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SHARPENING THE LYM INEQUALITY

PÉTER L. ERDŐS[1], P. FRANKL, D. J. KLEITMAN[2], M. E. SAKS[3] and L. A. SZÉKELY[4]

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The level sequence of a Sperner family $\mathcal{F}$ is the sequence $f(\mathcal{F})=\{f_i(\mathcal{F})\}$, where $f_i(\mathcal{F})$ is the number of i element sets of $\mathcal{F}$. The LYM inequality gives a necessary condition for an integer sequence to be the level sequence of a Sperner family on an n element set. Here we present an indexed family of inequalities that sharpen the LYM inequality.

1. Introduction

A collection $\mathcal{F}$ of subsets of a set X is a *Sperner family* of X if no member of $\mathcal{F}$ is a subset of another. Sperner theory is a rich area in combinatorial theory; the seminal result in the area is the well known:

Theorem 1. (Sperner) [9] *If $\mathcal{F}$ is a Sperner family of a set of cardinality n, then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

More detailed information about the structure of Sperner families can be obtained by considering their *level sequences*. The level sequence of a family $\mathcal{F}$, $f(\mathcal{F})=\{f_i(\mathcal{F})\}$, has $f_i(\mathcal{F})$ equal to the number of members of $\mathcal{F}$ with exactly i elements. Sperner's theorem asserts that $\sum_i f_i(\mathcal{F}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$. A stronger restriction on the level sequence was proved independently by Lubell, Yamamoto and Meshalkin:

Theorem 2. (The LYM inequality) [7], [10], [8] *If $\mathcal{F}$ is a Sperner family of an n-set then*

$$\sum_{i=0}^{n} \frac{f_i(\mathcal{F})}{\binom{n}{i}} \le 1.$$

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We say that a sequence $f = \{f_i : i \in \mathbb{Z}\}$ is *realizable* as a Sperner family of an n-set, or n-realizable for short, if there is a Sperner family $\mathcal{F}$ of an n-set such that $f = f(\mathcal{F})$. Theorem 2 gives an important necessary (but far from sufficient) condition for f to be n-realizable. On the other hand, Clements and Daykin et. al. gave necessary and sufficient conditions for n-realizability based in the notion of a highly structured class of Sperner families called canonical Sperner families (defined in section 2). Their result asserts:

Theorem 3. ([1], [2]) *For each n-realizable sequence f there is a unique canonical Sperner family $\mathcal{F}$ such that $f = f(\mathcal{F})$.*

In this note, we use Theorem 3 to derive a sequence of inequalities each of which strengthens the LYM inequality. The first of these strengthenings yields, for instance, an immediate proof of Sperner's stronger theorem that a maximum size Sperner family on an n-set consists of sets of the same size. It has also been used by Kleitman and Sha [6] to bound the number of linear extensions of the lattice of subsets of a set.

2. Preliminaries

To define the notion of canonical Sperner family (which appears in Theorem 3) requires some preliminary definitions. Assume that the base set X is totally ordered, $X = \{\alpha_1 < \alpha_2 < \ldots < \alpha_n\}$. This total ordering induces a total ordering on 2^X, called the antilexicographic order:

$$A <_{AL} B \Leftrightarrow \max\{\alpha_j : \alpha_j \in (A \setminus B) \cup (B \setminus A)\} \in A.$$

Let $X^{(i)}$ denote the collection of i element subsets of X and $AL(i)$ denote the restriction of AL to $X^{(i)}$. Also, let $AL(i,t)$ denote the first t subsets of $X^{(i)}$ under $AL(i)$.

A Sperner family $\mathcal{F}$ is *canonical* if for some integers $t_0, t_1, \ldots, t_n$, $\mathcal{F}$ consists of the minimal sets (with respect to inclusion) of $AL(0,t_0) \cup AL(1,t_1) \cup \ldots \cup AL(n,t_n)$. Generally this form is not unique, but if we suppose, that the condition $f_i(\mathcal{F}) = 0$ ($i = 0, \ldots, n$) implies that $t_i = 0$ then this form becomes unique. The connections between the parameters $f_i(\mathcal{F})$'s and t_i's are determined by the Kruskal–Katona Theorem ([4] and [5]).

