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Minimum Shadows in Uniform Hypergraphs and a Generalization of the Takagi Function

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The shadow function is closely related to the Kruskal–Katona Theorem. The Takagi function is a standard example of a nowhere differentiable continuous function. The purpose of this paper is to exhibit a rather surprising relationship between the shadow function and the Takagi function. Using this relationship, one can approximately compute the size of minimum shadows in uniform hypergraphs with a given number of edges. In order to describe the asymptotic behaviour of the size of shadows, we introduce a new, generalized Takagi function. The results explain the difficulties, often encountered when using the best possible bounds arising from the Kruskal–Katona Theorem.

1. INTRODUCTION

The shadow function is closely related to the Kruskal–Katona Theorem. The Takagi function is a standard example of a nowhere differentiable continuous function. The purpose of this paper is to exhibit a rather surprising relationship between the shadow function and the Takagi function.

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Let $\binom{\mathbf{N}}{k} = \{F \subset \mathbf{N}\colon |F| = k\}$. For a family $\mathscr{F} \subset \binom{\mathbf{N}}{k}$ and an integer $l < k$, we define the lth (lower) shadow of $\mathscr{F}$ by $\Delta_l(\mathscr{F}) = \{G \in \binom{\mathbf{N}}{l}\colon \exists F \in \mathscr{F}$ such that $G \subset F\}$. We define the colex order on $\binom{\mathbf{N}}{k}$ by $A <_{\text{colex}} B \Leftrightarrow \max\{a \in A - B\} < \max\{b \in B - A\}$ for $A, B \in \binom{\mathbf{N}}{k}$. The family of the first m elements in $\binom{\mathbf{N}}{k}$ with respect to the colex order is denoted by Colex(k, m). For example, Colex$(3, 5) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}\}$. Kruskal and Katona independently determined the minimum size of shadows in a uniform hypergraph with a given number of edges.

THEOREM 1 [4, 3]. *For all $\mathscr{F} \subset \binom{\mathbf{N}}{k}$ and $l < k$ one has*

$$\#\Delta_l(\mathscr{F}) \geq \#\Delta_l(\text{Colex}(k, |\mathscr{F}|)).$$

The complexity of the Colex$(k, |\mathscr{F}|)$ often makes the Kruskal–Katona Theorem awkward for concrete applications. The following version due to Lovász is more handy for computations.

THEOREM 2 [5]. *Suppose that $\mathscr{F} \subset \binom{\mathbf{N}}{k}$, $|\mathscr{F}| \geq \binom{x}{k}$ with $x \geq k$, real. Then*

$$\#\Delta_{k-1}(\mathscr{F}) \geq \binom{x}{k-1}$$

with equality holding if and only if x is an integer and $\mathscr{F} = \binom{X}{k}$ for some x-element set X.

This result shows the uniqueness of optimal families in the Kruskal–Katona Theorem for the case $|\mathscr{F}| = \binom{n}{k}$, $n \geq k$, integer. Applying the same result $k - l$ times proves $\#\Delta_l(\mathscr{F}) \geq \binom{n}{l}$ and uniqueness for all $1 \leq l < k$.

The values of $|\mathscr{F}| = m$ for given k and l such that Colex(k, m) is the only optimal family in the Kruskal–Katona Theorem were determined independently by Füredi and Griggs [1] and Mörs [7].

Combining the Kruskal–Katona Theorem with its Lovász version gives the following.

THEOREM 3. *Suppose that $\mathscr{F} \subset \binom{\mathbf{N}}{k}$, $|\mathscr{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{l+1}}{l+1} + \binom{x}{l}$ with $a_k, \ldots, a_{l+1}$ integers and x real, satisfying*

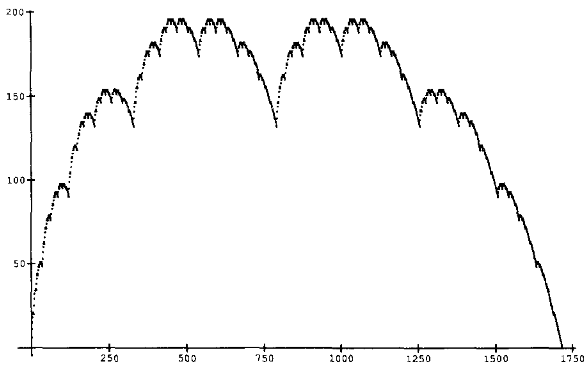
$a_k > \cdots > a_{l+1} \geq x + 1 \geq l + 1$, *then*

$$\#\Delta_{k-1}(\mathscr{F}) \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_{l+1}}{l} + \binom{x}{l-1}.$$

Now we define the Kruskal–Katona function by

$$K_l^k(m) = -m + \#\Delta_l(\mathrm{Colex}(k,m)).$$

The following picture is the graph of the Kruskal–Katona function for $k = 7$, $l = 6$:



One often encounters this function when dealing with intersecting families.

Example 1. Let two families $\mathscr{A} \subseteq \binom{[n]}{a}$ and $\mathscr{B} \subseteq \binom{[n]}{b}$ be cross-intersecting, i.e., $A \cap B \neq \varnothing$ holds for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Then by the Kruskal–Katona Theorem [4, 3], we have

$$|\mathscr{A}| + |\mathscr{B}| \leq |\mathscr{A}| + \binom{n}{b} - \#\Delta_b(\{A^c : A \in \mathscr{A}\}) \leq \binom{n}{b} - K_b^{n-a}(|\mathscr{A}|).$$

The shadow function S_k is defined by normalizing K_l^k, where $l := k - 1$:

$$\begin{aligned} S_k(x) &:= k\binom{k+l}{k}^{-1} K_l^k\left(\left\lfloor \binom{k+l}{k} x \right\rfloor\right) \\ &= k\binom{2k-1}{k}^{-1} K_{k-1}^k\left(\left\lfloor \binom{2k-1}{k} x \right\rfloor\right), \qquad \text{for } 0 \leq x \leq 1. \end{aligned}$$

Next, we introduce the Takagi function. In 1903, Takagi constructed a nowhere differentiable continuous function [8, 9]. It is called the Takagi function and denoted by T,

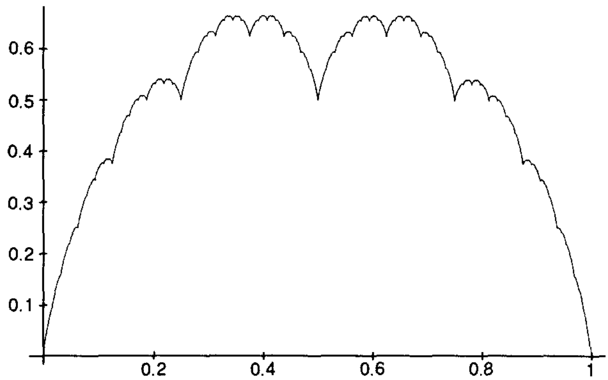
$$T(x) := \sum_{j=1}^{\infty} \varphi_j(x) 2^{-j} \qquad \text{for } 0 \le x \le 1,$$

where

$$\varphi_1(x) := \begin{cases} 2x, & \text{if } 0 \le x \le 1/2, \\ 2(1-x), & \text{if } 1/2 \le x \le 1, \end{cases}$$

$$\varphi_n(x) := \varphi_{n-1}(\varphi_1(x)).$$

The following picture is the graph of the Takagi function:



This function has many interesting properties including self-similarity (see the Appendix). The first result of the present paper describes how the shadow functions are approximated by the Takagi function.

