JOURNAL OF COMBINATORIAL THEORY, Series A **71**, 127–145 (1995)

Uniform Intersecting Families with Covering Number Four

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Communicated by the Managing Editors

Received February 24, 1993

We determine the maximum size of uniform intersecting families with covering number at least four. The unique extremal configuration turns out to be different from the one that was conjectured 12 years ago. At the same time it permits us to give a counterexample to a conjecture of Lovász.

1. INTRODUCTION

Let X be a finite set. We denote by $\binom{X}{k}$ the family of all k-element subsets of X. A family $\mathscr{F}$ satisfying $\mathscr{F} \subset \binom{X}{k}$ is called k-uniform. The vertex set of $\mathscr{F}$ is X and it is often denoted by $V(\mathscr{F})$. An element of $\mathscr{F}$ is also called an edge of $\mathscr{F}$. The family $\mathscr{F}$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every F, $G \in \mathscr{F}$.

A set $C \subset X$ is called a *cover* (or transversal set) of $\mathscr{F}$ if it intersects every edge of $\mathscr{F}$. A cover C is also called a t-cover if $|C| = t$. The set of all t-covers of $\mathscr{F}$ is denoted by $\mathscr{C}_t(\mathscr{F})$. The covering number of $\mathscr{F}$ is the minimum cardinality of the covers and is denoted by $\tau(\mathscr{F})$. By the definition, $\tau(\mathscr{F}) = \min\{t : \mathscr{C}_t(\mathscr{F}) \neq \emptyset\}$.

0097-3165/95 $12.00

For a family $\mathscr{A} \subset 2^X$ and vertices $x_1, ..., x_i, y_1, ..., y_j \in X$. we define

$$\mathscr{A}(x_1 \cdots x_i \bar{y}_1 \cdots \bar{y}_j) := \{A \in \mathscr{A} : x_1, ..., x_i \in A, y_1, ..., y_j \notin A\},$$

and for $Y \subset X$,

$$\mathscr{A}(Y) := \{A : Y \subset A \in \mathscr{A}\},$$

$$\mathscr{A}(\bar{Y}) := \{A \in \mathscr{A} : Y \cap A = \emptyset\}.$$

For fixed $|X|$ and k, the maximum size of an intersecting family $\mathscr{F} \subset \binom{X}{k}$ was determinhed by Erdős *et al.* [1]. The covering number of the extremal configuration is one (if $|X| > 2k$), which means that there exists a vertex $x \in X$ such that all edges of the family contain this vertex. Such families are called trivial. Hilton and Milner [9] determined the maximum size of non-trivial (i.e., the covering number is at least 2) intersecting families. Then, Frankl [3] determined the maximum size of intersecting families with covering number three.

The main purpose of the present paper is to determine the maximum size of intersecting families with covering number four. We also prove the uniqueness of the extremal configuration. This turns out to be completely different from the one conjectured in [3]. This new construction permits us to give a counterexample to a conjecture of Lovász.

Let us begin with an important example.

Example 1. We construct an intersecting family $\mathscr{F}_0 \subset \binom{X}{k}$ *with* $\tau(\mathscr{F}_0) = 4$ as follows. First, fix $1 + 3(k-1)$ vertices x_0, x_i, y_i, z_i $(1 \leqslant i \leqslant k-1)$ in X. For $i = 1, 2$ we define 6 edges

$$X_i := \{x_1, ..., x_{k-1}, y_i\},$$

$$Y_i := \{y_1, ..., y_{k-1}, z_i\},$$

$$Z_i := \{z_1, ..., z_{k-1}, x_i\},$$

and set $\mathscr{G}_0 := \{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$. Next, we define $\mathscr{B}_0 \subset \binom{X}{4}$ by

$$\begin{aligned}\mathscr{B}_0 := &\{\{x_0, x_i, y_j, z_l\} : 1 \leqslant i, j, l \leqslant k-1\}\\ &\cup \{\{x_0, x_1, x_2, y_i\} : 1 \leqslant i \leqslant k-1\}\\ &\cup \{\{x_0, y_1, y_2, z_i\} : 1 \leqslant i \leqslant k-1\}\\ &\cup \{\{x_0, z_1, z_2, x_i\} : 1 \leqslant i \leqslant k-1\}.\end{aligned}$$

Finally, $\mathscr{F}_0 \subset \binom{X}{k}$ is defined by

$$\mathscr{F}_0 := \left\{ F \in \binom{X}{k} : \exists B \in \mathscr{B}_0,\ B \subset F \right\} \cup \mathscr{G}_0.$$

Remark 1. Let us examine $\mathscr{F}_0$. By the definition it follows that $|\mathscr{G}_0| = 6$ and

$$|\mathscr{B}_0| = k^3 - 3k^2 + 6k - 4.$$

If $k = 4$, then both $\mathscr{B}_0$ and $\mathscr{G}_0$ are k-uniform, and so is $\mathscr{F}_0 = \mathscr{B}_0 \cup \mathscr{G}_0$ as well. In this case, $|\mathscr{F}_0| = 42$. Erdős and Lovász constructed k-uniform intersecting family with covering number k which has $\lfloor k!(e-1) \rfloor$ edges. Lovász conjectured that this is the exact maximum size. For the case of $k = 4$ their family has size 41. Thus, our example is a counterexample to the conjecture of Lovász.

For $k \geq 4$, we have

$$\mathscr{F}_0 \subset \left\{ B \cup A : B \in \mathscr{B}_0,\ A \in \binom{X - B}{k - 4} \right\} \cup \mathscr{G}_0$$

and

$$\mathscr{F}_0 \supset \left\{ B \cup A : B \in \mathscr{B}_0,\ A \in \binom{X - (\{x_0\} \cup V(\mathscr{G}_0))}{k - 4} \right\}.$$

Therefore, we have

$$|\mathscr{B}_0| \binom{n - (1 + 3k)}{k - 4} \leqslant |\mathscr{F}_0| \leqslant |\mathscr{B}_0| \binom{n - 4}{k - 4} + 6.$$

For the case of covering number four, Frankl [3] conjectured that $|\mathscr{F}| \leqslant (k^3 - 3k^2 + 3k + 1)\binom{n-4}{k-4} + o(n^{k-4})$ holds if $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with covering number four. Thus, Example 1 is a counterexample to his conjecture. The above example is important, because it gives the maximum size of intersecting families with covering number four and it is the unique extremal configuration, as is shown by our main result:

THEOREM 1. *Let $k \geqslant 9$, $n > n_0(k)$, and $|X| = n$. Suppose that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{F}) \geqslant 4$, then*

$$|\mathscr{F}| \leqslant |\mathscr{F}_0|$$

holds. Equality holds if and only if $\mathscr{F}$ is isomorphic to $\mathscr{F}_0$.

The essential part of our Proof of Theorem 1 is to prove the following result.

THEOREM 2. *Let $k \geqslant 9$ and $|X| = n$. Suppose that $\mathscr{G} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{G}) \geqslant 3$. Then,*

$$|\mathscr{C}_3(\mathscr{G})| \leqslant k^3 - 3k^2 + 6k - 4$$

holds. Equality holds if and only if $\mathscr{G}$ is isomorphic to $\mathscr{G}_0$.