In addition to Theorem 3, we will need an additional fact about canonical Sperner families. For a Sperner gamily $\mathcal{F}$ on $\{\alpha_1, \ldots, \alpha_n\}$ and for every pair k, i ($k \le n$ and $0 \le i \le 2^k - 1$) define $\mathcal{F}^{i,k}$ to be $\{A \in \mathcal{F} : A \cap \{\alpha_{n-k+1}, \ldots, \alpha_n\} = T_i\}$ where T_i is the i^{th} set of $\{\alpha_{n-k+1}, \ldots, \alpha_n\}$ in the AL order. Then for each $0 \le k \le n$, $\mathcal{F}^{0,k}, \mathcal{F}^{1,k}, \ldots, \mathcal{F}^{2^k-1,k}$ is a partition.

Proposition 4. *If $\mathcal{F}$ is a canonical Sperner family of $\{\alpha_1, \ldots, \alpha_n\}$ and $0 \le i < j \le 2^k - 1$ then every set in $\mathcal{F}^{i,k}$ has cardinality less than or equal to every set in $\mathcal{F}^{j,k}$.*

Proof. $\mathcal{F}$ is the set of minimal sets (with respect to inclusion) of $AL(0,t_0) \cup \ldots \cup AL(n,t_n)$ for some $t_0, \ldots, t_n$. Suppose $i < j$ and $A \in \mathcal{F}^{j,k}$ with $|A| = a$. Then $A \in AL(a,t_a)$ and by definition of AL order, $AL(a,t_a)$ contains all sets of size a whose intersection with $\{\alpha_{n-k+1},\dots,\alpha_n\}$ is T_i. Thus if $|B|>|A|$ and $B\cap\{\alpha_{n-k+1},\dots,\alpha_n\}=T_i$ then B has a subset in $AL(a,t_a)$ so $B\notin\mathcal{F}$. ∎

The following definitions concerning sequences of integers will be needed. For simplicity, all such sequences, $f=\{f_i\}$, are assumed to have index set $\mathbb{Z}$. The *support* of f, $\operatorname{supp}(f)=\{i: f_i\neq 0\}$. A sequence g is a *prefix* of f if there is an index j such that $g_i=f_i$ if $i<j$, $g_i=0$ if $i>j$ and $0\le g_j\le f_j$. There is a natural total ordering on prefixes of f with $g\le h$ if $g_i\le h_i$ for all i. Finally, define the operations $+$ and $-$ on prefixes componentwise.

An easy consequence of Proposition 4 is

Proposition 5. *Let $\mathcal{F}$ be a canonical Sperner family on $\{\alpha_1,\dots,\alpha_n\}$ with $f=f(\mathcal{F})$. For $k\le n$ and $0\le i\le 2^k-1$, let $f^{i,k}=f(\mathcal{F}^{i,k})$. Then*

(i) *$f^{0,k}$ is a prefix of f,*

(ii) *For $1\le i\le 2^k-2$, $f^{i,k}$ is a prefix of $f-(f^{0,k}+f^{1,k}+\ldots+f^{i-1,k})$,*

(iii) *$f^{2^k-1,k}=f-(f^{0,k}+\ldots+f^{2^k-2,k})$.* ∎

A sequence f satisfies the property LYM_n if

$$f_i=0 \text{ if } i<0 \text{ or } i>n;$$
$$\sum_{i=0}^{n}\frac{f_i}{\binom{n}{i}}\le 1.$$

The *shift sequence* ϱf of f is defined by $(\varrho f)_i=f_{i+1}$. For $k\ge 0$, $\varrho^k f$ is given by $(\varrho^k f)_i=f_{i+k}$.

3. A stronger version of LYM

In this section we prove the first strengthening of the LYM inequality.

Theorem 6. *Let f be a nonzero sequence with $\operatorname{supp}(f)\subseteq\{0,1,\dots,n\}$. Let g be the maximal prefix of f (with respect to the prefix ordering) such that ϱg satisfies LYM_{n-1} and let q be the largest index such that $g_q\neq 0$ (or $q=0$ if no such index exists). Then if f is n-realizable,*

$$\sum_{i=0}^{q}\frac{q}{i}\frac{f_i}{\binom{n}{i}}+\sum_{i=q+1}^{n}\frac{n-q}{n-i}\frac{f_i}{\binom{n}{i}}\le 1. \tag{1}$$

Remark: The theorem sharpens the LYM inequality since the coefficients of $f_i/\binom{n}{i}$ are at least 1.