THEOREM 4. *The shadow functions uniformly converge to the Takagi function, i.e.*,

$$\lim_{k \to \infty} \sup_{0 \le x \le 1} |S_k(x) - T(x)| = 0.$$

Sometimes an estimation of the Kruskal–Katona function needs heavy computations. This situation is explained to some extent by the fact that this function converges to a nowhere differentiable function.

The following inequality is a direct consequence of the Kruskal–Katona Theorem and Theorem 4.

THEOREM 5. *Suppose that* $\mathscr{F} \subset \binom{N}{k}$, $|\mathscr{F}| = \left\lfloor \binom{2k-1}{k} x \right\rfloor$ *with* $0 \leq x \leq 1$, *real. Then*

$$\#\Delta_{k-1}(\mathscr{F}) \geq |\mathscr{F}| + (T(x) + o(1)) \frac{1}{k} \binom{2k-1}{k} \qquad \text{as } k \to \infty.$$

The following is an easy application of Theorem 5.

EXAMPLE 2. Let $|X| = n = 2k+1$. Suppose that $\mathscr{A} \subset \binom{X}{k}$ and $\mathscr{B} \subset \binom{X}{k}$ are cross-intersecting families. Then,

$$\min_{0 \leq x \leq \binom{n}{k}} \max_{|\mathscr{A}| = x} (|\mathscr{A}| + |\mathscr{B}|) = \left(1 - \frac{2}{3k}(1 + o(1))\right) \binom{n}{k}.$$

For the analysis of the case $l = \lfloor ck \rfloor$, $0 < c < 1$, we need two more functions T_c and $S_{c,k}$ defined below. We define the generalized Takagi function T_c, which is different from the one defined in [2]. For a fixed real c with $0 < c \neq 1$, we can represent x, $0 \leq x \leq 1$ in $(1+c)$-nary form

$$x = \sum_{j \geq 0} c^{-j} \left(\frac{c}{1+c} \right)^{\beta_j},$$

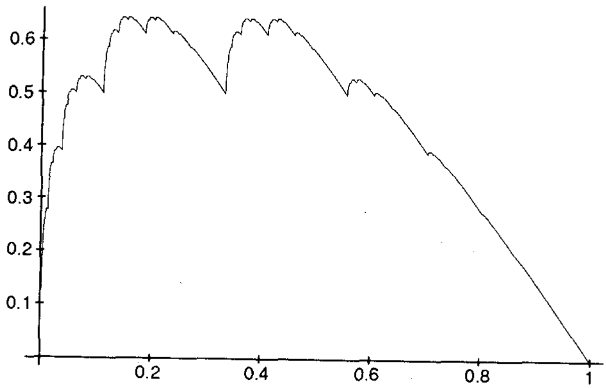
where $\{\beta_j\}$ is a strictly increasing sequence of positive integers. There is only one ambiguity of representation, namely,

$$\sum_{j=0}^{i-1} c^{-j} \left(\frac{c}{1+c} \right)^{\beta_j} + c^{-i} \left(\frac{c}{1+c} \right)^{\beta_i}$$
$$= \sum_{j=0}^{i-1} c^{-j} \left(\frac{c}{1+c} \right)^{\beta_j} + \sum_{j=i+1}^{\infty} c^{-j+1} \left(\frac{c}{1+c} \right)^{\beta_i + j - i}.$$

However, both representations give the same value in the following function T_c:

$$T_c(x) := \frac{1+c}{2(1-c)} \sum_{j \geq 0} \frac{c^{2j} - c^{\beta_j}}{c^j (1+c)^{\beta_j}}.$$

The following picture is the graph of the generalized Takagi function for $c = 1/2$:



This function is a generalization of the Takagi function in the following sense.

THEOREM 6. *The generalized Takagi functions converge uniformly to the Takagi function as* $c \to 1$, *i.e.*,

$$\lim_{c \to 1} \sup_{0 \le x \le 1} |T_c(x) - T(x)| = 0.$$

More precisely,

$$T(x) - T_c(x) \ll |c - 1| \left(\log \frac{1}{|c - 1|} \right)^3$$

holds.

Finally, we define the c-shadow function. Let c be a fixed real with $0 < c < 1$ and k be an integer. To consider lower shadows, set $l := \lfloor ck \rfloor$. The c-shadow function $S_{c,k}$ is defined by

$$S_{c,k}(x) := \frac{1 + c}{2(1 - c)} \binom{k + l}{k}^{-1} K_l^k \left(\left\lfloor \binom{k + l}{k} x \right\rfloor \right) \qquad \text{for } 0 \le x \le 1.$$

The following picture is the graph of the Kruskal–Katona function for

$k = 10$, $l = 5$. This is corresponding to the generalized Takagi function for $c = 5/10 = 1/2$:

The c-shadow functions are approximated by the generalized Takagi function as follows.

THEOREM 7. *The c-shadow functions uniformly converge to the generalized Takagi function, i.e.,*

$$\lim_{k\to\infty} \sup_{0\le x\le 1} |S_{c,k}(x) - T_c(x)| = 0$$

holds for all c, $0 < c < 1$.

In this case, the inequality corresponding to Theorem 5 is the following.

THEOREM 8. *Let* $0 < c < 1$ *and* $0 \le x \le 1$ *be reals. Suppose that* $\mathscr{F} \subset \binom{\mathbf{N}}{k}$, $|\mathscr{F}| = \left\lfloor \binom{k+l}{k} x \right\rfloor$ *with* $l = \lfloor ck \rfloor$. *Then*

$$\#\Delta_l(\mathscr{F}) \ge |\mathscr{F}| + (T_c(x) + o(1)) \frac{2(1-c)}{1+c} \binom{k+l}{k} \qquad \text{as } k \to \infty.$$

The authors believe that these results on uniform convergence are interesting on their own sake. However, it would be nicer to have some concrete applications of the theorems to extremal problems. We hope to return to this in some future paper.

2. Proof of Theorem 4

In this section, we assume that $0 \leq x \leq 1$. In this case x can be represented in the form $x = \sum_{j \geq 0} 2^{-\beta_j}$, where $\{\beta_j\}$ is a strictly increasing sequence of positive integers. For $x = \sum_{j=0}^{s} 2^{-\beta_j}$, we define $R(x) := \sum_{j=0}^{s} (\beta_j - 2j) 2^{-\beta_j}$, and for $x = \sum_{j=0}^{\infty} 2^{-\beta_j}$, we define $R(x) := \lim_{s \to \infty} \sum_{j=0}^{s} (\beta_j - 2j) 2^{-\beta_j}$.

Lemma 1. $T(x) = R(x)$.