The Proof of Theorem 2 is valid for a proof of the following.

THEOREM 3. *Let $k \geqslant 9$ and $|X| = n$. Suppose that $\mathscr{G} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{G}) = \tau \geqslant 3$. Then, for every $A \in \binom{X}{\tau - 3}$ we have*

$$\# \left\{ C \in \binom{X}{\tau} : A \subset C,\ C \in \mathscr{C}_\tau(\mathscr{G}) \right\} \leqslant k^3 - 3k^2 + 6k - 4.$$

2. THEOREM 2 IMPLIES THEOREM 1

In this section, we assume Theorem 2 and prove Theorem 1. Let $k \geqslant 9$, $n \geqslant n_0(k)$, and $|X| = n$. Suppose that $\mathscr{F} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{F}) = \tau \geqslant 4$.

Let $x \in F \in \mathscr{F}$. We define edge-shrinking (see [10]) $\phi(x, F, \mathscr{F})$ as the following operation on a family $\mathscr{F}$. If $\varnothing \neq F' := F - \{x\}$, and $\mathscr{F}' := (\mathscr{F} - \{F\} \cup \{F'\}$ is still intersecting, then we define $\phi(x, F, \mathscr{F}) := \mathscr{F}'$; otherwise $\phi(x, F, \mathscr{F}) := \mathscr{F}$. (If we obtain multiple edges in this operation, we replace them by a single edge.) We continue this operation until we get a family $\mathscr{F}'$ such that

$$\phi(x, F, \mathscr{F}') = \mathscr{F}' \qquad \text{for all} \quad x \in F \in \mathscr{F}'.$$

Of course, $\mathscr{F}'$ is not uniquely determined from $\mathscr{F}$ in general, it depends on the choice of operations. We fix one such shrink-invariant family $\mathscr{F}'$. $\mathscr{F}'$ is called a kernel of $\mathscr{F}$. By the construction, $\mathscr{F}'$ is intersecting and $\tau(\mathscr{F}') = \tau$. (Thus, $|F'| \geqslant \tau$ holds for every $F' \in \mathscr{F}'$.) Note also that for every $F \in \mathscr{F}$ there exists $F' \in \mathscr{F}'$ such that $F' \subset F$. Define

$$\mathscr{B} := \mathscr{F}' \cap \binom{X}{\tau},$$

which we call a *base* of $\mathscr{F}$. $\mathscr{B}$ is intersecting and every edge of $\mathscr{B}$ is a τ-cover of $\mathscr{F}$.

Let $\mathscr{G}$ be the set of edges $G \in \mathscr{F}$ such that $B \not\subset G$ for every $B \in \mathscr{B}$. Finally, we define $\mathscr{K} := \mathscr{B} \cup \mathscr{G}$. Clearly, we have

$$\mathscr{F} \subset \left\{ F \in \binom{X}{k} : \exists K \subset F,\ K \in \mathscr{K} \right\},$$

which implies

$$|\mathscr{F}| \leqslant \sum_{K \in \mathscr{K}} \binom{n-|K|}{k-|K|} \leqslant |\mathscr{B}| \binom{n-\tau}{k-\tau} + |\mathscr{G}|. \tag{1}$$

It is known that $|V(\mathscr{K})|$ is bounded by a function not depending on n, i.e., $|V(\mathscr{K})| \leqslant f(k, \tau)$. So, we have $|\mathscr{G}| \leqslant O(n^{k-\tau-1})$. Thus, in order to give an upper bound of $|\mathscr{F}|$ we estimate the size of the base $\mathscr{B}$. First we consider the covering number of $\mathscr{B}$. The following result is a slight extension of an inequality obtained in [2].

LEMMA 1. *Let* $s := \tau(\mathscr{B})$. *Then* $|\mathscr{B}| \leqslant s\tau^{s-1}k^{\tau-s}$.

Proof. For $A \subset X$, we define $\mathscr{B}(A) := \{B \in \mathscr{B} : A \subset B\}$. Since $\tau(\mathscr{B}) = s$. there exists an s-cover S of $\mathscr{B}$. So we can choose $x_1 \in S$ such that $|\mathscr{B}(X_1)| \geqslant |\mathscr{B}|/s$ where $X_1 := \{x_1\}$.

Suppose that we could define $X_i = \{x_1, ..., x_i\}$ $(i < s)$ such that

$$|\mathscr{B}(X_i)| \geqslant |\mathscr{B}|/(s\tau^{i-1}).$$

X_i is not a cover of $\mathscr{B}$, because $|X_i| < \tau(\mathscr{B})$. So there exists $B \in \mathscr{B}$ such that $X_i \cap B = \emptyset$. Since $\mathscr{B}$ is intersecting, every edge in $\mathscr{B}(X_i)$ meets the τ-element set B. Thus, we can find $x_{i+1} \in B$ such that

$$|\mathscr{B}(X_{i+1})| \geqslant |\mathscr{B}(X_i)|/\tau \geqslant |\mathscr{B}|/(s\tau^i),$$

where $X_{i+1} = X_i \cup \{x_{i+1}\}$. Continuing this way, we obtain an s-element set X_s such that

$$|\mathscr{B}(X_s)| \geqslant |\mathscr{B}|/(s\tau^{s-1}).$$

If $s < \tau$, X_s is not a cover of $\mathscr{F}$. So there exists $F \in \mathscr{F}$ such that $X_s \cap F = \emptyset$. Since $\mathscr{F}$ is intersecting, every edge in $\mathscr{B}(X_s)$ meets the k-element set F. Thus, we can find $x_{s+1} \in F$ such that

$$|\mathscr{B}(X_{s+1})| \geqslant |\mathscr{B}(X_s)|/k \geqslant |\mathscr{B}|/(s\tau^{s-1}k),$$

where $X_{s+1} = X_s \cup \{x_{s+1}\}$. Continuing this way, we finally get a τ-element set X_τ such that

$$|\mathscr{B}(X_\tau)| \geqslant |\mathscr{B}|/(s\tau^{s-1}k^{\tau-s}).$$

Clearly $|\mathscr{B}(X_\tau)| \leqslant 1$, and we have the desired inequality. ▮

The RHS of (1) attains its maximum when $\tau = 4$. So, from now on, we assume that $\tau(\mathscr{F}) = 4$. In this case, $\mathscr{B}$ consists of 4-covers of $\mathscr{F}$, and it follows that

$$|\mathscr{F}| \leqslant |\mathscr{B}| \binom{n-4}{k-4} + O(n^{k-5}). \tag{2}$$

Define $b(k) := |\mathscr{B}_0| = k^3 - 3k^2 + 6k - 4$.