Remark: The theorem implies the strict version of Sperner's theorem, i.e., the only Sperner families of cardinality $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ are $X^{(\lfloor\frac{n}{2}\rfloor)}$ and $X^{(\lceil\frac{n}{2}\rceil)}$. To see this, write (1) in the form $\sum(1+\Delta_i)(f_i/\lfloor\frac{n}{2}\rfloor)$. Then each Δ_i is nonnegative with $\Delta_i=0$ only if $i=q$ and $q=\lfloor\frac{n}{2}\rfloor$ of $\lceil\frac{n}{2}\rceil$. Thus if $\sum f_i=\binom{n}{\lfloor\frac{n}{2}\rfloor}$ the summation simplifies to $1+\sum(\Delta_i f_i)/(\binom{n}{\lfloor\frac{n}{2}\rfloor})$. The summation must be 0, which implies that $f_i=0$ unless $q=\lfloor\frac{n}{2}\rfloor$ or $\lceil\frac{n}{2}\rceil$.

Proof of Theorem 6.

Claim. *$f-g$ satisfies LYM_{n-1}.*

Proof. Let $\mathcal{F}$ be the canonical Sperner family with $f=f(\mathcal{F})$, which exists by Theorem 3. As defined in the previous section, $\mathcal{F}^{0,1}=\{A\in\mathcal{F}:\alpha_n\in A\}$ and $\mathcal{F}^{1,1}=\{A\in\mathcal{F}:\alpha_n\notin A\}$. Deleting α_n from each set in $\mathcal{F}^{0,1}$ yields a Sperner family on $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ with level sequence $\varrho f^{0,1}$. By Proposition 5, $f^{0,1}$ is a prefix of f so, by choice of g, $f^{0,1}\le g$, and $f^{1,1}=f-f^{0,1}\ge f-g$. Since $f^{1,1}$ satisfies LYM_{n-1}, $f-g$ does as well. ∎

By the claim and the fact that $g_i=0$ if $i>q$

$$\frac{(f-g)_q}{\binom{n-1}{q}}+\sum_{i>q}\frac{f_i}{\binom{n-1}{i}}\le 1. \tag{2}$$

By the choice of g,

$$\sum_{i<q}\frac{f_i}{\binom{n-1}{i-1}}+\frac{g_q}{\binom{n-1}{q-1}}\le 1. \tag{3}$$

Multiplying (2) by q/n and (3) by $(n-q)/n$ and summing yields

$$\sum_{i\le q-1}\frac{q}{i}\frac{f_i}{\binom{n}{i}}+\frac{f_q}{\binom{n}{q}}+\sum_{i>q}\frac{n-q}{n-i}\frac{f_q}{\binom{n}{i}}\le 1,$$

as required. ∎

4. A sequence of inequalities

We now present inequalities, one for each integer $k\le n$, that strengthen the LYM inequality, where the case $k=0$ in the LYM inequality itself and the case $k=1$ is the result of the previous section. First we need some definitions and preliminary lemmas.

For an integer i, let $B(i)$ be the set of nonzero bit positions in the binary expansion of i, i.e., $i=\sum_{j\in B(i)}2^j$, and let $b(i)=|B(i)|$. If f is a sequence with support in $\{0,\dots,n\}$ and $0\le k\le n$, the (k,n)-segmentation of f is the collection $f^0,f^1,\dots,f^{2^k-1}$ of sequences where

(i) f^0 is maximal prefix of f such that $\varrho^k f^0$ satisfies LYM_{n-k}

(ii) For each $1\le i\le 2^k-2$, f^i in the maximal prefix of $f-f^0-f^1-\dots-f^{i-1}$ such that $\varrho^{k-b(i)}f^i$ satisfies LYM_{n-k}.

(iii) $f^{2^k-1}=f-f^0-f^1-\dots-f^{2^k-2}$.

Note that each of $f^0,f^0+f^1,f^0+f^1+f^2,\dots$ are prefixes of f. Let $q_0=0$ and for $1\le i\le 2^k-1$, let q_i be the largest index in $\operatorname{supp}(f^0+f^1+\dots+f^{i-1})$ (or 0 if $\operatorname{supp}(f^0+f^1+\dots+f^{i-1})=\emptyset$). Let $q_{2^n}=n$. Note that all nonzero terms of f^i have indices between q_i and q_{i+1}.