Proof. Let $x = \sum_{j=0}^{\infty} a_j 2^{-j}$, $a_j = 0, 1$. By the definition of φ_j, we have

$$\varphi_j(x) 2^{-j} = \begin{cases} x - \sum_{i<j} a_i 2^{-i}, & \text{if } a_j = 0, \\ 2^{-j+1} - \left(x - \sum_{i<j} a_i 2^{-i} \right), & \text{if } a_j = 1. \end{cases} \qquad (\clubsuit)$$

First we assume that $x = \sum_{j=0}^{s} 2^{-\beta_j}$. We prove $T(x) = R(x)$ by induction on s. Since this clearly holds for $s = 0$, we assume that $s > 0$ and define $x' = \sum_{j=0}^{s-1} 2^{-\beta_j}$. Then, by ($\clubsuit$), we have

$$\varphi_j(x) 2^{-j} = \begin{cases} \varphi_j(x') 2^{-j} + 2^{-\beta_s}, & \text{if } a_j = 0 \text{ and } j \leq \beta_{s-1}, \\ \varphi_j(x') 2^{-j} - 2^{-\beta_s}, & \text{if } a_j = 1 \text{ and } j \leq \beta_{s-1}, \\ 2^{-\beta_s}, & \text{if } \beta_{s-1} < j \leq \beta_s, \\ 0, & \text{if } j > \beta_s. \end{cases}$$

So, we obtain

$$\begin{aligned} T(x) &= T(x') + (\beta_{s-1} - s) 2^{-\beta_s} - s 2^{-\beta_s} + (\beta_s - \beta_{s-1}) 2^{-\beta_s} \\ &= R(x') + (\beta_s - 2s) 2^{-\beta_s} \qquad \text{(by the induction hypothesis)} \\ &= R(x). \end{aligned}$$

Next, we consider the case $x = \sum_{j=0}^{\infty} 2^{-\beta_j}$. By the definition of R and the continuity of T, we have

$$R(x) = \lim_{s \to \infty} R\left(\sum_{j=0}^{s} 2^{-\beta_s} \right) = \lim_{s \to \infty} T\left(\sum_{j=0}^{s} 2^{-\beta_s} \right) = T(x). \quad \blacksquare$$

For $x = \sum_{j \geq 0} 2^{-\beta_j}$, we define $x_s := \sum_{\beta_j \leq s} 2^{-\beta_j}$, and $x_s(k) := \binom{2k-1}{k}^{-1} \sum_{\beta_j \leq s} \binom{2k - \beta_j - 1}{k - j}$. Note that $\#\{x_s\colon 0 \leq x \leq 1\}$ is finite for any fixed s. To prove Theorem 4 we need a rather technical lemma.

LEMMA 2. (1) *For every positive* ε *there exists* $s_0 = s_0(\varepsilon)$ *such that* $\max_{0 \le x \le 1} |x - x_s| < \varepsilon$ *holds for* $s \ge s_0$.

(2) *For every positive* ε *and every* s *there exists* $k_0 = k_0(\varepsilon, s)$ *such that* $\max_{0 \le x \le 1} |x_s(k) - x_s| < \varepsilon$ *holds for* $k \ge k_0$.

(3) *For every positive* ε *and every* s *there exists* $k_1 = k_1(\varepsilon, s)$ *such that* $\sup_{0 \le x \le 1} |S_k(x_s(k)) - R(x_s)| < \varepsilon$ *holds for* $k \ge k_1$.

Proof. (1) This follows from $|x - x_s| \le 2^{-s_0+1}$.

(2) For fixed x_s, we have

$$|x_s(k) - x_s| = \left| \sum_{\beta_j \le s} \left(\frac{\{k \cdots (k-j+1)\}\{(k-1) \cdots (k-\beta_j+j)\}}{(2k-1) \cdots (2k-\beta_j)} - 2^{-\beta_j} \right) \right| < \varepsilon$$

if $k \ge k(x_s)$. For a given s, x_s assumes only finitely many values for $0 \le x \le 1$. So we may choose $k_0 := \max_{x_s}\{k(x_s)\}$.

(3) For fixed x_s, we have

$$\begin{aligned} S_k(x_s(k)) &= k\binom{2k-1}{k}^{-1} K_{k-1}^k \left(\sum_{\beta_j \le s} \binom{2k-\beta_j-1}{k-j} \right) \\ &= k\binom{2k-1}{k}^{-1} \sum_{\beta_j \le s} \left\{ -\binom{2k-\beta_j-1}{k-j} + \binom{2k-\beta_j-1}{k-j-1} \right\} \\ &= \sum_{\beta_j \le s} (\beta_j - 2j) \frac{k}{k-\beta_j+j} \binom{2k-1}{k}^{-1} \binom{2k-\beta_j-1}{k-j}. \end{aligned}$$

Now it is easy to see that for $k \to \infty$ the limit of the RHS is

$$\sum_{\beta_j \le s} (\beta_j - 2j) 2^{-\beta_j} = R(x_s).$$

This means $|S_k(x_s(k)) - R(x_s)| < \varepsilon$ holds if $k \ge k(x_s)$. Since x_s assumes only a finite number of values, setting $k_1 := \max_{x_s}\{k(x_s)\}$, (3) is proved. ∎

The proof of the following lemma is rather involved and will be presented at the end of this section.

LEMMA 3. *The family of shadow functions* $\{S_k\}$ *is "uniformly equicontinuous"; i.e., for every positive* ε *there exist* $k_2 = k_2(\varepsilon)$ *and a positive* $\delta = \delta(\varepsilon)$ *such that* $|S_k(x) - S_k(x')| < \varepsilon$ *holds for* $k \ge k_2$, $|x - x'| < \delta$.

We assume the lemma above, and prove the theorem.

Proof of Theorem 4. Since the Takagi function is uniformly continuous, $|R(x) - R(x')| < \varepsilon$ holds for some positive $\delta_0 = \delta(\varepsilon)$ whenever $|x - x'| < \delta_0$. We take $k_2, \delta > 0$ from Lemma 3, and define $\delta_1 := \min\{\delta_0, \delta\}$. By Lemma 2 (1), there exists $s_0 = s_0(\delta_1)$ such that $\max_{0 \le x \le 1}|x - x_s| < \delta_1$ holds for $s \ge s_0$. Similarly, by Lemma 2 (2), there exists $k_0 = k_0(\delta_1)$ such that $\max_{0 \le x \le 1}|x_{s_0}(k) - x_{s_0}| < \delta_1$ holds for $k \ge k_0$. Finally, by Lemma 2 (3), there exists $k_1 = k_1(\varepsilon, s_0)$ such that $\sup_{0 \le x \le 1}|S_k(x_{s_0}(k)) - R(x_{s_0})| < \varepsilon$ holds for $k \ge k_1$. Define $k_3 := \max\{k_0, k_1, k_2\}$. Then for $k \ge k_3$, $0 \le x \le 1$, we have

$$\begin{aligned}|S_k(x) - R(x)| \le& \left|S_k(x) - S_k(x_{s_0})\right| + \left|S_k(x_{s_0}) - S_k(x_{s_0}(k))\right| \\ &+ \left|S_k(x_{s_0}(k)) - R(x_{s_0})\right| + \left|R(x_{s_0}) - R(x)\right| \\ \le& \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon. \quad \blacksquare\end{aligned}$$

Since we could not prove the uniform equicontinuity directly, we introduce a function $f_{k,s}$. For $0 \le x \le 1$, there exists an (essentially) unique sequence $\{\vec{\beta}_j\}$ which satisfies $\left\lfloor \binom{2k-1}{k} x \right\rfloor = \sum_{j \ge 0} \binom{2k - \beta_j - 1}{k - j}$. Using this sequence, we define the s's approximate of x as

$$\operatorname{appr}_{k,s}(x) := \binom{2k-1}{k}^{-1} \sum_{\beta_j \le s} \binom{2k - \beta_j - 1}{k - j}.$$

Further, we define

$$f_{k,s}(x) := S_k(\operatorname{appr}_{k,s}(x)) = k\binom{2k-1}{k}^{-1} \sum_{\beta_j \le s} \frac{\beta_j - 2j}{k - \beta_j + j} \binom{2k - \beta_j - 1}{k - j}.$$

We need two more lemmas.

