $W = \{w_0, w_1, w_2\}$.

Construction 10. Let us construct a $(4an, 4a, \{0, 1, a, a+1, 2a, 2a+1, 3a+1\})$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^4)$ and $\text{IS}(\mathcal{F})$ is generated by

$$\mathcal{I}_5^* = \{XYZw_0, XYw_1, \dots, XYw_{a-1}, XZw_1, \dots, XZw_{a-1}, \\ YZw_1, YZw_{a-1}, XW, YW, ZW\},$$

where $[4a] = X \cup Y \cup Z \cup W$, $|X| = |Y| = |Z| = |W| = a$, $W = \{w_0, \dots, w_{a-1}\}$.

\mathcal{F} will be a $4a$ -partite with partition $\bigcup_{i \in X \cup Y \cup Z \cup W} V_i$ ($|V_i| = n$). For $A \subset [4a]$ define $\tilde{A} := \bigcup_{i \in A} V_i$. We construct $(an, a, \{0\})$ -systems $\mathcal{A} = \{A_1, \dots, A_n\}$ on \tilde{X} , $\mathcal{B} = \{B_1, \dots, B_n\}$ on \tilde{Y} , and $\mathcal{C} = \{C_1, \dots, C_n\}$ on \tilde{Z} . Set $V_{w_0} = \{p_1, \dots, p_n\} = \mathbf{Z}/n\mathbf{Z}$. On $\bigcup_{i \in W - \{w_0\}} V_i$ we construct an $((a-1)n, a-1, 01)$ -system $\mathcal{D} = \bigcup_{i=1}^{cn} \mathcal{D}_i$ using Construction 4. Set $\mathcal{D}_i = \{D(i, 1), \dots, D(i, cn)\}$. Finally define

$$\mathcal{F} := \{A_i \cup B_j \cup C_l \cup \{p_{i+j+l}\} \cup D(i+j+l, m) : 1 \leq i, j, l, m \leq cn\}.$$

3.5. $\alpha(10, 0136) = 2.5$

By computer search, we found that $\text{rank}(10, 0136) = 3$ and there is a unique IS

$$\mathcal{I}_6 = \left\{ \binom{A}{2} : A \subset [5], A \neq [5] \right\}$$

of rank 3. For $i \in [5]$, define $I(\bar{i}) := \{e : i \notin e \in \binom{[5]}{2}\}$, e.g., $I(\bar{5}) = \{12, 13, 23, 14, 24, 34\} = \binom{[4]}{2}$. Then \mathcal{I}_6 is generated by

$$\mathcal{I}_6^* = \{I(\bar{1}), I(\bar{2}), I(\bar{3}), I(\bar{4}), I(\bar{5})\}.$$

THEOREM 11. *Let \mathcal{F} be $(10n, 10, 0136)$ -system. Suppose that $\text{IS}(\mathcal{F}) = \mathcal{I}_6$, then*

$$|\mathcal{F}| = O(n^{2.5}).$$

Proof. We may assume that \mathcal{F} is 10-partite with partition $[10n] = \bigcup_{e \in \binom{[5]}{2}} V_e$, $|V_e| = n$. For $A \subset [5]$ define $\tilde{A} := \bigcup_{e \in \binom{A}{2}} V_e$ and $\mathcal{F}(A) := \{F \cap \tilde{A} : F \in \mathcal{F}\}$.

Case 1. $|\mathcal{F}(A)| \leq n^{1.5}$ holds for some $A \in \binom{[5]}{3}$. Assume $|\mathcal{F}(123)| \leq n^{1.5}$. By the IS, if $\{12, 13, 23, 45\} \subset J$ then $J \notin \mathcal{I}_6$. Thus $|\mathcal{F}| \leq |\mathcal{F}(123)||\mathcal{F}(45)| \leq n^{1.5} |V_{45}| = n^{2.5}$.

Case 2. $|\mathcal{F}(A)| > n^{1.5}$ holds for all $A \in \binom{[5]}{3}$. In general,

$$\frac{|\mathcal{F}|}{|\mathcal{F}(1234)|} \leq \frac{|\mathcal{F}(1235)|}{|\mathcal{F}(123)|}$$

holds (see [7] for a proof). In \mathcal{I}_6 , only $I(\bar{5})$ contains both 12 and 34. Thus,

$$|\mathcal{F}(1234)| \leq |\mathcal{F}(12)||\mathcal{F}(34)| \leq |V_{12}||V_{34}| = n^2.$$

