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Some Best Possible Inequalities Concerning Cross-Intersecting Families

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Let $\mathscr{A}$ be a non-empty family of a-subsets of an n-element set and $\mathscr{B}$ a non-empty family of b-subsets satisfying $A \cap B \neq \emptyset$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Suppose that $n \geqslant a+b$, $b \geqslant a$. It is proved that in this case $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{b} - \binom{n-a}{b} + 1$ holds. Various extensions of this result are proved. Two new proofs of the Hilton–Milner theorem on non-trivial intersection families are given as well.

1. INTRODUCTION

Let $X := \{1, 2, ..., n\}$ be an n-element set. For an integer k, $0 \leqslant k \leqslant n$, we denote by $\binom{X}{k}$ the set of all k-element subsets of X. A family $\mathscr{F} \subset \binom{X}{k}$ is called intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathscr{F}$. One of the best known results in extremal set theory is the following.

THEOREM [EKR]. *Let $\mathscr{F} \subset \binom{X}{k}$ be an intersecting family with $n = |X| \geqslant 2k$. Then, $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$.*

Two families $\mathscr{A} \subset \binom{X}{a}$ and $\mathscr{B} \subset \binom{X}{b}$ are said to be cross-intersecting if and only if $A \cap B \neq \emptyset$ holds for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Recall the following result of Hilton and Milner.

THEOREM A [HM]. *Let $\mathscr{A} \subset \binom{X}{a}$ and $\mathscr{B} \subset \binom{X}{a}$ be non-empty cross-intersecting families with $n = |X| \geqslant 2a$. Then, $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{a} - \binom{n-a}{a} + 1$.*

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Recently, Simpson [S] rediscovered this theorem. In this paper, we generalize the above result in various ways. Probably the following is the most natural extension of Theorem A.

THEOREM 1. *Let* $\mathscr{A} \subset \binom{X}{a}$ *and* $\mathscr{B} \subset \binom{X}{b}$ *be non-empty cross-intersecting families with* $n = |X| \geqslant a+b$, $a \leqslant b$. *Then the following hold*:

(i) $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{b} - \binom{n-a}{b} + 1$.

(ii) *If* $|\mathscr{A}| \geqslant \binom{n-1}{n-a}$, *then*

$$|\mathscr{A}| + |\mathscr{B}| \leqslant \begin{cases} \binom{n}{a} - \binom{n-a}{a} + 1 & \text{if } a = b \geqslant 2 \\ \binom{n-1}{a-1} + \binom{n-1}{b-1} & \text{otherwise.} \end{cases}$$

Putting restrictions on the size of $\mathscr{A}$ we can obtain stronger bounds.

THEOREM 2. *Let* $\mathscr{A} \subset \binom{X}{a}$ *and* $\mathscr{B} \subset \binom{X}{b}$ *be non-empty cross-intersecting families with* $n = |X| \geqslant a+b$, $a \leqslant b$. *Suppose taht* $\binom{\alpha}{n-a} \leqslant |\mathscr{A}| \leqslant \binom{n-1}{n-a}$ *holds for some real number* α *with* $n-a \leqslant \alpha \leqslant n-1$. *Then the following holds*:

$$|\mathscr{A}| + |\mathscr{B}| \leqslant \begin{cases} \binom{n}{b} - \binom{\alpha}{b} + \binom{\alpha}{n-a} & \text{if } a < b \text{ or } \alpha \leqslant n-2 \\ 2\binom{n-1}{a-1} & \text{if } a = b \text{ and } \alpha \geqslant n-2. \end{cases}$$

The next result is of similar flavor, and it will be used for one of the new proofs for the Hilton–Milner theorem (see Section 4).

THEOREM 3. *Let* $\mathscr{A} \subset \binom{Y}{a}$, $\mathscr{B} \subset \binom{Y}{a-1}$ *be non-empty cross-intersecting families with* $m = |Y| \geqslant 2a-1$. *Suppose that* $|\mathscr{A}| < \binom{m-1}{a-1}$, *then* $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{m}{a-1} - \binom{m-a}{a-1} + 1$.