LEMMA 4. *For every positive* ε *there exists* $s_0 = s_0(\varepsilon)$ *such that* $\sup_{0 \le x \le 1}|S_k(x) - f_{k,s}(x)| < \varepsilon$ *holds for every* k *and every* $s \ge s_0$.

Proof. We choose s_0 such that $\sum_{n>s_0} n(3/4)^n < \varepsilon$. Suppose that $s \ge s_0$. We have

$$\begin{aligned}|S_k(x) - f_{k,s}(x)| &= \left| k\binom{2k-1}{k}^{-1} \sum_{\beta_j > s} \frac{\beta_j - 2j}{k - \beta_j + j}\binom{2k-\beta_j-1}{k-j} \right| \\ &= \left| \sum_{\beta_j > s} (\beta_j - 2j) \frac{\{k \cdots (k-j+1)\}\{k \cdots (k-\beta_j+j+1)\}}{(2k-1)\cdots(2k-\beta_j)} \right|.\end{aligned}$$

Note that for fixed k and β,

$$\begin{aligned}&\{k \cdots (k-j+1)\}\{k \cdots (k-\beta+j+1)\} \\ &\quad = \frac{(k!)^2}{(2k-\beta)!}\binom{2k-\beta}{k-j} \\ &\quad \le \begin{cases} \{k \cdots (k-m+1)\}^2, & \text{if } \beta = 2m, \\ \{k \cdots (k-m+1)\}^2 (k-m), & \text{if } \beta = 2m+1. \end{cases}\end{aligned}$$

Case 1. $\beta_j = 2m$. In this case, we have $\beta_j \le 2k-2$ and $m \le k-1$. Thus,

$$\begin{aligned}&\frac{\{k \cdots (k-m+1)\}^2}{(2k-1)\cdots(2k-2m)} \\ &\quad = \prod_{j=0}^{m-1} \frac{(k-j)^2}{(2k-2j-1)(2k-2j-2)} \le \prod_{j=0}^{m-1}\left(\frac{k-j}{2k-2j-2}\right)^2 \\ &\quad \le \left(\frac{k-m+2}{2k-2m+2}\right)^{2m-2}\left(\frac{k-m+1}{2k-2m}\right)^2 \le \left(\frac{3}{4}\right)^{2m-2}\cdot 1^2 = \left(\frac{3}{4}\right)^{\beta_j-2}.\end{aligned}$$

Case 2. $\beta_j = 2m+1$. In this case, we have $\beta_j \le 2k-1$ and $m \le k-1$. Thus,

$$\begin{aligned}&\frac{\{k \cdots (k-m+1)\}^2 (k-m)}{(2k-1)\cdots(2k-2m-1)} \\ &\quad = \left\{\frac{k-m}{2k-2m-1}\prod_{j=0}^{m-1}\frac{k-j}{2k-2j-1}\right\}\left\{\frac{k-m+1}{2k-2m}\prod_{j=0}^{m-2}\frac{k-j}{2k-2j-2}\right\} \\ &\quad \le \left\{1\cdot\left(\frac{3}{4}\right)^m\right\}\left\{1\cdot\left(\frac{3}{4}\right)^{m-1}\right\} = \left(\frac{3}{4}\right)^{\beta_j-2}.\end{aligned}$$

Consequently, in both cases we have

$$|S_k(x) - f_{k,s}(x)| \le \sum_{\beta_j > s} |\beta_j - 2j| \left(\frac{3}{4}\right)^{\beta_j - 2}$$
$$\le \sum_{\beta_j > s} 2\beta_j \left(\frac{3}{4}\right)^{\beta_j - 2} \le \frac{32}{9} \sum_{n>s} n \left(\frac{3}{4}\right)^n \le \frac{32}{9}\varepsilon. \quad ∎$$

For $\vec{\beta} = (\beta_0, \beta_1, \dots, \beta_i)$, $1 \le \beta_0 \le \beta_1 \le \cdots \le \beta_i \le 2k - 1$, we define

$$z(\vec{\beta}) := \binom{2k-1}{k}^{-1} \sum_{j=0}^{i} \binom{2k - \beta_j - 1}{k - j} \quad \text{and} \quad b(\vec{\beta}) := \sum_{j=0}^{i} 2^{-\beta_i}.$$

Note that $b(\vec{\beta}) < b(\vec{\beta}')$ holds iff $z(\vec{\beta}) < z(\vec{\beta}')$.

LEMMA 5. *For every positive ε there exists s_1 such that for $s \ge s_1$ one can choose $k_1 = k_1(\varepsilon)$, $\delta = \delta(\varepsilon)$ for which $|f_{k,s}(x) - f_{k,s}(x')| < \varepsilon$ holds whenever $k \ge k_1$ and $|x - x'| < \delta$.*

Proof. Take $\delta_0 > 0$ such that $|x - x'| < \delta_0$ implies $|R(x) - R(x')| < \varepsilon$. Choose s_1 such that $2^{-s_1} < \delta_0$. Suppose that $s \ge s_1$. Let $B := \{\vec{\beta} = (\beta_0, \beta_1, \dots, \beta_i)$: $1 \le i \le s$, $1 \le \beta_0 \le \beta_1 \le \cdots \le \beta_i \le s\}$. For every $\vec{\beta} \in B$, there exists $k(\vec{\beta}) > s$ such that

$$k \ge k(\vec{\beta}) \Rightarrow \left|S_k(z(\vec{\beta})) - R(b(\vec{\beta}))\right| < \varepsilon.$$

In fact, for $x = \sum_{j=0}^{i} 2^{-\beta_i}$ we have $x_{\beta_i} = x = b(\vec{\beta})$ and $x_{\beta_i}(k) = z(\vec{\beta})$, which imply

$$\lim_{k \to \infty} S_k(z(\vec{\beta})) = \lim_{k \to \infty} S_k(x_{\beta_i}(k)) = R(x_{\beta_i}).$$

The last equality follows from Lemma 2 (3). Define $k_0 := \max_{\vec{\beta} \in B}\{k(\vec{\beta})\}$. Then,

$$\max_{\vec{\beta} \in B} \left|S_k(z(\vec{\beta})) - R(b(\vec{\beta}))\right| < \varepsilon$$

holds if $k \ge k_0$.

Next we define $\delta := \min_{\vec{\beta}, \vec{\beta}' \in B,\ \vec{\beta} \ne \vec{\beta}'} |z(\vec{\beta}) - z(\vec{\beta}')|$. Suppose that $0 \le x < x' < 1$, $|x - x'| < \delta$. We define $\vec{\beta}$ and $\vec{\beta}'$ by $\mathrm{appr}_{k,s}(x) = z(\vec{\beta})$ and $\mathrm{appr}_{k,s}(x') = z(\vec{\beta}')$. Then $\vec{\beta}$ and $\vec{\beta}'$ are adjacent in B with respect to the order given by

$$\vec{\beta} < \vec{\beta}' \Leftrightarrow z(\vec{\beta}) < z(\vec{\beta}') \Leftrightarrow b(\vec{\beta}) < b(\vec{\beta}').$$

Hence $|b(\vec{\beta}) - b(\vec{\beta}')| < 2^{-s} < \delta_0$ and $|R(b(\vec{\beta})) - R(b(\vec{\beta}'))| < \varepsilon$ hold. Thus,

$$\begin{aligned} &|f_{k,s}(x) - f_{k,s}(x')| \\ &\quad = |S_k(\mathrm{appr}_{k,s}(x)) - S_k(\mathrm{appr}_{k,s}(x'))| = \left|S_k\left(z(\vec{\beta})\right) - S_k\left(z(\vec{\beta}')\right)\right| \\ &\quad \le \left|S_k\left(z(\vec{\beta})\right) - R\left(b(\vec{\beta})\right)\right| + \left|R\left(b(\vec{\beta})\right) - R\left(b(\vec{\beta}')\right)\right| \\ &\qquad + \left|S_k\left(z(\vec{\beta}')\right) - R\left(b(\vec{\beta}')\right)\right| \\ &\quad < 3\varepsilon. \quad \blacksquare \end{aligned}$$

Finally, we prove Lemma 3.