In the same way, we have $|\mathcal{F}(1235)| \leq n^2$. Therefore, $|\mathcal{F}|/n^2 \leq n^2/n^{1.5}$, that is $|\mathcal{F}| \leq n^{2.5}$. ■

Construction 11. Let us construct an $(n, 10, 0136)$ -system \mathcal{F} such that $|\mathcal{F}| = \Theta(n^{2.5})$ and $\text{IS}(\mathcal{F}) = \mathcal{I}_6$. Let $G = K_m$ be a complete graph on m vertices. The 10-uniform family \mathcal{F} with vertex set $E(G)$ is defined by

$$\mathcal{F} := \left\{ \binom{A}{2} : A \in \binom{V(G)}{5} \right\}.$$

Since \mathcal{F} has $n := \binom{m}{2}$ vertices and $\binom{m}{5}$ edges, we obtain $|\mathcal{F}| = \Theta(n^{2.5})$.

3.6. $\alpha(12, 02358) = 2.5$

By computer search, we found that $\text{rank}(12, 02358) = 3$ and there is a unique IS \mathcal{I}_7 of rank 3. Let $\mathcal{I}_6^* \subset 2^{[10]}$ be the generator set defined in Section 3.5, then $\mathcal{I}_7 \subset 2^{[12]}$ is generated by

$$\mathcal{I}_7^* = \{I \cup \{11, 12\} : I \in \mathcal{I}_6^*\}.$$

THEOREM 12. *Let \mathcal{F} be $(12n, 12, 02358)$ -system. Suppose that $\text{IS}(\mathcal{F}) = \mathcal{I}_7$, then*

$$|\mathcal{F}| = O(n^{2.5}).$$

Proof. We may assume that \mathcal{F} is 12-partite with partition $[12n] = \bigcup_{i=1}^{12} V_i$, $|V_i| = n$. Let $\mathcal{F}_0 := \{F \cap (\bigcup_{i=1}^{10} V_i) : F \in \mathcal{F}\}$. Note that if $F \in \mathcal{F}_0$, $G, H \in V_{11} \cup V_{12}$, and $F \cup G, F \cup H \in \mathcal{F}$ then $G = H$. This means $|\mathcal{F}| = |\mathcal{F}_0|$. Further, $\text{IS}(\mathcal{F}_0) = \mathcal{I}_6$ implies $|\mathcal{F}_0| \leq n^{2.5}$ by Theorem 11. ■

3.7. $\alpha(10, 0134) = 3$

Construction 12. Let \mathcal{F}_0 be the set of 2-dimensional affine subspaces of a d -dimensional vector space over $GF(3)$, and let $n = 3^d$. Then \mathcal{F}_0 is an $(n, 9, 013)$ -system of size $\Theta(n^3)$. It is not hard to see that one can divide \mathcal{F}_0 as

$$\mathcal{F}_0 = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{cn},$$

where each \mathcal{F}_i is an $(n, 9, 03)$ -system. Now define

$$\mathcal{F} := \bigcup_{i=1}^{cn} \{F \cup \{x_i\} : F \in \mathcal{F}_i\},$$

where x_1, \dots, x_{cn} are new vertices. Then \mathcal{F} is a $((1+c)n, 10, 0134)$ -system of size $\Theta(n^3)$.

 3.8. $\alpha(12, 0134) = 3$

Construction 13. Let \mathcal{F}_0 be the set of 2-dimensional affine subspaces of a d -dimensional vector space over $GF(3)$, and let $n = 3^d$. Then \mathcal{F}_0 is an $(n, 9, 013)$ -system of size $\Theta(n^3)$. Divide \mathcal{F}_0 as

$$\mathcal{F}_0 = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{cn},$$

where each \mathcal{F}_i is an $(n, 9, 01)$ -system. Now define

$$\mathcal{F} := \bigcup_{i=1}^{cn} \{F \cup \{x_i, y_i, z_i\} : F \in \mathcal{F}_i\},$$

where $x_1, y_1, z_1, \dots, x_{cn}, y_{cn}, z_{cn}$ are new vertices. Then \mathcal{F} is a $((1+3c)n, 12, 0134)$ -system of size $\Theta(n^3)$.

 3.9. $\alpha(11, 012457) = 4$

By Reduction 11, one has

$$\alpha(11, 012457) \leq \max\{\alpha(11, 01245), \alpha(7, 01245) + \alpha(4, 0)\}.$$

By Reduction 10, one has $\alpha(11, 01245) \leq 4$. (Actually, we can prove $\alpha(11, 01245) \leq 3$, but we do not need this bound here.) On the other hand, by Reductions 6 and 5, it follows that

$$\alpha(7, 01245) \leq \alpha(7, 012456) \leq 3.$$

Finally, we have $\alpha(11, 012457) \leq 4$.