LEMMA 2. *For every $x, y \in X$, we have $|\mathscr{B}(xy)| \leqslant k^2 - k + 1$.*

Proof. Suppose $\tau(\mathscr{F}(\bar{x}\bar{y})) = 1$ and $z \in \mathscr{C}_1(\mathscr{F}(\bar{x}\bar{y}))$. Then $\{x, y, z\}$ is a cover of $\mathscr{F}$, which contradicts $\tau(\mathscr{F}) = 4$. So $\tau(\mathscr{F}(\bar{x}\bar{y})) \geqslant 2$ must hold. Using Proposition 1 (see Appendix), we have

$$|\mathscr{C}_2(\mathscr{F}(\bar{x}\bar{y}))| \leqslant k^2 - k + 1.$$

If $\{x, y, z, w\} \in \mathscr{B}$, then this edge is a cover of $\mathscr{F}$, which implies

$$\{z, w\} \in \mathscr{C}_2(\mathscr{F}(\bar{x}\bar{y})).$$

Thus, we have

$$|\mathscr{B}(xy)| \leqslant |\mathscr{C}_2(\mathscr{F}(\bar{x}\bar{y}))| \leqslant k^2 - k + 1. \quad ▮$$

The next lemma settles the case $\tau(\mathscr{B}) \geqslant 2$.

LEMMA 3. *If $s := \tau(\mathscr{B}) \geqslant 2$ then $|\mathscr{B}| < b(k)$.*

Proof. By Lemma 1, we have

$$|\mathscr{B}| \leqslant 48k \quad \text{if } s = 3$$

$$|\mathscr{B}| \leqslant 256 \quad \text{if } s = 4.$$

These upper bounds are less than $b(k)$ if $k \geqslant 9$. For $s = 2$, we have $|\mathscr{B}| \leqslant 8k^2 < b(k)$ if $k \geqslant 11$.

Finally, we settle the case $9 \leqslant k \leqslant 10$ and $s=2$. Suppose $\{1, 2\} \in \mathscr{C}_2(\mathscr{B})$. Since $\{1\}$ is not a cover of $\mathscr{B}$, we may suppose $\{2, 3, 4, 5\} \in \mathscr{B}$. Every edge in $\mathscr{B}(1\bar{2})$ meets $\{3, 4, 5\}$. Thus, using Lemma 2, we have

$$|\mathscr{B}(1\bar{2})| \leqslant |\mathscr{B}(13)| + |\mathscr{B}(14)| + |\mathscr{B}(15)| \leqslant 3(k^2-k+1).$$

In the same way, we also have

$$|\mathscr{B}(\bar{1}2)| \leqslant 3(k^2-k+1).$$

Therefore, we have

$$|\mathscr{B}| \leqslant |\mathscr{B}(12)| + |\mathscr{B}(1\bar{2})| + |\mathscr{B}(\bar{1}2)| \leqslant 7(k^2-k+1) < b(k). \quad \blacksquare$$

The next lemma shows that $|\mathscr{B}| = b(k)$ must hold to attain $|\mathscr{F}| \geqslant |\mathscr{F}_0|$.

LEMMA 4. *If* $|\mathscr{B}| \leqslant b(k)-1$, *then* $|\mathscr{F}| < |\mathscr{F}_0|$.

Proof. Using the inequality (2), we obtain

$$|\mathscr{F}| \leqslant (b(k)-1)\binom{n-4}{k-4} + O(n^{k-5}).$$

By the construction of $\mathscr{F}_0$ (see Remark 1), we have

$$b(k)\binom{n-(3k+1)}{k-4} < |\mathscr{F}_0|.$$

If $n > n_0(k)$, we have

$$(b(k)-1)\binom{n-4}{k-4} + O(n^{k-5}) < b(k)\binom{n-(3k+1)}{k-4},$$

because this is equivalent to

$$\binom{n-4}{k-4}\Big/\binom{n-(3k+1)}{k-4} + O(n^{-1}) < 1 + \frac{1}{b(k)-1}.$$

Consequently, we get $|\mathscr{F}| < |\mathscr{F}_0|$. $\blacksquare$

Now we return to the proof of Theorem 1. By Lemma 3 and Lemma 4, we have $|\mathscr{F}| < |\mathscr{F}_0|$ if $\tau(\mathscr{B}) \geqslant 2$. Thus, we may assume that $\tau(\mathscr{B}) = 1$. Let $\{x_0\}$ be a 1-cover of $\mathscr{B}$. Then, we have

$$\mathscr{G} = \{G \in \mathscr{F} : x_0 \notin G\}.$$

$\mathscr{G}$ is an intersecting family with $\tau(\mathscr{G}) = 3$. Using Theorem 2, we have $|\mathscr{C}_3(\mathscr{G})| \leqslant b(k)$.

If $\{x_0\} \cup C \in \mathscr{B}$ (and $x_0 \notin C$), then $C \in \mathscr{C}_3(\mathscr{G})$. So, we have $|\mathscr{C}_3(\mathscr{G})| \geqslant |\mathscr{B}|$. Hence we have $|\mathscr{B}| \leqslant b(k)$. If $|\mathscr{B}| < b(k)$, we have $|\mathscr{F}| < |\mathscr{F}_0|$ by Lemma 4. Thus, we may suppose $|\mathscr{B}| = b(k)$. Then, by Theorem 2, $\mathscr{B} = \mathscr{B}_0$ and $\mathscr{G} = \mathscr{G}_0$ hold. That is, $\mathscr{F} \subset \mathscr{F}_0$. This completes a proof of Theorem 1 assuming Theorem 2.

3. Proof of Theorem 2

Throughout this section, we assume that $k \geqslant 9$ and $|X| = n$. Suppose that $\mathscr{G} \subset \binom{X}{k}$ is an intersecting family with $\tau(\mathscr{G}) = 3$. Recall the definition of $\mathscr{G}_0$ and $\mathscr{B}_0$ (see Example 1). Let $\mathscr{C}_0 := \{B - \{x_0\} : B \in \mathscr{B}_0\}$; i.e.,

$$\begin{aligned}\mathscr{C}_0 := &\{\{x_i, y_j, z_l\} : 1 \leqslant i, j, l \leqslant k-1\}\\ &\cup \{\{x_1, x_2, y_i\} : 1 \leqslant i \leqslant k-1\}\\ &\cup \{\{y_1, y_2, z_i\} : 1 \leqslant i \leqslant k-1\}\\ &\cup \{\{z_1, z_2, x_i\} : 1 \leqslant i \leqslant k-1\}.\end{aligned}$$

Let $\mathscr{C} := \mathscr{C}_3(\mathscr{G})$. The destination of this section is to prove $|\mathscr{C}| \leqslant |\mathscr{C}_0| = k^3 - 3k^2 + 6k - 4$. We also determine the unique extremal configuration.

For $x \in G \in \mathscr{G}$, we define

$$\begin{aligned}\alpha(x, G) &:= \#\{C \in \mathscr{C}(x) : |C \cap G| = 1\},\\ \beta(x, G) &:= \#\{C \in \mathscr{C}(x) : |C \cap G| = 2\},\\ \gamma(x, G) &:= \#\{C \in \mathscr{C}(x) : |C \cap G| = 3\},\\ c(x, G) &:= \alpha(x, G) + \tfrac{1}{2}\beta(x, G) + \tfrac{1}{3}\gamma(x, G).\end{aligned}$$

$c(x, G)$ is called a contribution of x for G, because a simple enumeration shows the following.

Lemma 5. *For any $G \in \mathscr{G}$, $|\mathscr{C}| = \sum_{x \in G} c(x, G)$ holds.*

The following inequality was implicitly proved by Frankl [3]. (We include a proof in the Appendix for self-completeness. We recommend the reader see this proof first, because it is short but contains several basic ideas for our lengthy proof of Theorem 2.)