Lemma 7. *f^{2^k-1} satisfies LYM_{n-k}.*

Proof. Let $\mathcal{F}$ be the canonical family with level sequence f. Let T_i denote the i^{th} subset of $\{\alpha_{n-k+1},\dots,\alpha_n\}$ in AL order; then deleting T_i from each member of $\mathcal{F}^{i,k}$ yields a Sperner family on $\{\alpha_1,\dots,\alpha_{n-k}\}$ and thus $\varrho^{k-b(i)}f^{i,k}$ satisfies LYM_{n-k} for each i. Furthermore, Proposition 5 implies that $f^{0,k}, f^{0,k}+f^{1,k}, f^{0,k}+f^{1,k}+f^{2,k},\dots$ are prefixes of f.

By the definition of f^i, it is easy to prove by induction that $f^0+f^1+\dots+f^i \ge f^{0,k}+f^{1,k}+\dots+f^{i,k}$ for all $i \le 2^k-2$. Hence

$$\begin{aligned} f^{2^k-1,k} &= f-(f^{0,k}+f^{1,k}+\dots+f^{2^k-1,k}) \\ &\ge f-(f^0+f^1+\dots+f^{2^k-2}) \\ &= f^{2^k-1}. \end{aligned}$$

Since $f^{2^k-1,k}$ satisfies LYM_{n-k}, so does f^{2^k-1}. ∎

Subsequently we use the notation $(x)_j$ for the *falling factorial polynomial* $x(x-1)\cdots(x-j+1)$ where $(x)_0=1$.

Corollary 8. *Let f be n-realizable and $k \le n$. Then for any choice of nonnegative $\lambda_0,\lambda_1,\dots,\lambda_{2^n-1}$ with $\sum\lambda_i \le 1$, we have*

$$\sum_{j=0}^{2^k-1}\sum_{i=q_j}^{q_{j+1}} \lambda_j \frac{(n)_k}{(i)_{k-b(j)}(n-i)_{b(j)}} \frac{f_i^j}{\binom{n}{i}} \le 1. \tag{4}$$

Proof. By part (ii) of the definition of the (n,k)-segmentation and Lemma 6, $\varrho^{k-b(j)}f^j$ satisfies LYM_{n-k} for all $0 \le j \le 2^k-1$. Multiplying the j^{th} such inequality by λ_j and summing on j yields (4). ∎

To get a strengthening of the LYM inequality, we want to choose $\lambda_0,\dots,\lambda_{2^n-1}$ in the corollary so that $\lambda_j(n)_k/(i)_{k-b(j)}(n-i)_{b(j)} \ge 1$ for each j and for $q_j \le i \le q_{j+1}$. Such a selection of λ_j is given by the following. Let $A=\{1,\dots,n\}$, $B=\{0,\dots,k-1\}$ and let I be the set of injections from B to A, so $|I|=(n)_k$. For $0 \le j \le 2^k-1$, let I_j denote the set of injections r that map the integers in $B(j)$ to a number bigger than q_j and each integer in $B \backslash B(j)$ to a number less than or equal to q_{j+1}. We will prove:

Claim 1. $|I_0|+\dots+|I_{2^k-1}| \le (n)_k$.

Claim 2. $|I_j| \ge (i)_{k-b(j)}(n-i)_{b(j)}$ *for* $q_j \le i \le q_{j+1}$.

Claim 3. $c_j \equiv |I_j| = \sum_{m=0}^{b(j)} \binom{b(j)}{m}(q_{j+1}-q_j)_m(n-q_{j+1})_{b(j)-m}(q_{j+1}-m)_{k-b(j)}$.

From these claims, we get that $\lambda_j = c_j/(n)_k$ is an appropriate choice and the following strengthening of LYM is obtained.

Theorem 9. *Let f be a sequence and $f^0, f^1, \dots, f^{2^k-1}$ be the (n,k)-segmentation of f. Then if f is n-realizable,*

$$\sum_{j=0}^{2^k-1} \sum_{i=q_j}^{q_{j+1}} \frac{c_j}{(i)_{k-b(j)}(n-i)_{b(j)}} \frac{f_i^j}{\binom{n}{i}} \le 1.$$

To prove the theorem it is enough to prove the claims.