$k = 8$

									7	7	7	7	7	7	7	7	
				6	6	6	6					6	6	6	6		
		5	5			5	5		5	5				5	5		
		4	4	4	4	4	4	4	4	4	4	4	4	4	4		
0		1	2	1	2	1	2	2	3	1	2	1	2	2	2	3	4
0 1		2	2	2	3	2	2	2	3	2	2	2	3	2	2	3	4
0 2		2	3	2	3	2	4	2	4	2	3	2	3	2	4	3	4
0 1 2		3	4	3	4	3	4	3	4	3	4	3	4	3	4	3	4
0 3		1	2	2	3	1	2	2	4	1	2	2	3	2	2	3	5
0 1 3		2	2	2	3	2	2	3	4	2	2	2	3	2	2	3	5
0 2 3		2	3	2	4	2	4	3	5	2	3	2	4	2	4	3	6
0 1 2 3		4	5	4	6	4	5	4	7	4	5	4	6	4	5	4	8

 $k = 9$

										8	8	8	8	8	8	8	8
					7	7	7	7					7	7	7	7	
			6	6			6	6		6	6				6	6	
		5	5	5	5	5	5	5	5	5	5	5	5	5	5		
0		1	1	1	2	1	2	2	3	1	1	1	2	2	2	3	4
0 1		2	2	2	2	2	2	3	2	2	2	2	2	2	2	3	4
0 2		1	2	1	2	2	2	3	1	2	1	2	2	2	3	4	
0 1 2		3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	
0 3		2	2	3	3	2	3	3	3	2	2	3	3	2	3	3	4
0 1 3		3	3	3	3	3	4	3	4	3	3	3	3	3	4	3	4
0 2 3		2	3	3	3	2	3	3	3	2	3	3	3	2	3	3	4
0 1 2 3		4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
0 4		1	2	1	3	1	2	2	4	1	2	1	3	2	2	3	5
0 1 4		2	3	2	3	2	3	2	4	2	3	2	3	2	3	3	5
0 2 4		1	2	1	3	2	3	2	4	1	2	1	3	2	3	3	5
0 1 2 4		3	3	3	4	3	3	3	4	3	3	3	4	3	3	3	5
0 3 4		2	3	3	4	2	3	3	5	2	3	3	4	2	3	3	6
0 1 3 4		3	3	3	4	3	4	4	5	3	3	3	4	3	4	4	6
0 2 3 4		3	4	3	5	3	4	3	6	3	4	3	5	3	4	3	7
0 1 2 3 4		5	6	5	7	5	6	5	8	5	6	5	7	5	6	5	9

Let $A \subset \{6, 8, 9, 10\}$. By computer search, we found that

$$\text{rank}(11, \{0, 1, 2, 3, 5\} \cup A) = 5.$$

The Steiner system $S(11, 5, 4)$ is one of the ISs of rank 5. (Probably this is the unique one.) We know that $4 \leq \alpha(11, \{0, 1, 2, 3, 5\} \cup A) \leq 5$, but we were not able to determine the exact exponent.