Lemma 6. *For any $x \in X$, $|\mathscr{C}(x)| \leqslant k^2 - k + 1$ holds.*

Lemma 7. *Let $x \in G \in \mathscr{G}$. Then $\alpha(x, G) \leqslant k^2 - 3k + 3$ holds.*

Proof. Choose $G_1 \in \mathscr{G}(\bar{x})$, $x_1 \in G \cap G_1$, and $G_2 \in \mathscr{G}(\bar{x}\bar{x}_1)$. Set $a := |G \cap G_1|$, $b := |G \cap G_2|$, $c := |(G_1 \cap G_2) - G|$. Then we have

$$\alpha(x, G) \leqslant (k-a-c)(k-b-c) + c(k-1). \tag{3}$$

As a function of c, the RHS attains its maximum when $a = b = 1$. So,

$$\begin{aligned}\alpha(x, G) &\leqslant (k-c-1)^2 + c(k-1)\\ &= c^2 - (k-1)c + (k-1)^2.\end{aligned}$$

Case 1. $1 \leqslant c \leqslant k-2$. In this case, $\alpha(x, G)$ attains the maximum when $c = 1$ or $c = k-2$, which implies

$$\alpha(x, G) \leqslant (k-1)^2 - (k-1)c + (k-1)^2 = k^2 - 3k + 3.$$

Case 2. $c = 0$. Since $G_1 \cap G_2 \neq \varnothing$, we have $a \geqslant 2$. So the RHS of (3) takes maximum when $a = 2$ and $b = 1$. Thus,

$$\alpha(x, G) \leqslant (k-2)(k-1) < k^2 - 3k + 3.$$

Now we may assume that

$$G_1 \cap G_2 = G_1 - \{x_1\}, \qquad G \cap G_1 \cap G_2 = \varnothing$$

hold for every $G_2 \in \mathscr{G}(\bar{x}\bar{x}_1)$. Choose $y \in G_1 \cap G_2$ and $G_3 \in \mathscr{G}(\bar{x}\bar{y})$. Since $G_1 \cup \{x_1\} \not\subset G_3$, we have $G_3 \notin \mathscr{G}(\bar{x}\bar{x}_1)$ and so $x_1 \in G_3$. Hence we have $|(G_3 \cap G_2) - G| \leqslant k-2$, because $x_2, y \notin (G_3 \cup G_2) - G$. Thus, we can apply Case 1 again (replace G_1 by G_3). ∎

LEMMA 8. *Let $x \in G \in \mathscr{G}$. Then $c(x, G) \leqslant k^2 - 2k + 2$.*

Proof. Using Lemma 6 and Lemma 7,

$$\begin{aligned}c(x, G) &\leqslant \alpha(x, G) + \tfrac{1}{2}(|\mathscr{C}(x)| - \alpha(x, G))\\ &= \tfrac{1}{2}\{(k^2-k-1) + (k^2-3k+3)\}\\ &= k^2 - 2k + 2. \quad ∎\end{aligned}$$

LEMMA 9. *If $5 \leqslant |A \cap B| \leqslant k-3$ holds for some $A, B \in \mathscr{G}$, then $|\mathscr{C}| < |\mathscr{C}_0|$ holds.*

Proof. Suppose that $5 \leqslant a := |A \cap B| \leqslant k-3$. If $x \in A - B$, then we have

$$\begin{aligned}&\alpha(x, A) \leqslant (k-a)(k-1),\\ &c(x, A) \leqslant \tfrac{1}{2}\{(k^2-k+1) + (k-a)(k-1)\}.\end{aligned}$$

If $x \in A \cap B$, by Lemma 8, we have

$$c(x, A) \leqslant k^2 - 2k + 2.$$

Using Lemma 5, we have

$$\begin{aligned}|\mathscr{C}| &= \sum_{x \in A} c(x, A) \\ &= (k-a) \times \tfrac{1}{2}\{(k^2-k+1)+(k-a)(k-1)\} + a(k^2-2k+2) \\ &= \tfrac{1}{2}(a-k)\{(k-1)a-2k^2+2k-1\} + a(k^2-2k+2) \\ &=: f(a).\end{aligned}$$

A simple computation shows that $f(a)$ attains the maximum when $a = k-3$. Thus,

$$\begin{aligned}|\mathscr{C}| &\leqslant f(k-3) \\ &= |\mathscr{C}_0| - \tfrac{1}{2}\{(k-a)(k-1)+1\} < |\mathscr{C}_0|. \quad \blacksquare\end{aligned}$$

LEMMA 10. *If* $|A \cap B|$, $|B \cap C|$, $|C \cap A| \leqslant 4$ *holds for some* $A, B, C \in \mathscr{G}$, *then one of the following holds.*

(i) $|\mathscr{C}| < |\mathscr{C}_0|$.

(ii) $|A \cap B| = |B \cap C| = |C \cap A| = 1$ *and* $A \cap B \cap C = \varnothing$.

(iii) $|A \cap B| = |B \cap C| = |C \cap A| = |A \cap B \cap C| = 1$.

Proof. Fix $A, B, C \in \mathscr{G}$ such that each of the pairwise intersections consists of at most four vertices. We define

$$D := A \cap B \cap C,$$

$$U_A := (B \cap C) - A, \qquad U_B := (C \cap A) - B, \qquad U_C := (A \cap B) - C,$$

$$W := U_A \cup U_B \cup U_C \cup D,$$

$$A' := A - W, \qquad B' := B - W, \qquad C' := C - W,$$

$$a := |U_A|, \qquad b := |U_B|, \qquad c := |U_C|, \qquad d := |D|.$$

We distinguish three types of 3-covers in $\mathscr{C}$. Let $\mathscr{C}_1 := \bigcup_{v \in D} \mathscr{C}(v)$, $\mathscr{C}_2 := \bigcap_{w \in W} \mathscr{C}(\bar{w})$, and $\mathscr{C}_3 := \mathscr{C} - \mathscr{C}_1 - \mathscr{C}_2$. By Lemma 5,

$$|\mathscr{C}_1| \leqslant \sum_{v \in D} |\mathscr{C}(v)| \leqslant d(k^2-k+1). \tag{4}$$

Since every 3-cover in $\mathscr{C}_2$ consists of three vertices each from A', B', and C', we have

$$|\mathscr{C}_2| \leqslant |A'|\,|B'|\,|C'| = (k-d-a-b)(k-d-b-c)(k-d-c-a). \tag{5}$$

Now, we want to estimate the size of $\mathscr{C}_3$. By the definition, each 3-cover $T \in \mathscr{C}_3$ contains some vertex in $U_A \cup U_B \cup U_C$ and no vertex in D. If T

contains a vertex in U_A, it must also contain a vertex in $A-D=A'\cup U_B\cup U_C$. We define "contributions" of pairs of vertices to the size of $\mathscr{C}_3$. For $u_1\in U_A$ and $x\in A'$, we define

$$c(u_1, x) := \#\{T\in\mathscr{C}_3(u_1x) : |T\cap(A—D)|=1\} + \tfrac{1}{2}\#\{\mathscr{T}\in\mathscr{C}_3(u_1x) : |T\cap(A-D)|=2\}.$$