2. PROOF OF THEOREM 1

To prove the theorem, we start with an easy inequality.

LEMMA 1. *Let* a, b, *and* n *be integers. Suppose that* $n \geqslant a+b$ *and* $a \leqslant b$. *Then, it follows that*

$$\binom{n-1}{a-1} + \binom{n-1}{b-1} \leqslant \binom{n}{b} - \binom{n-a}{b} + 1,$$

or equivalently,

$$\binom{n-1}{n-a}-\binom{n-1}{b}\leqslant\binom{n-a}{n-a}-\binom{n-a}{b}.$$

Proof. To prove the above inequality, it suffices to show that

$$\binom{x}{n-a}-\binom{x}{b}\leqslant\binom{x-1}{n-a}-\binom{x-1}{b}$$

holds for all real numbers x, $n-a+1\leqslant x\leqslant n-1$. This is equivalent to

$$\binom{x-1}{n-a-1}\leqslant\binom{x-1}{b-1}$$

$$\leftrightarrow (x-b)\cdot\cdots\cdot(x-n+a+1)\leqslant(n-a-1)\cdot\cdots\cdot b$$

$$\leftrightarrow x-b\leqslant n-a-1\leftrightarrow x\leqslant n-1+(b-a).$$

The above inequality follows from $x\leqslant n-1$ and $a\leqslant b$. ∎

Proof of Theorem 1. We prove the theorem by induction on n. Since the theorem clearly holds for $n=a+b$, we assume that $n>a+b$. Further, by the Kruskal–Katona theorem [Kr, Ka1], we may assume that $\mathscr{A}^c:=\{X-A : A\in\mathscr{A}\}$ is the collection of the smallest $|\mathscr{A}|$ sets in $\binom{X}{n-a}$ with respect to the colex order (see Appendix). Let us define

$$\mathscr{A}(n):=\{A-\{n\} : n\in A\in\mathscr{A}\}\subset\binom{X-\{n\}}{a-1},$$

$$\mathscr{A}(\bar{n}):=\{A : n\notin A\in\mathscr{A}\}\subset\binom{X-\{n\}}{a}.$$

We also define $\mathscr{B}(n)$ and $\mathscr{B}(\bar{n})$ in the same way.

Proof of (i). Since the RHS of the inequality in (ii) does not exceed the RHS of that of (i) we may suppose that $|\mathscr{A}|<\binom{n-1}{a-1}$ and therefore $\mathscr{A}(\bar{n})=\varnothing$.

Case 1. $\mathscr{B}(\bar{n})\neq\varnothing$. By the induction hypothesis, we have

$$|\mathscr{A}(n)|+|\mathscr{B}(\bar{n})|\leqslant\binom{n-1}{b}-\binom{(n-1)-(a-1)}{b}+1.$$

This, together with $|\mathscr{B}(n)|\leqslant\binom{n-1}{b-1}$, gives

$$|\mathscr{A}|+|\mathscr{B}|=|\mathscr{A}(n)|+|\mathscr{B}(\bar{n})|+|\mathscr{B}(n)|\leqslant\binom{n}{b}-\binom{n-a}{b}+1.$$

Case 2. $\mathscr{B}(\bar{n}) = \varnothing$. In this case, we have

$$|\mathscr{A}| + |\mathscr{B}| = |\mathscr{A}(n)| + |\mathscr{B}(n)| \leqslant \binom{n-1}{a-1} + \binom{n-1}{b-1}.$$

Using Lemma 1, we obtain the desired inequality.

Proof of (ii). Since the theorem holds if $|\mathscr{A}| = \binom{n-1}{a-1}$, we assume that $|\mathscr{A}(n)| = \binom{n-1}{a-1}$ and $|\mathscr{A}(\bar{n})| > 0$. Note that $|\mathscr{A}(n)| = \binom{n-1}{a-1}$ implies $|\mathscr{B}(\bar{n})| = 0$.