Proof of Lemma 3. Fix $\varepsilon > 0$. Choose s_0 from Lemma 4 and s_1 from Lemma 5, and define $s := \max\{s_0, s_1\}$. By Lemma 4, $\sup_{0 \le x \le 1}|S_k(x) - f_{k,s}(x)| < \varepsilon$. By Lemma 5, $k \ge k_1$ and $|x - x'| < \delta$ imply $|f_{k,s}(x) - f_{k,s}(x')| < \varepsilon$. Therefore,

$$\begin{aligned} |S_k(x) - S_k(x')| &\le |S_k(x) - f_{k,s}(x)| + |f_{k,s}(x) - f_{k,s}(x')| \\ &\quad + |f_{k,s}(x') - S_k(x')| < 3\varepsilon. \quad \blacksquare \end{aligned}$$

This completes the proof of Theorem 4.

3. Proof of Theorem 6

First we establish a modulus of continuity for $T(x)$. Let $0 \le x < y \le 1$, $y - x = \varepsilon$. We want to estimate $T(x) - T(y)$.

Let $x = \sum 2^{-\beta_j}$, $y = \sum 2^{-\gamma_j}$. Let k be the smallest number for which $\beta_k > \gamma_k$. We have $\beta_k \ge \gamma_k + 1$, hence $\beta_{k+j} \ge \gamma_k + j + 1$ for all j. Suppose first that there exists a smallest number m satisfying

$$\beta_{k+m} \ge \gamma_k + m + 2. \tag{$\heartsuit$}$$

We have

$$y - x \ge 2^{-\gamma_{k+1}} + 2^{-\gamma_k - m - 1}.$$

In $T(x) - T(y)$, the terms up to $j = k - 1$ cancel each other. We shall compare the term with γ_k to the terms with $\beta_k, \beta_{k+1}, \ldots$. To this end we use the identity

$$(\gamma - 2k)2^{-\gamma} = \sum_{j=0}^{\infty} (\gamma + j + 1 - 2(k+j))2^{-\gamma-j-1}.$$

This gives

$$\begin{aligned} T(y) - T(x) &\\ &= \sum_{j=k+1}^{\infty} (\gamma_j - 2j)2^{-\gamma_j} \\ &\quad + \sum_{j=k+m}^{\infty} \left\{(\gamma_k + j - k + 1 - 2j)2^{-\gamma_k - j + k - 1} - (\beta_j - 2j)2^{-\beta_j}\right\}. \end{aligned}$$

If there is no m satisfying (♥), then the second summand is missing and our task is simpler.

We have $|\gamma_j - 2j| \leq \gamma_j$, hence

$$|\gamma_j - 2j|2^{-\gamma_j} \ll \gamma_j 2^{-\gamma_j}.$$

Now for $j \geq k+1$ we have $\gamma_j \geq \gamma_{k+1} + (j-k-1)$ and the function $x2^{-x}$ is decreasing, thus with $\gamma = \gamma_{k+1}$ we have

$$\begin{aligned} \sum_{j=k+1}^{\infty} (\gamma_j - 2j)2^{-\gamma_j} &\ll \sum (\gamma + j + k + 1)2^{-\gamma-j+k} \\ &\ll \gamma 2^{-\gamma} \ll \varepsilon \log(1/\varepsilon). \end{aligned}$$

The second term can be estimated similarly and we obtain

$$(k+m)2^{-(k+m)} \ll \varepsilon \log(1/\varepsilon)$$

again.

Now we compare $T(x)$ and $T_c(x)$. Write $c = 1 + \delta$, where $|\delta| < 1/2$. Let

$$x = \sum_{j \geq 0} 2^{-\beta_j} = \sum_{j \geq 0} c^{-j}\left(\frac{c}{1+c}\right)^{\gamma_j},$$

so that

$$T(x) = \sum (\beta_j - 2j)2^{-\beta_j}$$

and

$$T_c(x) = \frac{1+c}{2(1-c)} \sum_{j \geq 0} \frac{c^{2j} - c^{\gamma_j}}{c^j(1+c)^{\gamma_j}}.$$

We consider also the number $y = \sum 2^{-\gamma_j}$.

Let us estimate $|x-y|$. We have

$$|x-y| \leq \sum \left| 2^{-\gamma_j} - c^{-j}\left(\frac{c}{1+c}\right)^{\gamma_j} \right|.$$

Now consider the function

$$F(j,\gamma) = 2^{-\gamma} - c^{-j}\left(\frac{c}{1+c}\right)^{\gamma}.$$

For a fixed γ this is a monotone function of j (either increasing or decreasing), hence for every $0 \leq j \leq \gamma$ we have

$$|F(j,\gamma)| \leq \max(|F(0,\gamma)|, |F(\gamma,\gamma)|) \leq |F(0,\gamma)| + |F(\gamma,\gamma)|.$$

Now if $c < 1$, then $F(0,\gamma)$ is positive and $F(\gamma,\gamma)$ is negative, while for $c > 1$ it is just the opposite. In any case we have

$$|F(0,\gamma)| + |F(\gamma,\gamma)| = \operatorname{sgn}(c-1)\frac{(c^{\gamma}-1)}{(1+c)^{\gamma}}.$$

This is an upper bound for $|F(j,\gamma)|$ independent of j, hence

$$|x-y| \leq \sum |F(j,\gamma_j)| \leq \sum \frac{(c^{\gamma_j}-1)\operatorname{sgn}(c-1)}{(1+c)^{\gamma_j}}.$$

The γ_j are different positive integers, so we get an upper bound if we extend this sum for all integers, not just the γ_j's. This yields

$$|x-y| \leq \sum_{n=0}^{\infty} \frac{(c^n-1)\operatorname{sgn}(c-1)}{(1+c)^n} = \frac{(c^2-1)\operatorname{sgn}(c-1)}{c}$$
$$= \frac{|c^2-1|}{c} \leq 3|c-1|$$

if $c \geq 1/2$.

This implies $T(x) - T(y) \ll |\delta|\log(1/|\delta|)$. Since we have

$$|T(x) - T_c(x)| \leq |T(x) - T(y)| + |T(y) - T_c(x)|,$$

it is now sufficient to estimate

$$|T(y) - T_c(x)| \leq \sum \left| (\gamma_j - 2j)2^{-\gamma_j} - \frac{1+c}{2(1-c)} \frac{c^{2j} - c^{\gamma_j}}{c^j(1+c)^{\gamma_j}} \right|. \quad (\spadesuit)$$

It is possible to do the same as above and maximize each term in j for a fixed γ_j, but now this does not easily lead to a nice result. Instead we can just do a direct attack as follows.