For $u_1\in U_A$ and $u_2\in U_B$, we define

$$c(u_1, u_2) := \#\{T\in\mathscr{C}_3(u_1u_2) : |T\cap(A\cup B)|=2\} + \tfrac{1}{2}\#\{T\in\mathscr{C}_3(u_1u_2) : T\subset(A-B)\cup(B-A)\} + \tfrac{1}{3}\#\{T\in\mathscr{C}_3(u_1u_2) : T\cap U_C\neq\varnothing\}.$$

We also define the contributions of the other pairs of vertices, symmetrically. Then, by the above argument, we can show the following.

$$|\mathscr{C}_3| = \sum_{u_1\in U_A,\, x\in A'} c(u_1, x) + \sum_{u_2\in U_B,\, y\in B'} c(u_2. y) + \sum_{u_3\in U_C,\, z\in C'} c(u_3, z) + \sum_{u_1\in U_A,\, u_2\in U_B} c(u_1, u_2) + \sum_{u_2\in U_B,\, u_3\in U_C} c(u_2, u_3) + \sum_{u_3\in U_C,\, u_1\in U_A} c(u_3, u_1)$$

From now on, we estimate the contribution of each pair of vertices. Fix $u_1\in U_A$ and $x\in A'$. Take an edge $G\in\mathscr{G}(\bar{u}_1\bar{x})$. We note that $G\cap A\neq\varnothing$, and that every $T\in\mathscr{C}_3(u_1x)$ contains a vertex in G. Therefore, $c(u_1, x)\leqslant |G-A|+\frac{1}{2}|G\cap(A-D)|\leqslant k-\frac{1}{2}$.

Next, we fix $u_1\in U_A$ and $u_2\in U_B$, and take an edge $G\in\mathscr{G}(\bar{u}_1\bar{u}_2)$. We note that $G\cap A\neq\varnothing$ and $G\cap B\neq\varnothing$. If $G\cap(A\cap B)\neq\varnothing$, then it is easy to see that $c(u_1, u_2)\leqslant k-\frac{2}{3}$. Otherwise, we have $G\cap(A-B)\neq\varnothing$ and $G\cap(B-A)\neq\varnothing$, and hence $c(u_1, u_2)\leqslant k-1$. Thus, we can estimate $c(u_1, u_2)\leqslant k-\frac{2}{3}$.

Adding up these contributions, we get

$$|\mathscr{C}_3|\leqslant a(k-d-b-c)(k-\tfrac{1}{2})+b(k-d-c-a)(k-\tfrac{1}{2}) + c(k-d-a-b)(k-\tfrac{1}{2})+(ab+bc+ca)(k-\tfrac{2}{3}). \tag{6}$$

By three inequalities (4), (5), and (6), we have

$$|\mathscr{C}|\leqslant d(k^2-k+1)+(k-d-a-b)(k-d-b-c)(k-d-c-a) + a(k-d-b-c)(k-\tfrac{1}{2})+b(k-d-c-a)(k-\tfrac{1}{2}) + c(k-d-a-b)(k-\tfrac{1}{2})+(ab+bc+ca)(k-\tfrac{2}{3}) =: q(k). \tag{7}$$

Here $q(k)$ is a cubic polynomial of k, where the coefficients of k^3 and k^2 are 1 and $-(a+b+c+2d)$, respectively. Hence if $a+b+c+2d \geqslant 4$ and k is sufficiently large, then $q(k)$ is much less than $|'\mathscr{C}_0| = k^3 - 3k^2 + 6k - 4$. So, we must check the cases where $a+b+c+2d \leqslant 3$. When $d=0$, since A, B, and C are pairwise intersecting, we have $a=b=c=1$, and (ii) follows. When $d=1$, we have $a+b+c \leqslant 1$. If $a=b=c=0$, then (iii) follows. So, we may assume that $a=1$ and $b=c=0$. Then, the RHS of (7) is equal to $k^3 - 3k^2 + \frac{11}{k}k - \frac{5}{2}$, and is less than $|\mathscr{C}_0|$.

For small value of k, one can check directly. Recall that for $1 \leqslant a+d$, $b+d$, $c+d \leqslant 4$ there are only finitely many possibilities for choosing a, b, c, d (164 ways). Checking them one by one (of course by computer), one can show that the $q(k)$ is less than $|\mathscr{C}_0|$ if $k \geqslant 9$, except for the following two cases that imply (ii) or (iii):

$$a=b=c=1 \quad \text{and} \quad d=0$$

or

$$a=b=c=0 \quad \text{and} \quad d=1. \quad \blacksquare$$

LEMMA 11. *If* $|A \cap B| \geqslant k-2$ *holds for every* A, $B \in \mathscr{G}$ $(A \neq B)$, *then* $|\mathscr{C}| \leqslant |\mathscr{C}_0|$ *holds.*

Proof. Fix $G \in \mathscr{G}$. For every $x \in G$, we have

$$\alpha(x, G) \leqslant 2 \times 2 = 4,$$
$$c(x, G) \leqslant \tfrac{1}{2}\{(k^2-k+1)+4\} = \tfrac{1}{2}(k^2-k+5).$$

Thus,

$$|\mathscr{C}| \leqslant k \times \tfrac{1}{2}(k^2-k+5) < |\mathscr{C}_0|. \quad \blacksquare$$

LEMMA 12. *Suppose that* $|A \cap B| \leqslant 4$ *holds for some* A, $B \in \mathscr{G}$. *If* $|G \cap A| \geqslant k-2$ *or* $|G \cap B| \geqslant k-2$ *holds for every* $G \in \mathscr{G}$, *then* $|\mathscr{C}| < |\mathscr{C}_0|$ *holds.*

Proof. Set $a := |A \cap B|$ $(1 \leqslant a \leqslant 4)$. If $x \in A - B$, then we have

$$\alpha(x, A) \leqslant (k-a)\left(2 + \frac{k-a-1}{2}\right) = \frac{1}{2}\{k^2 - (2a-3)k + a^2 - 3a\},$$

$$c(x, A) \leqslant \frac{1}{2}\{(k^2-k+1) + \frac{1}{4}(k^2 - (2a-3)k + a^2 - 3a)\}$$
$$= \frac{1}{4}\{3k^2 - (2a-1)k + a^2 - 3a + 2\}.$$

If $x \in A \cap B$, we use $c(x, A) \leqslant k^2 - 2k + 2$ by Lemma 8. Thus,

$$|\mathscr{C}| \leqslant \tfrac{1}{4}(k-a)\{3k^2 - (2a-1)k + a^2 - 3a + 2\} + a(k^2 - 2k + 2).$$

The RHS is less than $|\mathscr{C}_0|$ when $k \geqslant 9$ and $1 \leqslant a \leqslant 4$. ▮

LEMMA 13. *If* $2 \leqslant |A \cap B| \leqslant k-3$ *holds for some* $A, B \in \mathscr{G}$, *then* $|\mathscr{C}| < |\mathscr{C}_0|$ *holds.*

Proof. Fix $A, B \in \mathscr{G}$ such that $2 \leqslant |A \cap B| \leqslant k-3$. By Lemma 9, we may assume that $2 \leqslant |A \cap B| \leqslant 4$. By Lemma 12, we may assume that there exists $G \in \mathscr{G}$ such that $|G \cap A|$, $|G \cap B| \leqslant 4$. We use Lemma 10. In this situation, neither (ii) nor (iii) can happen. Then $|\mathscr{C}| < |\mathscr{C}_0|$ follows. ▮

From now on, we may assume that $|A \cap B| \in \{1, k-2, k-1\}$ holds for every $A, B \in \mathscr{G}$ $(A \neq B)$.

