ELSEVIER
Discrete Mathematics 140 (1995) 265–267
DISCRETE MATHEMATICS

Note

On the section of a convex polyhedron

Peter Frankl[a], Hiroshi Maehara[b], Junichiro Nakashima[c]

[a] CNRS, Paris, France
[b] Ryukyu University, Okinawa, Japan
[c] Tokyo University, Tokyo, Japan

Received 1 February 1993

Abstract

Let P be a convex polyhedron in R^3, and E be a plane cutting P. Then the section $P_E = P \cap E$ is a convex polygon. We show a sharp inequality

(the perimeter of $P_E) < \frac{2}{3} L(P)$,

where $L(P)$ denotes the sum of the edge-lengths of P.

For a polyhedron (or a polygon) X, $L(X)$ denotes the sum of the edge-lengths of X. Thus if X is a polygon then $L(X)$ is the perimeter of X.

Let P be the *surface* of a convex polyhedron in R^3, and E be a plane cutting P. Then the section $P_E = P \cap E$ is a convex polygon, see Fig. 1. We prove the following theorem.

Theorem. $L(P_E) < \frac{2}{3} L(P)$.

We use the following lemma.

Lemma. *If X is a convex polygon contained in another convex polygon Y, then* $L(X) \leqslant L(Y)$.

Proof of the theorem. Let us denote by $f: P \to E$ the orthogonal projection into E. First we consider the case where the plane E satisfies that

($*$) for every face F of $P, f(F)$ has positive area. Since $f(P)$ is a convex polygon containing P_E, we have $L(P_E) \leqslant L(f(P))$ by the above lemma. Therefore, it is sufficient to show that $L(f(P)) < \frac{2}{3} L(P)$.

SSDI 0012-365X(93)E0172-Z

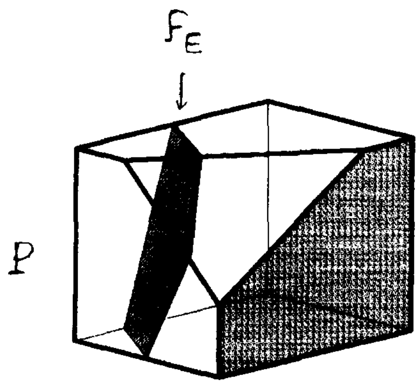

Fig. 1.

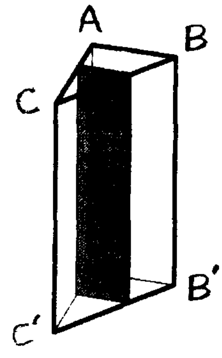

Fig. 2.

(1) For each edge b of $f(P)$, there is exactly one edge e of P such that $f(e)=b$.

(2) The interior of $f(P)$ is doubly covered by the projections $f(F)$ of the faces F of P.

(3) The polygon $f(P)$ is partitioned into small polygons Q by the projections $f(e)$ of the edges e of P.

(4) No small polygon Q is incident to more than one (boundary) edge b of the polygon $f(P)$.

To see (4), suppose that a small polygon Q is incident to two boundary edges $b_1=f(e_1)$ and $b_2=f(e_2)$ of $f(P)$. Connect the midpoint of b_1 and the midpoint of b_2 by a line segment γ. Then γ is contained in Q. By (2), there are exactly two faces F, F' of P such that $f(F)\cap f(F')$ contains Q. Hence the inverse image $f^{-1}(\gamma)$ consists of two *distinct* line segments. However, these two distinct line segments must have the same endpoints, the midpoint of e_1 and the midpoint of e_2. This is a contradiction.

(5) $2L(f(P))<\sum_Q L(Q)$ (sum is over all Q). To see this, let $b_1, \ldots, b_n$ be the edges of the polygon $f(P)$, and let Q_i be the small polygon incident to b_i. Then $Q_1, \ldots, Q_n$ are all distinct by (4). Since

$$2\ (\text{the length of } b_i)<L(Q_i),$$

we have

$$2L(f(P))<L(Q_1)+\cdots+L(Q_n)\leqslant\sum_Q L(Q).$$

Therefore we have (5).

Adding $L(f(P))$ to both sides of (5), we have

$$3L(f(P))<L(f(P))+\sum_Q L(Q).$$

On the right-hand side, the length of $f(e)$ appears exactly twice for every edge e of P. Hence the right-hand side is equal to

$$\sum_F L(f(F))=L(f(P))+\sum_Q L(Q).$$

Therefore, we have

$$(\#)\, 3L(f(P)) < \sum_F L(f(F)),$$

and since $L(f(F)) \leqslant L(F)$, we have

$$3L(f(P)) < \sum_F L(F) = 2L(P).$$

Now, let us consider the case where the plane E does not satisfy $(*)$. Denote by $L(f(F))$ the sum of the length of $f(e)$ for the edges e of a face F even if $f(F)$ degenerates to a line segment. Note that if we move the plane E continuously, then both $L(P_E)$ and $\sum_F L(f(F))$ change their values continuously. Let $\delta = \sum_F L(F) - \sum_F L(f(F))$. Since there must be an edge of P which is mapped by f to a shorter line segment, δ is positive. Hence it is possible to move the plane E so that (i) it comes to a position satisfying $(*)$ and (ii) neither $L(P_E)$ nor $\sum_F L(f(F))$ changes its value more than $\delta/2$. Then since $(\#)$ holds for the plane in the new position, the theorem holds for the plane at the original position. □

Remark. Fig. 2 shows that the inequality of the theorem is best possible.

Corollary. *For any tetrahedron T contained in a convex polytope P, $L(T) < \frac{4}{3} L(P)$.*

Proof. Let F_1, F_2, F_3, F_4 be the four faces of a tetrahedron T contained in a convex polyhedron P. Since F_i is a convex polygon contained in the section of P by the plane determined by F_i, we have $L(F_i) < \frac{2}{3} L(P)$. Hence,

$$2L(T) = L(F_1) + L(F_2) + L(F_3) + L(F_4) < \tfrac{8}{3} L(P),$$

and the corollary follows. □

Remark. The inequality of the corollary is best possible. To see this consider the tetrahedron $ABB'C'$ contained in the triangular prism in Fig. 2.