Since $0 \leq j \leq \gamma_j$, each term is smaller than (writing γ for γ_j)

$$2\gamma 2^{-\gamma} + \frac{1+c}{2(1-c)} \frac{1+c^\gamma}{(1+c)^\gamma}.$$

The sum of these terms for $\gamma > L$ gives a contribution $\ll L2^{-L} + |\delta|^{-1}(3/5)^L = o(|\delta|)$ say with $L = K\log(|\delta|^{-1})$ with a suitable absolute constant K. For the terms with $\gamma \leq L$ we apply a Taylor expansion. We have

$$c^k = 1 + k\delta + O(k^2\delta^2) = 1 + k\delta + O(\gamma^2\delta^2)$$

uniformly for $|k| \leq \gamma \leq 1/|\delta|$, and also

$$(1+c)^\gamma = 2^\gamma(1 + O(\gamma\delta)).$$

These yield

$$\frac{c^{2j} - c^{\gamma_j}}{c^j(1+c)^{\gamma_j}} = \delta 2^{-\gamma_j}\left((2j - \gamma_j) + O\left(\gamma_j^2\delta\right)\right).$$

Hence each term in ($\spadesuit$) is $O(\gamma^2\delta)$, and summing these errors for all the summands with $\gamma \leq L$ we get a total error of size

$$L^3\delta \ll |\delta|(\log(1/|\delta|))^3.$$

4. Proof of Theorem 7

In this section, we assume that $0 \leq x \leq 1$ and $0 < c < 1$. We always represent x in the form

$$x = \sum_{j \geq 0} c^{-j}\left(\frac{c}{1+c}\right)^{\beta_j},$$

where $\{\beta_j\}$ is a strictly increasing sequence of positive integers. Recall that

$T_c(x)$ is defined by

$$T_c(x) := \frac{1+c}{2(1-c)} \sum_{j \geq 0} \frac{c^{2j} - c^{\beta_j}}{c^j (1+c)^{\beta_j}}.$$

Further, we set $l := \lfloor ck \rfloor$ and define the c-shadow function $S_{c,k}$ by

$$S_{c,k}(x) := \frac{1+c}{2(1-c)} \binom{k+l}{k}^{-1} K_l^k\left(\left\lfloor \binom{k+l}{k} x \right\rfloor\right), \qquad \text{for } 0 \leq x \leq 1.$$

For $x = \sum_{j \geq 0} c^{-j}(c/(1+c))^{\beta_j}$, and a positive integer s, we define $x_s := \sum_{\beta_j \leq s} c^{-j}(c/(1+c))^{\beta_j}$, and $x_s(k) := \binom{k+l}{l}^{-1} \sum_{\beta_j \leq s} \binom{k+l-\beta_j}{k-j}$. Note that $\#\{x_s\colon 0 \leq x \leq 1\}$ is finite for fixed s.

LEMMA 6. (1) *For every positive ε there exists $s_0 = s_0(\varepsilon)$ such that $\max_{0 \leq x \leq 1} |x - x_s| < \varepsilon$ holds for $s \geq s_0$.*

(2) *For every positive ε and every s there exists $k_0 = k_0(\varepsilon, s)$ such that $\max_{0 \leq x \leq 1} |x_s(k) - x_s| < \varepsilon$ holds for $k \geq k_0$.*

(3) *For every positive ε and every s there exists $k_1 = k_1(\varepsilon, s)$ such that $\sup_{0 \leq x \leq 1} |S_{c,k}(x_s(k)) - T_c(x_s)| < \varepsilon$ holds for $k \geq k_1$.*

Proof. (1) This follows from

$$|x - x_s| = \sum_{\beta_j > s} c^{-j} \left(\frac{c}{1+c}\right)^{\beta_j} = \sum_{\beta_j > s} c^{\beta_j - j} \left(\frac{1}{1+c}\right)^{\beta_j}$$
$$< \sum_{\beta_j > s} \left(\frac{1}{1+c}\right)^{\beta_j} \underset{s \to \infty}{\to} 0.$$

(2) For fixed x_s, we have

$$|x_s(k) - x_s|$$
$$= \left| \sum_{\beta_j \leq s} \left(\binom{k+l-\beta_j}{k-j} \binom{k+l}{l}^{-1} - c^{-j} \left(\frac{c}{1+c}\right)^{\beta_j} \right) \right|$$
$$= \left| \sum_{\beta_j \leq s} \left(\frac{\{k \cdots (k-j+1)\}\{l \cdots (l-\beta_j+j+1)\}}{(k+l) \cdots (k+l-\beta_j+1)} - \frac{-c^{\beta_j - j}}{(1+c)^{\beta_j}} \right) \right| < \varepsilon$$

if $k \geq k(x_s)$. So we may choose $k_0 := \max_{x_s}\{k(x_s)\}$.

(3) For fixed x_s, we have

$$\begin{aligned} S_{c,k}(x_s(k)) & \\ &= \frac{1+c}{2(1-c)}\binom{k+l}{k}^{-1} K_l^k\left(\sum_{\beta_j \le s}\binom{k+l-\beta_j}{k-j}\right) \\ &= \frac{1+c}{2(1-c)}\binom{k+l}{k}^{-1} \sum_{\beta_j \le s}\left\{-\binom{k+l-\beta_j}{k-j} + \binom{k+l-\beta_j}{l-j}\right\} \\ &\underset{k\to\infty}{\to} \frac{1+c}{2(1-c)} \sum_{\beta_j \le s} \frac{c^j - c^{\beta_j - j}}{(1+c)^{\beta_j}} = T_c(x_s). \end{aligned}$$

This means $|S_{c,k}(x_s(k)) - T_c(x_s)| < \varepsilon$ holds if $k \ge k(x_s)$. So we put $k_1 := \max_{x_s}\{k(x_s)\}$. ∎

The proof of the following lemma is rather involved and will be presented at the end of this section.

LEMMA 7. *The c-shadow functions* $\{S_{c,k}\}$ *are uniformly equicontinuous; i.e., for every positive* ε *there exist* $k_2 = k_2(\varepsilon)$ *and a positive* $\delta = \delta(\varepsilon)$ *such that* $|S_{c,k}(x) - S_{c,k}(x')| < \varepsilon$ *holds for* $k \ge k_2$, $|x - x'| < \delta$.

Assuming validity of the above lemma, we prove the theorem.

Proof of Theorem 7. Since T_c is uniformly continuous, $|T_c(x) - T_c(x')| < \varepsilon$ holds for some positive $\delta_0 = \delta(\varepsilon)$ whenever $|x - x'| < \delta_0$. We take $k_2, \delta > 0$ from Lemma 7, and define $\delta_1 := \min\{\delta_0, \delta\}$. By Lemma 6 (1), there exists $s_0 = s_0(\delta_1)$ such that $\max_{0 \le x \le 1}|x - x_s| < \delta_1$ holds for $s \ge s_0$. Similarly, by Lemma 6 (2), there exists $k_0 = k_0(\delta_1)$ such that $\max_{0 \le x \le 1}|x_{s_0}(k) - x_{s_0}| < \delta_1$ holds for $k \ge k_0$. Finally, by Lemma 6 (3), there exists $k_1 = k_1(\varepsilon, s_0)$ such that $\sup_{0 \le x \le 1}|S_{c,k}(x_{s_0}(k)) - T_c(x_{s_0})| < \varepsilon$ holds for $k \ge k_1$. Define $k_3 := \max\{k_0, k_1, k_2\}$. Then for $k \ge k_3$, $0 \le x \le 1$, we have

$$\begin{aligned} |S_{c,k}(x) - T_c(x)| &\le |S_{c,k}(x) - S_{c,k}(x_{s_0})| + |S_{c,k}(x_{s_0}) - S_{c,k}(x_{s_0}(k))| \\ &\quad + |S_{c,k}(x_{s_0}(k)) - T_c(x_{s_0})| + |T_c(x_{s_0}) - T_c(x)| \\ &\le \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon. \end{aligned}$$ ∎