LEMMA 14. *If* $|\mathscr{C}| \geqslant |\mathscr{C}_0|$ *then there exists* $A, B, C \in \mathscr{G}$ *such that* $A \cap B \cap C = \varnothing$ *and* $|A \cap B| = |B \cap C| = |C \cap A| = 1$.

Proof. By Lemma 11, we can choose $G_1, G_2 \in \mathscr{G}$ such that $|G_1 \cap G_2| = 1$. By Lemma 12, we can choose $G_3 \in \mathscr{G}$ such that $|G_1 \cap G_3| = |G_2 \cap G_3| = 1$. If $G_1 \cap G_2 G_3 = \varnothing$ then these are the desired edges.

Let $\{x\} = G_1 \cap G_2 \cap G_3$. Choose $A \in \mathscr{G}(\bar{x})$. Note that $\#\{i : |A \cap G_i| = k-2\} \leqslant 1$. So, we may assume that $|A \cap G_2| = |A \cap G_3| = 1$. Then, A, G_2, and G_3 are the desired edges. ▮

From now on, we fix $A, B, C \in \mathscr{G}$ such that $A \cap B = \{z\}$, $B \cap C = \{x\}$, and $C \cap A = \{y\}$ $(x \neq y \neq z \neq x)$.

LEMMA 15. *If* $|\mathscr{C}| \geqslant |\mathscr{C}_0|$ *then, for every* $G \in \mathscr{G}$, G *contains* $A - \{y, z\}$ *or* $B - \{z, x\}$ *or* $C - \{x, y\}$.

Proof. Fix any $G \in \mathscr{G} - \{A, B, C\}$.

Case 1. $|G \cap A| = |G \cap B| = |G \cap C| = 1$. Let $G \cap A = \{x'\}$, $G \cap B = \{y'\}$, $G \cap C = \{x'\}$. (We does not assume that x', y', and z' are distinct.) For every $K \in \mathscr{C}$ we have $K \cap \{x, y, z, x', y', z'\} \neq \varnothing$. Thus,

$$|\mathscr{C}| \leqslant 6(k^2 - k + 1) < |\mathscr{C}_0|$$

holds if $k \geqslant 8$.

Case 2. Otherwise. By symmetry, we may assume that $|G \cap A| \geqslant k-2$. In this case, $|G \cap B| = |G \cap C| = 1$ holds. Suppose that $G \cap A \not\supset A - \{y, z\}$.

Choose $v \in A - \{y, z\} - G$. Since $|G \cap A| \geqslant k-2$, we have $y \in G$ or $z \in G$. We may assume $z \in G$, i.e., $G \cap B = \{z\}$. Then, we have

$$|\mathscr{C}(\bar{x}\bar{y}\bar{z}\bar{v})| \leqslant |A - \{y, z, v\}|\,|B - \{z, x\}|\,|C - \{x, y\}| = (k-2)^2(k-3),$$
$$|\mathscr{C}(\bar{x}\bar{y}\bar{z}v)| \leqslant |B - \{z, x\}|\,|C \cap G| = k-2.$$

By Lemma 6, both $|\mathscr{C}(x)|$, $|\mathscr{C}(y)|$, and $|\mathscr{C}(z)|$ are at most $k^2 - k + 1$. Therefore, we can estimate

$$|\mathscr{C}| \leqslant (k-2)^2(k-3) + (k-2) + 3(k^2 - k + 1)$$
$$= |\mathscr{C}_0| - (k-1)(k-7) < |\mathscr{C}_0|. \quad \blacksquare$$

LEMMA 16. (i) $|\mathscr{C}(x\bar{y}\bar{z})| \leqslant k^2 - 3k + 3$.

(ii) *If* $G \cap A = \{y\}$ *or* $\{z\}$ *holds for some* $G \in \mathscr{G}(\bar{x})$, *then* $|\mathscr{C}(x\bar{y}\bar{z})| \leqslant k^2 - 3k + 2$.

Proof. For $u \in A - \{y, z\}$, we define

$$\alpha'(u) := \#\{K \in \mathscr{C}(xu\bar{y}\bar{z}) : |K \cap A| = 1\},$$
$$\beta'(u) := \#\{K \in \mathscr{C}(xu\bar{y}\bar{z}) : |K \cap A| = 2\},$$
$$c'(u) := \alpha(u) + \tfrac{1}{2}\beta(u).$$

Note that $|\mathscr{C}(x\bar{y}\bar{z})| = \sum_{u \in A - \{y, z\}} c'(u)$. We estimate $c'(u)$ for each $u \in A - \{y, z\}$. We use

$$\alpha'(u) + \beta'(u) = |\mathscr{C}(xu\bar{y}\bar{z})| \leqslant |\mathscr{C}(xu)| \leqslant k.$$

We also use the fact that for every $u \in A - \{y, z\}$ we have $\alpha'(u) \leqslant k-1$ and $c'(u) \leqslant k - \frac{1}{2}$. This follows from the fact that we can choose $G \in \mathscr{G}(\bar{x}\bar{u})$ and so $\alpha'(u) \leqslant |G - A| \leqslant k-1$.

Case 1. There exists $G \in \mathscr{G}(\bar{x})$ such that $G \cap A = \{y\}\rangle$ or $\{z\}$. In this case, for every $u \in A - \{y, z\}$ we have $\beta'(u) = 0$ which implies that

$$c'(u) = \alpha'(u) \leqslant k-1.$$

Thus, $|\mathscr{C}(x\bar{y}\bar{z})| = \sum_{u \in A - \{y, z\}} c'(u) \leqslant (k-2)(k-1)$.

Case 2. Otherwise. Choose $G_1 \in \mathscr{G}(\bar{x}\bar{u})$. Using Lemma 15 and $u \notin G_1$, we have $|G_1 \cap A| = 1$. Let $\{v_1\} = G_1 \cap A \subset A - \{y, z\}$. Choose $G_2 \in \mathscr{G}(\bar{x}\bar{v}_1)$ then $|G_2 \cap A| = 1$. Let $\{v_2\} = G_2 \cap A \subset A - \{y, z\}$. For $u \in A - \{y, z, v_1, v_2\}$, we have $\beta'(u) = 0$, and so $c'(u) = \alpha'(u) \leqslant k-1$. For v_1, v_2, we estimate $c'(v_i) \leqslant k - 1/2$. Thus,

$$|\mathscr{C}(x\bar{y}\bar{z})| = \sum_{u \in A - \{y, z\}} c'(u) \leqslant (k-4)(k-1) + 2(k-1/2)$$
$$= k^2 - 3k + 3. \quad \blacksquare$$

Remark 2. With the above assumptions, we have the following.