$k = 12$

	6	7	8	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10												
0	1	2	1	2	1	2	2	3	1	2	2	2	3	4	1	2	1	2	2	3	2	2	2	3	3	3	4	5			
0 1	2	2	2	3	2	2	2	3	2	2	2	3	2	2	3	4	2	2	2	3	2	3	2	2	2	3	3	4	5		
0 2	2	2	2	2	2	3	2	3	2	2	2	2	2	3	3	4	2	2	2	2	2	3	2	2	2	3	3	4	5		
0 12	3	3	3	4	3	3	3	4	3	3	3	4	3	3	4	3	3	3	4	3	3	3	4	3	3	3	4	5			
0 3	2	3	2	3	2	3	2	3	2	4	2	4	2	4	3	4	2	3	2	3	2	3	2	3	2	4	4	5			
0 1 3	2	3	2	3	2	3	2	3	2	4	2	4	2	4	3	4	2	3	2	3	2	3	2	3	3	4	4	5			
0 2 3	2	3	2	3	2	3	2	3	2	4	2	4	2	4	3	4	2	3	2	3	2	3	2	3	3	4	4	5			
0 1 2 3	4	4	4	4	4	5	4	4	4	4	4	4	4	5	4	4	4	4	4	4	4	5	4	4	4	4	4	5			
0 4	2	2	2	3	3	3	3	2	2	2	2	3	3	3	4	2	2	2	2	3	4	3	4	2	2	3	3	4	5		
0 1 4	2	2	2	3	3	3	3	2	2	2	3	3	3	3	4	2	2	2	3	3	4	3	4	2	2	2	3	4	5		
0 2 4	3	4	3	4	3	5	3	5	3	4	3	5	3	5	3	4	3	6	3	6	3	6	3	4	3	4	3	6	6		
0 1 2 4	3	4	3	4	3	5	3	5	3	4	3	5	3	5	3	4	3	6	3	6	3	6	3	4	3	4	4	6	6		
0 3 4	2	3	2	3	3	3	3	2	4	2	4	2	4	3	4	2	3	2	3	3	4	3	4	2	4	2	4	4	5		
0 1 3 4	3	3	3	3.5	3	3	3.5	3	4	3	4	3	4	3	4	3	4	3	4	3	4	3	4	3	4	3	5	3	4	5	
0 2 3 4	3	4	3	4	3	5	4	5	3	4	3	4	3	5	4	5	3	4	3	4	3	6	4	6	3	4	3	6	6		
0 1 2 3 4	5	5	5	5	5	6	5	5	5	5	6	5	5	6	5	5	5	6	5	6	5	6	5	5	5	5	6	5	6		
0 5	1	2	2	3	1	3	2	4	1	2	2	3	2	3	3	5	1	2	2	3	2	3	2	2	2	3	3	3	4	6	
0 1 5	2	2	2	3	2	3	4	2	2	2	3	2	3	3	5	2	2	2	3	2	3	3	4	2	2	2	3	3	4	6	
0 2 5	2	2	2	3	2	3	2	4	2	2	2	3	2	3	5	2	2	3	2	3	3	4	2	2	2	3	3	3	4	6	
0 1 2 5	3	3	3	4	3	3	4	3	3	4	3	4	3	4	5	3	3	4	3	3	4	3	3	4	3	3	4	3	4	6	
0 3 5	2	3	2	3	2	3	2	4	2	2	4	2	4	3	5	2	3	2	3	2	3	3	4	2	4	2	4	3	4	6	
0 1 3 5	2	3	2	3	2	3	4	2	4	2	4	2	4	3	5	2	3	2	3	3	3	4	3	4	2	4	3	4	4	6	
0 2 3 5	2	3	2	3	2.5	3	3	4	2	4	2	4	3	4	5	2	3	3	3	2.5	3	3	4	3	4	3	4	3	4	6	
0 1 2 3 5	4	4	4	4	4	5	4	4	4	4	4	4	4	5	4	4	4	4	4	5	4	4	4	4	4	4	4	4	5	6	
0 4 5	2	3	2	4	3	4	3	5	2	3	2	4	3	4	5	2	3	2	4	3	4	3	5	2	3	3	4	3	4	7	
0 1 4 5	2.5	3	2.5	4	3	4	3	5	2.5	3	2.5	4	4	4	6	2.5	3	2.5	4	3	4	3	5	2.5	3	3	4	4	4	7	
0 2 4 5	3	4	3	4	3	5	3	4	3	4	3	5	3	6	3	4	3	4	3	6	3	6	3	4	3	4	3	6	4	7	
0 1 2 4 5	3	4	3	4	3	5	3	4	3	4	5	4	6	3	4	3	4	3	6	3	6	3	6	3	4	3	4	4	6	7	
0 3 4 5	3	4	3	5	3	4	3	6	3	4	3	5	3	4	7	3	4	3	5	3	4	3	6	3	4	3	5	3	4	8	
0 1 3 4 5	3	4	3	5	3	4	3	6	3	4	3	5	4	4	7	3	4	3	5	3	4	3	6	3	4	3	5	4	4	8	
0 2 3 4 5	4	5	4	6	4	5	4	7	4	5	4	6	4	5	8	4	5	4	6	4	6	4	6	4	7	4	5	4	6	9	
0 1 2 3 4 5	6	7	6	8	6	7	6	9	6	7	6	8	6	7	10	6	7	6	8	6	7	6	9	6	7	6	8	6	7	6	11

Let $A \subset \{7, 8, 9, 10, 11\}$ and $\{7, 8\}, \{8, 10\} \not\subset A$. By computer search, we found that

$$\text{rank}(12, \{0, 1, 2, 3, 4, 6\} \cup A) = 6.$$

The Steiner system $S(12, 6, 5)$ is one of the ISs of rank 6. (Probably this is the unique one.) We know that $5 \leq \alpha(12, \{0, 1, 2, 3, 4, 6\} \cup A) \leq 6$, but we were unable to determine the exact exponent.

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