To prove Lemma 7, we introduce a function $g_{k,s}$. For $0 \le x \le 1$, $\left\lfloor \binom{k+l}{k} x \right\rfloor = \sum_{j \ge 0}\binom{k+l-\beta_j}{k-j}$, we define the s's approximate of x as

$$\mathrm{appr}_{k,s}(x) := \binom{k+l}{l}^{-1} \sum_{\beta_j \le s}\binom{k+l-\beta_j}{k-j}.$$

Further, we define

$$\begin{aligned} g_{k,s}(x) &:= S_{c,k}(\mathrm{appr}_{k,s}(x)) \\ &= \binom{k+l}{l}^{-1} \sum_{\beta_j \le s} \left\{ \binom{k+l-\beta_j}{l-j} - \binom{k+l-\beta_j}{k-j} \right\}. \end{aligned}$$

We need two more lemmas.

LEMMA 8. *For every positive ε there exist $s_0 = s_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that*

$$\sup_{0 \le x \le 1} |S_{c,k}(x) - g_{k,s}(x)| < \varepsilon$$

holds whenever $s \ge s_0$ and $k \ge k_0$.

Proof. First note that for every $k \ge 2$ we have

$$\begin{aligned} &\binom{k+l-\beta}{l-j}\binom{k+l}{l}^{-1} \\ &= \frac{l \cdots (l-j+1)}{(k+l) \cdots (k+l-j+1)} \cdot \frac{k \cdots (k-\beta+j+1)}{(k+l-j) \cdots (k+l-\beta+j)} \\ &\le \left(\frac{l}{k+l}\right)^j \left(\frac{k}{k+l-j}\right)^{\beta-j} \le \left(\frac{l}{k+l}\right)^j = \left(\frac{\lfloor ck \rfloor}{k+\lfloor ck \rfloor}\right)^j \\ &\le \left(\frac{ck}{k+ck-1}\right)^j = \left(\frac{c}{1+c-(1/k)}\right)^j \le \left(\frac{c}{c+(1/2)}\right)^j. \end{aligned}$$

Next, fix a positive real ε with $0 < \varepsilon < c$. Then there exists k_0 such that $c - (1/k) > \varepsilon$ holds for all $k \ge k_0$. Thus, we have

$$\begin{aligned} &\binom{k+l-\beta}{k-j}\binom{k+l}{l}^{-1} \\ &= \frac{k \cdots (k-j+1)}{(k+l) \cdots (k+l-j+1)} \cdot \frac{l \cdots (l-\beta+j+1)}{(k+l-j) \cdots (k+l-\beta+j)} \\ &\le \left(\frac{k}{k+l}\right)^j \left(\frac{l}{k+l-j}\right)^{\beta-j} \le \left(\frac{k}{k+l}\right)^j \\ &\le \left(\frac{1}{1+c-(1/k)}\right)^j \le \left(\frac{1}{1+\varepsilon}\right)^j. \end{aligned}$$

Define $j_0 := \min\{j: \beta_j > s\}$. Then, we have

$$\begin{aligned} &|S_{c,k}(x) - g_{k,s}(x)| \\ &= \left| \binom{k+l}{l}^{-1} \sum_{\beta_j > s} \left\{ \binom{k+l-\beta}{l-j} - \binom{k+l-\beta_j}{k-j} \right\} \right| \\ &\le \binom{k+l}{l}^{-1} \sum_{\beta_j > s} \left\{ \binom{k+l-\beta}{l-j} + \binom{k+l-\beta_j}{k-j} \right\} \\ &\le 2 \sum_{j \ge j_0} \left(\frac{1}{1+\varepsilon} \right)^j = \frac{2}{\varepsilon} \left(\frac{1}{1+\varepsilon} \right)^{j_0 - 1} \underset{j_0 \to \infty}{\to} 0. \quad \blacksquare \end{aligned}$$

For $\vec{\beta} = (\beta_0, \beta_1, \ldots, \beta_i)$, $1 \le \beta_0 \le \beta_1 \le \cdots \le \beta_i \le k + l$, we define

$$z(\vec{\beta}) := \binom{k+l}{l}^{-1} \sum_{j=0}^{i} \binom{k+l-\beta_j}{k-j}$$

and

$$b(\vec{\beta}) := \sum_{j=0}^{i} c^{-j} \left(\frac{c}{1+c} \right)^{\beta_j}.$$

Note that $b(\vec{\beta}) < b(\vec{\beta}')$ holds iff $z(\vec{\beta}) < z(\vec{\beta}')$.

LEMMA 9. *For every positive ε there exists s_1 such that for $s \ge s_1$ one can choose $k_1 = k_1(\varepsilon)$, $\delta = \delta(\varepsilon)$ for which $|g_{k,s}(x) - g_{k,s}(x')| < \varepsilon$ holds whenever $k \ge k_1$ and $|x - x'| < \delta$.*

Proof. Take $\delta_0 > 0$ such that $|x - x'| < \delta_0$ implies $|T_c(x) - T_c(x')| < \varepsilon$. Choose s_1 such that $(1/(1+c))^{s_1} < \delta_0$. Suppose that $s \ge s_1$. Let $B := \{\vec{\beta} = (\beta_0, \beta_1, \ldots, \beta_i): 1 \le i \le s,\ 1 \le \beta_0 \le \beta_1 \le \cdots \le \beta_i \le s\}$. For every $\vec{\beta} \in B$, there exists $k(\vec{\beta}) > s$ such that

$$k \ge k(\vec{\beta}) \Rightarrow \left| S_{c,k}\left(z(\vec{\beta})\right) - T_c\left(b(\vec{\beta})\right) \right| < \varepsilon.$$

In fact, for $x = \sum_{j=0}^{i} c^{-j} (c/(1+c))^{\beta_j}$ we have $x_{\beta_i} = x = b(\vec{\beta})$ and $x_{\beta_i}(k) = z(\vec{\beta})$, which imply

$$\lim_{k \to \infty} S_{c,k}\left(z(\vec{\beta})\right) = \lim_{k \to \infty} S_{c,k}\left(x_{\beta_i}(k)\right) = T_c(x_{\beta_i}) = T_c\left(z(\vec{\beta})\right).$$

Define $k_0 := \max_{\vec{\beta} \in B}\{k(\vec{\beta})\}$. Then, $\max_{\vec{\beta} \in B} |S_{c,k}(z(\vec{\beta})) - T_c(b(\vec{\beta}))| < \varepsilon$ holds if $k \ge k_0$.