$$|\mathscr{C}(\bar{x}\bar{y}\bar{z})| \leqslant (k-2)^3,$$
$$|\mathscr{C}(x\bar{y}\bar{z})|,\ |\mathscr{C}(\bar{x}y\bar{z})|,\ |\mathscr{C}(\bar{x}\bar{y}z)| \leqslant k^2 - 3k + 3,$$
$$|\mathscr{C}(xy\bar{z})|,\ |\mathscr{C}(\bar{x}yz)|,\ |\mathscr{C}(x\bar{y}z)| \leqslant k,$$
$$|\mathscr{C}(xyz)| \leqslant 1.$$

Thus, we get

$$|\mathscr{C}| \leqslant k^3 - 3k^2 + 6k + 2 = |\mathscr{C}_0| + 6.$$

We shall improve this bound. To reduce the size of 3*-covers by six more edges, we need more precise discussion as we will see in the following.*

LEMMA 17. *If* $\mathscr{G}(\bar{x}\bar{y}\bar{z}) \neq \emptyset$ *then* $|\mathscr{C}| < |\mathscr{C}_0|$ *holds.*

Proof. Fix $G \in \mathscr{G}(\bar{x}\bar{y}\bar{z})$. By Lemma 15, we may assume that $G \cap A = A - \{y, z\}$. Let $G \cap B = \{w_1\}$ and $G \cap C = \{w_2\}$. Fix $u \in A - \{y, z\}$ and $G_1 \in \mathscr{G}(\bar{x}\bar{u})$. Using Lemma 15, and by symmetry, we may assume that $G_1 \cap B \supset B - \{z, x\}$. Since $|G \cap G_1| = 1$ and $G_1 \cap A \neq \emptyset$, we have $|\{y, z\} \cap G_1| = 1$.

Case 1. $y \in G_1$. We have

$$|\mathscr{C}(x\bar{y}\bar{z})| \leqslant k^2 - 3k + 2 \qquad \text{(by Lemma 16(ii))},$$
$$|\mathscr{C}(\bar{x}\bar{y}z)| \leqslant |C - \{x, y, w_2\}|\ |G \cap G_1| + |G_1 - \{y\}|$$
$$= (k-3) \times 1 + (k-1) = 2k - 4,$$
$$|\mathscr{C}(x\bar{y}z)| \leqslant 1,$$
$$|\mathscr{C}(xyz)| = 0.$$

This together with Remark 2, we get $|\mathscr{C}| \leqslant |\mathscr{C}_0| - (k^2 - 4k + 2) < |\mathscr{C}_0|$.

Case 2. $z \in G_1$. In this case, we have

$$|\mathscr{C}(x\bar{y}\bar{z})| \leqslant k^2 - 3k + 2 \qquad \text{(by Lemma 16(ii))},$$
$$|\mathscr{C}(xy\bar{z})| \leqslant 1,$$
$$|\mathscr{C}(xyz)| = 0.$$

Thus, $|\mathscr{C}| \leqslant |\mathscr{C}_0| - (k-5) < |\mathscr{C}_0|$. $\blacksquare$

From now on, we may assume that $|G \cap \{x, y, z\}| = 1$ holds for every $G \in \mathscr{G} - \{A, B, C\}$. Edges G in $\mathscr{G} - \{A, B, C\}$ are classified into two types.

(i) Type I: $|G \cap ((A \cup B \cup C) - \{x, y, z\})| = k - 2$. For example, $G = (A - \{y, z\}) \cup \{x\} \cup \{w\}$, $w \notin A \cup B \cup C$.

(ii) Type II: $|G \cap ((A \cup B \cup C) - \{x, y, z\})| = k - 1$. For example, $G = (A - \{z\}) \cup \{z'\}$, $z' \in C - \{x, z\}$.

LEMMA 18. *If there exists a type-I edge* $G \in \mathscr{G} - \{A, B, C\}$, *then* $|\mathscr{C}| < |\mathscr{C}_0|$ *holds.*

Proof. By symmetry, we may assume that $G = (A - \{y, z\}) \cup \{x\} \cup \{w\}$, $w \notin A \cup B \cup C$. Choose $u \in A - \{y, z\}$ and $G_1 \in \mathscr{G}(\bar{x}\bar{u})$. Using Lemma 15, and by symmetry, we may assume that $G_1 \supset B - \{z, x\}$. If $z \in G_1$ then $G \cap G_1 = \emptyset$, a contradiction. So G_1 is type I, which implies $G_1 = (B - \{z, x\}) \cup \{y, w\}$. Then, we have

$$|\mathscr{C}(x\bar{y}\bar{z})|,\ |\mathscr{C}(\bar{x}y\bar{z})| \leqslant k^2 - 3k + 2 \qquad \text{(by Lemma 16(ii))},$$

$$|\mathscr{C}(\bar{x}\bar{y}z)| \leqslant |C - \{x, y\}|\ |G \cap G_1| = k - 2,$$

$$|\mathscr{C}(xy)| \leqslant k,$$

$$|\mathscr{C}(\bar{x}yz)|,\ |\mathscr{C}(x\bar{y}z)| \leqslant k - 1.$$

Thus, $|\mathscr{C}| \leqslant |\mathscr{C}_0| - (k^2 - 4k + 4) < |\mathscr{C}_0|$. ∎

Now we are in the final stage. From now on, we may assume that all edges in $\mathscr{G} - \{A, B, C\}$ are type II. Choose $G_1 = (A - \{z\}) \cup \{z_1\}$, $z_1 \in B - \{z, x\}$. Choose $u \in A - \{y, z\}$ and $G_2 \in \mathscr{G}(\bar{x}\bar{u})$. (Of course, G_2 is also type II.)

Case 1. $G_2 \supset C - \{x, y\}$. Choose $x_1 \in B - \{x, z, z_1\}$ and $G_2 = (C - \{x\}) \cup \{x_1\}$. Then, we have

$$|\mathscr{C}(x\bar{y}\bar{z})|,\ |\mathscr{C}(\bar{x}\bar{y}z)| \leqslant k^2 - 3k + 2 \qquad \text{(by Lemma 16(ii))},$$

$$|\mathscr{C}(xy\bar{z})|,\ |\mathscr{C}(\bar{x}yz)| \leqslant k - 1,$$

$$|\mathscr{C}(xz)| \leqslant |G_1 \cap G_2| = 1.$$

Thus, $|\mathscr{C}| \leqslant |\mathscr{C}_0| - (k - 2) < |\mathscr{C}_0|$.

Case 2. $G_2 \supset B - \{z, x\}$. Choose $x_1 \in C - \{x, y\}$ and $G_2 = (B - \{x\}) \cup \{x_1\}$. Choose $v \in B - \{z, x\}$ and $G_3 \in \mathscr{G}(\bar{y}\bar{v})$. Applying the same argument in Case 1 to G_2 and G_3, we can choose $y_1 \in A - \{y, z\}$ and $G_3 = (C - \{y\}) \cup \{y_1\}$. Then, we have

$$|\mathscr{C}(x\bar{y}\bar{z})|,\ |\mathscr{C}(\bar{x}\bar{y}z)|,\ |\mathscr{C}(\bar{x}y\bar{z})| \leqslant k^2-3k+2 \qquad \text{(by Lemma 16(ii))},$$

$$|\mathscr{C}(xy\bar{z})|,\ |\mathscr{C}(\bar{x}yz)|,\ |\mathscr{C}(x\bar{y}z)| \leqslant k-1.$$

By putting this together with Remark 2, we have $|\mathscr{C}| \leqslant |\mathscr{C}_0|$.