Case 1. $a < b$. By the induction hypothesis, we have

$$|\mathscr{A}(\bar{n})| + |\mathscr{B}(n)| \leqslant \binom{n-1}{b-1} - \binom{(n-1)-a}{b-1} + 1.$$

So we obtain

$$\begin{aligned}|\mathscr{A}| + |\mathscr{B}| &= |\mathscr{A}(n)| + |\mathscr{A}(\bar{n})| + |\mathscr{B}(n)| \\ &\leqslant \left\{\binom{n-1}{a-1} + \binom{n-1}{b-1}\right\} + \left\{1 - \binom{n-a-1}{b-1}\right\} \\ &< \binom{n-1}{a-1} + \binom{n-1}{b-1}.\end{aligned}$$

Using Lemma 1, we obtain the desired inequality.

Case 2. $a = b$. By the induction hypothesis, we have

$$|\mathscr{A}(\bar{n})| + |\mathscr{B}(n)| \leqslant \binom{n-1}{a} - \binom{(n-1)-(a-1)}{a} + 1.$$

This, together with $|\mathscr{A}(n)| = \binom{n-1}{a-1}$, gives

$$|\mathscr{A}| + |\mathscr{B}| = |\mathscr{A}(n)| + |\mathscr{A}(\bar{n})| + |\mathscr{B}(n)| \leqslant \binom{n}{a} - \binom{n-a}{a} + 1.$$

This completes the proof of (ii). ∎

3. Proofs of Theorem 2 and Theorem 3

In this section, we use Lovász' version of the Kruskal–Katona theorem, and so we need the following technical lemma.

Lemma 2. *Let s, t, and n be integers with $n > s + t$. Define a real valued function $f(x) := \binom{n}{s} - \binom{x}{s} + \binom{x}{n-t}$. Then, the following hold:*

(i) *Suppose that* $1+(n-s-t)v/s(v-n+t+1)<\binom{v}{s}/\binom{v}{n-t}$, *then* $f'(x)<0$ *holds for all real numbers* $x\leqslant v$.

(ii) *Let* u, v *be real numbers with* $u<v$, $u<n-t+s$. *Suppose that* $f'(u)<0$ *and* $f(u)\geqslant f(v)$, *then* $f(u)\geqslant f(x)$ *holds for all real numbers* x, $u\leqslant x\leqslant v$.

Proof. Proof of (i). Since

$$f'(x)=-\binom{x}{s}\sum_{j=0}^{s-1}\frac{1}{x-j}+\binom{x}{n-t}\sum_{j=0}^{n-t-1}\frac{1}{x-j},$$

the inequality $f'(x)<0$ is equivalent to

$$\left(\sum_{j=0}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right)<\binom{x}{s}\Bigg/\binom{n}{n-t}$$
$$=\frac{(n-t)\cdot\cdots\cdot(s+1)}{(x-s)\cdot\cdots\cdot(x-n+t+1)}. \tag{1}$$

By simple estimation, we have

$$\text{LHS}=1+\left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Bigg/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right)\leqslant 1+\frac{n-t-s}{x-n+t+1}\cdot\frac{x}{s}.$$

Thus, to prove (1), it suffices to show that

$$(x-s)\cdot\cdots\cdot(x-n+t+1)\left(1+\frac{n-t-s}{x-n+t+1}\cdot\frac{x}{s}\right)<(n-t)\cdot\cdots\cdot(s+1). \tag{2}$$

Since the LHS of (2) is increasing with x, it suffices to show (2) for $x=v$, that is,

$$1+\frac{n-t-s}{v-n+t+1}\cdot\frac{v}{s}<\binom{v}{s}\Bigg/\binom{v}{n-t}.$$

This was exactly our assumption.