Next we define $\delta := \min_{\vec{\beta}, \vec{\beta}' \in B,\ \vec{\beta} \neq \vec{\beta}'} |z(\vec{\beta}) - z(\vec{\beta}')|$. Suppose that $0 \leq x < x' < 1$, $|x - x'| < \delta$. We define $\vec{\beta}$ and $\vec{\beta}'$ by $\mathrm{appr}_{k,s}(x) = z(\vec{\beta})$ and $\mathrm{appr}_{k,s}(x') = z(\vec{\beta}')$. Then $\vec{\beta} = \vec{\beta}'$, or $\vec{\beta}$ and $\vec{\beta}'$ are adjacent in B, i.e., $|b(\vec{\beta}) - b(\vec{\beta}')| = (c/(1+c))^s < \delta_0$. So $|T_c(b(\vec{\beta})) - T_c(b(\vec{\beta}'))| < \varepsilon$ holds. Thus,

$$
\begin{aligned}
&|g_{k,s}(x) - g_{k,s}(x')| \\
&= |S_{c,k}(\mathrm{appr}_{k,s}(x)) - S_{c,k}(\mathrm{appr}_{k,s}(x'))| = \left|S_{c,k}\left(z\left(\vec{\beta}\right)\right) - S_{c,k}\left(z\left(\vec{\beta}'\right)\right)\right| \\
&\leq \left|S_{c,k}\left(z\left(\vec{\beta}\right)\right) - T_c\left(b\left(\vec{\beta}\right)\right)\right| + \left|T_c\left(b\left(\vec{\beta}\right)\right) - T_c\left(b\left(\vec{\beta}'\right)\right)\right| \\
&\quad + \left|S_{c,k}\left(z\left(\vec{\beta}'\right)\right) - T_c\left(b\left(\vec{\beta}'\right)\right)\right| \\
&< 3\varepsilon. \quad \blacksquare
\end{aligned}
$$

Finally, we prove Lemma 7.

Proof. Fix $\varepsilon > 0$. Choose s_0, k_0 from Lemma 8 and s_1 from Lemma 9, and define $s := \max\{s_0, s_1\}$. By Lemma 8, $\sup_{0 \leq x \leq 1} |S_{c,k}(x) - g_{k,s}(x)| < \varepsilon$ holds if $k \geq k_0$. Choose k_1 and δ from Lemma 9. By Lemma 9, $k \geq \max\{k_0, k_1\}$ and $|x - x'| < \delta$ imply $|g_{k,s}(x) - g_{k,s}(x')| < \varepsilon$. Therefore,

$$
\begin{aligned}
|S_{c,k}(x) - S_{c,k}(x')| \leq |S_{c,k}(x) - g_{k,s}(x)| + |g_{k,s}(x) - g_{k,s}(x')| \\
+ |g_{k,s}(x') - S_{c,k}(x')| < 3\varepsilon. \quad \blacksquare
\end{aligned}
$$

This completes the proof of Theorem 7.

APPENDIX

It is known that Takagi function is not fractal; i.e., the Hausdorff dimension of the graph of the Takagi function is one (see [2]). However, Mandelbrot [6] treats curves like the Takagi function as borderline cases. The following example shows the self-similarity in the Takagi function.

EXAMPLE 3. Define $\mu, \lambda: \mathbf{R}^2 \to \mathbf{R}^2$ by

$$
\mu(x, y) = \left(\frac{1}{4} + \frac{x}{4}, \frac{1}{2} + \frac{y}{4}\right),
$$

$$
\lambda(x, y) = \left(\frac{1}{2} + \frac{x}{4}, \frac{1}{2} + \frac{y}{4}\right).
$$

Further, define

$$C := \{(x, T(x)) \in \mathbf{R}^2 : 0 \le x \le 1\},$$
$$C_0 := \{(x, y) \in C : 0 \le x < 1/4, 3/4 < x \le 1\}.$$

Then $C = \mu(C) \cup \lambda(C) \cup C_0$ holds. In human language, this says that enlarging the graph of the Takagi function on $[1/4, 1/2]$ by a factor of 4, one gets back the graph of the original Takagi function. The same holds for $[1/2, 3/4]$ as well.

The shadow function has a property in the same flavor, that is,

$$S_k\left(\frac{k-1}{2(2k-1)} + \frac{kx}{2(2k-1)}\right) = S_k\left(\frac{k-1}{2k-1} + \frac{kx}{2(2k-1)}\right)$$
$$= \frac{k-1}{2k-1}\left(1 + \frac{S_{k-1}(x)}{2}\right).$$

Let us define $U_\alpha : [0,1] \to [0,1]$ by

$$U_\alpha\left(\sum_{j \ge 0} 2^{-\beta_j}\right) := \sum_{j \ge 0} \alpha^{\beta_j}\left(\frac{1-\alpha}{\alpha}\right)^j \qquad \text{for } 0 < \alpha < 1.$$

Note that

$$U_{c/(1+c)}\left(\sum_{j \ge 0} 2^{-\beta_j}\right) = \sum_{j \ge 0} c^{-j}\left(\frac{c}{1+c}\right)^{\beta_j}$$

holds for $0 < c < 1$. This means $U_\alpha(x)$ gives the $(1+c)$-nary expansion of x when $\alpha = c/(1+c)$. $U_\alpha(x)$ is a kind of "Lebesgue singular function." In fact, it is a strictly increasing continuous function of bounded variation whose derivative vanishes almost everywhere if $\alpha \ne 1/2$. The Takagi function and U_α have the following relation [2]:

$$\frac{1}{2}\frac{\partial}{\partial \alpha} U_\alpha(x)\bigg|_{\alpha = 1/2} = T(x).$$

The generalized Takagi function has the following self-similarity.

EXAMPLE 4. Let c be a fixed real with $0 < c < 1$. Define $\mu_c, \lambda_c : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\mu_c(x, y) = \left(\left(\frac{c}{1+c}\right)^2 + \frac{cx}{(1+c)^2}, \frac{1}{2} + \frac{cy}{(1+c)^2}\right),$$
$$\lambda_c(x, y) = \left(\frac{c}{1+c} + \frac{cx}{(1+c)^2}, \frac{1}{2} + \frac{cy}{(1+c)^2}\right).$$

Further, define

$$C := \{(x, T_c(x)) \in \mathbf{R}^2 : 0 \le x \le 1\},$$

$$C_0 := \{(x, y) \in C : 0 \le x < (c/(1+c))^2, c(2+c)/(1+c)^2 < x \le 1\}.$$

Then $C = \mu_c(C) \cup \lambda_c(C) \cup C_0$ holds.

The generalized Takagi function can be expanded into series in the following way.

EXAMPLE 5. Let us define

$$z_j^i := U_{c/(1+c)}\left(\frac{i}{2^j}\right),$$

$$p(n) := n - \sum_{j \ge 1} \left\lfloor \frac{n}{2^j} \right\rfloor,$$

$$h_{i,j} := \frac{c^{p(2i+1)}}{2c(1+c)^j},$$

$$q_{c,j,i}(x) := \begin{cases} 0, & \text{for } 0 \le x \le z_j^i, \\ \dfrac{h_{i,j}}{z_{j+1}^{2i+1} - z_j^i}(x - z_j^i), & \text{for } z_j^i \le x \le z_{j+1}^{2i+1}, \\ \dfrac{-h_{i,j}}{z_j^{i+1} - z_{j+1}^{2i+1}}(x - z_j^{i+1}), & \text{for } z_{j+1}^{2i+1} \le x \le z_j^{i+1}, \\ 0, & \text{for } x \le z_j^{i+1} \le 1, \end{cases}$$

$$\varphi_{c,j}(x) := \sum_{i=0}^{2^j-1} q_{c,j,i}(x) \qquad \text{for } 0 \le x \le 1.$$

Then, the generalized Takagi function satisfies

$$T_c(x) = \sum_{j=0}^{\infty} \varphi_{c,j}(x).$$

Note that $T(x) = \sum_{j=0}^{\infty} \varphi_{1,j}(x)$.

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