Proof of Claim 1. This follows from the fact that the I_i's are disjoint. Suppose $0 \le j < j' \le 2^k - 1$. Then $t \in B(j') \setminus B(j)$ for some $0 \le t \le k-1$, which implies that $r(t) \le q_{j+1}$ if $r \in I(j)$ and $r(t) > q_{j'} \ge q_j$ if $r \in I(j')$ which means that I_j and $I_{j'}$ are disjoint. ∎

Proof of Claim 2. I_j contains the set on injections that map $B \setminus B(j)$ to $\{1, \dots, i\}$ and $B(j)$ to $\{i+1, \dots, n\}$, and there are $(i)_{k-b(j)}(n-i)_{b(j)}$ to these. ∎

Proof of Claim 3. The members of I_j can be constructed exactly once as follows. Select the number m of elements of $B(j)$ that are mapped to $\{q_j+1, q_j+2, \dots, q_{j+1}\}$, where $0 \le m \le b(j)$. For each such m, there are $\binom{b(j)}{m}$ ways to select these m elements, $(q_{j+1}-q_j)_m$ ways to map them, $(n-q_{j+1})_{b(j)-m}$ ways to map the remaining elements of $B(j)$ and $(q_{j+1}-m)_{k-b(j)}$ ways to map the elements of $B - B_j$. ∎

Remark. The set of coefficients $\{c_j\}$ occurring on Theorem 9 are not unique. For $k=2$, we give another set of coefficients.

Theorem 10. *Suppose $k=2$, and $q_3 < n$ or $q_2 = q_3 = n$, then Theorem 2 holds with the following coefficients:*

$$c_0 = (q_1)_2, \ c_1 = q_2(n-q_1), \ c_2 = q_3(n-q_2), \ c_3 = (n-q_3)_2.$$

Proof. It is sufficient to prove that $\sum_{j=0}^{3} c_j \le (n)_2$. After elementary algebra we see, that this is equivalent to

$$(n - q_3 - 1)(q_3 - q_2) + (q_1 - 1)(q_2 - q_1) \ge 0.$$

This proves the theorem since $q_1 > 0$ because $f_0 = 0$. ∎

Note that Theorem 10 sharpens Theorem 9 for $k=2$ in almost every cases since the coefficients in Theorem 2 are

$$c_0 = (q_1)_2, \ c_1 = q_2(n-q_1) - (q_2 - q_1), \ c_2 = q_3(n-q_2) - (q_3 - q_2), \ c_3 = (n-q_3)_2.$$

References

[1] G. C. Clements: A minimization problem concerning subsets of finite sets, *Discrete Mathematics* **4** (1973), 123–128.

[2] D. E. Daykin, J. Godfrey, and A. J. W. Hilton: Existence theorems for Sperner families, *J. Combinatorial Theory* (A) **17** (1974), 245–251.

[3] C. Greene, and A. J. W. Hilton: Some results on Sperner families, *J. Comb. Theory* (A) **26** (1979), 202–209.

[4] G. O. H. Katona: A theorem of finite sets, *Theory of graphs,* Proc. Coll. Tihany, 1966, Akadémiai Kiadó, 1966, 187–207.

[5] J. B. Kruskal: The number of simplices in a complex, *Mathematical Optimization Techniques,* Univ. of Calif. Press, Berkeley and Los Angeles (1963), 251–278.

[6] D. J. Kleitman, J. Sha: The number of linear extensions of subset ordering, *Discrete Mathematics* **63** (1978), 271–278.

[7] D. Lubell: A short proof of Sperner's lemma, *J. Combinatorial Theory* **1** (1966), 299.

[8] L. D. Meshalkin: A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Verojatnost i Primen.* **8** (1963), 219–220, in Russian.

[9] E. Sperner: Ein Satz über Untermengen einer endliche Menge, *Math. Z.* **27** (1928), 544–548.

[10] K. Yamamoto: Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343–353.

P. L. Erdős

Mathematical Institute of the Hungarian Academy of Sciences, Budapest and Institute of Operations Research, University of Bonn

erdos@cwi.nl

P. Frankl

*University of Paris VII
2 Place Jussieu
Paris 75 005, France*

combinatorics@cs.meiji.ac.jo

D. J. Kleitman

*Massachusetts Institute of Technology,
Cambridge, MA 02139
U.S.A.*

djk@math.mit.edu

M. E. Saks

*University of California,
San Diego*

saks@ucsd.edu

L. A. Székely

*Eötvös University,
Budapest
and
University of New Mexico,
Albuquerque*

szekely@gauss.unm.edu