Proof of (ii). Suppose on the contrary that $f(u)<f(x)$ holds for some x, $x>u$. Then, we may assume that there exist p, q which satisfy

$$u<p<q\leqslant v,$$
$$f'(p)=f'(q)=0,$$
$$f(p)<f(u)<f(q).$$

If $f'(x)=0$, it follows that

$$\binom{x}{s}=\binom{x}{n-t}\left\{1+\left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Big/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right)\right\}.$$

Substituting this into $f(x)$, we define a new function:

$$g(x):=\binom{n}{s}-\binom{x}{n-t}\left(\sum_{j=s}^{n-t-1}\frac{1}{x-j}\right)\Big/\left(\sum_{j=0}^{s-1}\frac{1}{x-j}\right).$$

Note that $g(x)=f(x)$ holds if $f'(x)=0$. Thus, $f(u)<g(q)$ must hold. We derive a contradiction by showing that $f(u)\geqslant g(x)$, or equivalently.

$$\left\{\binom{u}{s}-\binom{u}{n-t}\right\}\sum_{j=0}^{s-1}\frac{1}{x-j}\leqslant\binom{x}{n-t}\sum_{j=s}^{n-t-1}\frac{1}{x-j}$$

holds for all $x\geqslant p$. Since $u<n-t+s$, $\binom{u}{s}-\binom{u}{n-t}$ is positive, and so the LHS is decreasing with x. On the other hand, the RHS is increasing with x. Therefore, it suffices to check the inequality for $x=p$, that is $f(u)\geqslant g(p)=f(p)$, This was our assumption. ∎

Using the above lemma, we prove Theorem 2, which contains Theorem 1 (i).

Proof of Theorem 2. Since the theorem holds for $n=a+b$, we assume that $n>a+b$. Let $|\mathscr{A}|=\binom{x}{n-a}$, $n-a\leqslant\alpha\leqslant x\leqslant n-1$. Then, by the Kruskal–Katona theorem we have $|\mathscr{B}|\leqslant\binom{n}{b}-\binom{x}{b}$. Define $f(x):=\binom{n}{b}-\binom{x}{b}+\binom{x}{n-a}$.

Case 1. $a<b$. In this case, we prove that $f'(x)<0$ holds for $n-a\leqslant x\leqslant n-1$. By Lemma 2 (i), it suffices to show that

$$1+\frac{(n-a-b)(n-1)}{ba}<\binom{n-1}{b}\Big/\binom{n-1}{n-a}. \tag{1}$$

This holds for $n=a+b+1$. So we may assume that $n\geqslant a+b+2$. Then,

$$\begin{aligned}\text{RHS}&=\frac{(n-a)\cdot\cdots\cdot(n-b)}{b\cdot\cdots\cdot a}\\&=\frac{(n-a)(n-a-1)}{a(a+1)}\cdot\frac{(n-a-2)\cdot\cdots\cdot(n-b)}{b\cdot\cdots\cdot(a+2)}\\&\geqslant\frac{(n-a)(n-a-1)}{a(a+1)},\end{aligned}$$

$$\text{LHS}=1+\frac{n-a-(a+1)}{a+1}\cdot\frac{n-1}{a}.$$

To prove (1), it suffices to show that

$$1+\frac{(n-2a-1)(n-1)}{a(a+1)}<\frac{(n-a)(n-a-1)}{a(a+1)},$$

or equivalently, $n>2a+1$, and this was our assumption.

Case 2. $a=b$.

Subcase 2.1. $\alpha \leqslant n-2$. In this case, we prove that $f'(x)<0$ holds for $n-a \leqslant x \leqslant n-2$. By Lemma 2 (i), it suffices to show that

$$1+\frac{(n-2a)(n-2)}{a(a-1)}<\binom{n-2}{a}\Big/\binom{n-2}{n-a}. \tag{2}$$

This holds for $n=2a+1$. So we assume that $n \geqslant 2a+2$. Then,

$$(2) \leftrightarrow 1+\frac{(n-2a)(n-2)}{a(a-1)}<\frac{(n-a)(n-a-1)}{a(a-1)} \leftrightarrow n>2a.$$

This was our assumption.