Here, we determine the extremal configuration. Suppose that $|\mathscr{C}| = |\mathscr{C}_0|$. Then all equalities must hold in the above eight inequalities. Let

$$A_1 = A - \{y, z\}, \qquad B_1 = B - \{z, x\}, \qquad C_1 = C - \{x, y\}.$$

In this situation, we have

$$\begin{aligned}
\mathscr{C} = \{\{a, b, c\} &: a \in A_1,\ b \in B_1,\ c \in C_1\} \\
\cup \{\{x, a, b\} &: a \in A_1,\ b \in B \cup \{x_1\}\} \\
\cup \{\{y, b, c\} &: b \in B_1,\ c \in C \cup \{y_1\}\} \\
\cup \{\{z, c, a\} &: c \in C_1,\ a \in A \cup \{z_1\}\} \\
\cup \{\{x, y, b\} &: b \in B \cup \{x_1\}\} \\
\cup \{\{y, z, z\} &: c \in C \cup \{y_1\}\} \\
\cup \{\{x, y, z\}\}&.
\end{aligned}$$

This is isomorphic to $\mathscr{C}_0$.

Finally, we consider $\mathscr{G}$. At this point, we know that $\mathscr{G} \supset \{A, B, C, G_1, G_2, G_3\} =: \mathscr{H}$ and $\mathscr{H}$ is isomorphic to $\mathscr{G}_0$. Suppose that $G \in \mathscr{G} - \mathscr{H}$ exists. If $G \supset C - \{x\}$, then $G = C \cup \{x_2\}$, $x_2 \in B - \{x, x_1, z\}$. This case is impossible, because there exists $K = \{x, x_1, a\} \in \mathscr{C}$, $a \in A_1$, which satisfies $G \cap K = \varnothing$. By the same argument, we may assume that G does not contain $A - \{y\}$, $B - \{z\}$, or $C - \{x\}$. But in this case, there exists $K = \{a, b, c\} \in \mathscr{C}$, $a \in A - \{y\}$, $b \in B - \{z\}$, and $c \in C - \{x\}$ such that $G \cap K = \varnothing$, a contradiction. Therefore, $\mathscr{G} = \mathscr{H} \cong \mathscr{G}_0$ must hold.

Consequently, we have $|\mathscr{C}| \leqslant |\mathscr{C}_0|$ and equality holds if and only if $\mathscr{C}$ is isomorphic to $\mathscr{C}_0$ and $\mathscr{G}$ is isomorphic to $\mathscr{G}_0$. This completes the proof of Theorem 2.

Appendix: Proof of Lemma 6

First we prove the following proposition.

Proposition 1. *Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family with $\tau(\mathscr{F}) \geqslant 2$. Let $E := \mathscr{C}_2(\mathscr{F})$. Then, $|E| \leqslant k^2 - k + 1$ holds.*

Proof. Let $x \in F \in \mathscr{F}$. We define

$$\alpha(x, F) := \#\{xy \in E : y \notin F\},$$
$$\beta(x, F) := \#\{xy \in E : y \in F\},$$
$$c(x, F) := \alpha(x, F) + \tfrac{1}{2}\beta(x, F),$$

$c(x, F)$ is considered as a contribution of x for F, because $|E| = \sum_{x \in F} c(x, F)$ holds. Since $\mathscr{F}$ is non-trivial intersecting, we have

$$\alpha(x, F) + \beta(x, F) \leqslant k \qquad \forall x \in F,$$
$$\alpha(x, F) \leqslant k-1 \qquad \forall x \in F.$$

If $\beta(x, F) = 0$ then we have $c(x, F) = \alpha(x, F) \leqslant k-1$. If $\beta(x, F) \geqslant 2$ then

$$c(x, F) = \tfrac{1}{2}\{\alpha(x, F) + \beta(x, F)\} + \tfrac{1}{2}\alpha(x, F)$$
$$\leqslant \tfrac{1}{2}(k-1) + \tfrac{1}{2}(k-1) = k-1.$$

Thus, if $\beta(x, F) \neq 1$ holds for every $x \in F$, we obtain

$$|E| \leqslant \sum_{x \in F} c(x, F) \leqslant k(k-1) < k^2 - k + 1.$$

So we may assume that $\beta(x, F) = 1$ holds for some $x \in F$. In this case,

$$c(x, F) \leqslant \alpha(x, F) + \tfrac{1}{2}\beta(x, F) \leqslant (k-1) + \tfrac{1}{2} = k - \tfrac{1}{2}.$$

Let us define

$$A :\# \{x \in F : \alpha(x, F) = k-1 \text{ and } \beta(x, F) = 1\}.$$

Then, we have

$$|E| \leqslant |A| \left(k - \frac{1}{2}\right) + (k - |A|)(k-1) = k^2 - k + \frac{|A|}{2}.$$

Thus, in order to attain $|E| \geqslant k^2 - k + 1$, we need $|A| \geqslant 2$. Let $F = \{x_1, ..., x_n\}$ and suppose that $x_1, x_2 \in A$. Define the neighborhood of x_i by $N(x_i) := \{y : x_i y \in E\}$. Note that $N(x_1), N(x_2) \in \mathscr{F}$.

Case 1. $x_1 x_2 \in E$. If $y \in F - \{x_1, x_2\}$ and $yz \in E$, then $z \in N(x_1) \cap N(x_2)$. This means $x_1 x_2$ is the only edge which is contained in F. Thus, $\beta(x_i, F) = 0$ holds if $i \geqslant 3$. Therefore, we have

$$|E| \leqslant \sum_{i=1,2} c(x_i, F) + \sum_{i \geqslant 3} c(x_i, F)$$
$$\leqslant 2(k - 1/2) + (k-2)(k-1) = k^2 - k + 1.$$

Case 2. $x_1 x_2 \notin E$. Let $x_2 x_3 \in E$. If $x_2 z \in E$ then $z \in N(x_1)$, which means $N(x_1) = N(x_2)$. Further, $N(x_i) \subset N(x_1)$ holds for every $i \geqslant 3$. Thus, every edge which meets F has x_3 as an endpoint. Therefore, we have

$$\begin{aligned} |E| &\leqslant \sum_{i \neq 3} c(x_i, F) + \{\alpha(x_3, F) + \tfrac{1}{2}\beta(x_3, F)\} \\ &\leqslant (k-1)^2 + (k-2) + \tfrac{1}{2} \times 2 = k^2 - k + 1. \quad \blacksquare \end{aligned}$$

Now we prove Lemma 6. Since $\mathscr{G} = \mathscr{G}(x) \cup \mathscr{G}(\bar{x})$ has covering number 3, we see that $\mathscr{G}(\bar{x})$ is an intersecting family with $\tau(\mathscr{G}(\bar{x})) \geqslant 2$. Let $E := \mathscr{C}_2(\mathscr{G}(\bar{x}))$. Then we have

$$\mathscr{C}(x) = \{x\} \cup E,$$

and $|\mathscr{C}(x)| \leqslant k^2 - k + 1$ follows from the proposition.

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