Subcase 2.2. $\alpha \geqslant n-2$. Note that $f(n-2)=f(n-1)=2\binom{n-1}{a-1}$. So by Lemma 2 (ii), $f(x)<2\binom{n-1}{a-1}$ holds for $n-2<x \leqslant n-1$. ∎

Next we prove Theorem 3, which will be used to prove the Hilton–Milner theorem.

Proof of Theorem 3. Since the theorem clearly holds for $m=2a-1$, we assume that $m \geqslant 2a$. We distinguish two cases according to the size of $\mathscr{A}$.

Case 1. $1 \leqslant |\mathscr{A}| \leqslant \binom{m-2}{m-a}$. Let $|\mathscr{A}|=\binom{x}{m-a}$, $m-a \leqslant x \leqslant m-2$. Then, by the Kruskal–Katona theorem we have $|\mathscr{B}| \leqslant \binom{m}{a-1}-\binom{x}{a-1}$. Define $f(x):=\binom{m}{a-1}-\binom{x}{a-1}+\binom{x}{m-a}$. First we prove that $f'(x)<0$ holds for all x, $m-a \leqslant x \leqslant m-3$. By Lemma 2(i), it suffices to show that

$$1+\frac{(m-2a+1)(m-3)}{(a-1)(a-2)}<\binom{m-3}{a-1}\Big/\binom{m-3}{m-a}.$$

This holds for $m=2a$, and if $m>2a$ this is equivalent to $m>2a-1$ which is our assumption.

Next, we prove that $f(m-2) \geqslant f(x)$ holds for all x, $m-3 \leqslant x \leqslant m-2$. By Lemma 2 (i), it suffices to show that $f(m-a) \geqslant f(m-2)$, or equivalently,

$$\binom{(m-1)-1}{(m-1)-(a-1)}-\binom{(m-1)-1}{a-1}$$
$$\leqslant\binom{(m-1)-(a-1)}{(m-1)-(a-1)}-\binom{(m-1)-(a-1)}{a-1}.$$

This follows from Lemma 1.

Case 2. $|\mathscr{A}|>\binom{m-2}{m-a}$. In this case, we have $|\mathscr{B}|<\binom{m}{a-1}-\binom{m-2}{a-1}=\binom{m-1}{m-(a-1)}+\binom{m-2}{m-(a-1)-1}$. Let $|\mathscr{B}|=\binom{m-1}{m-(a-1)}+\binom{x}{m-(a-1)-1}$, $m-a\leqslant x<m-2$. Then, by the Kruskal–Katona theorem, we have $|\mathscr{A}|\leqslant\binom{m}{a}-\binom{m-1}{a}-\binom{x}{a-1}$. Define $f(x):=\binom{m-1}{m-(a-1)}+\binom{x}{m-(a-1)-1}+\binom{m}{a}-\binom{m-1}{a}-\binom{x}{a-1}=\binom{m}{a-1}-\binom{x}{a-1}+\binom{x}{m-a}$. By arguments in Case 1, $f(x)\leqslant f(m-a)$ holds for all x, $m-a\leqslant x\leqslant m-2$. ∎

4. Application

Using results of earlier sections, we give two new proofs of the Hilton–Milner theorem. Let us mention that other short proofs were given in [FF, M]. Recall that an intersecting family $\mathscr{F}$ is called non-trivial if $\bigcap_{F\in\mathscr{F}}F=\varnothing$ holds.

THEOREM [HM]. *Let $\mathscr{F}\subset\binom{X}{k}$ be a non-trivial intersecting family with $n=|X|\geqslant 2k$, Then $|\mathscr{F}|\leqslant\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$.*

Proof I. Suppose that $|\mathscr{F}|$ is maximal with respect to the conditions. First we deal with an important special case. Suppose that there exists $A:=\{a,b\}\in\binom{X}{2}$ such that $A\cap F\neq\varnothing$ holds for all $F\in\mathscr{F}$. By the maximality of $|\mathscr{F}|$, $\{G:A\subset G\in\binom{X}{2}\}\subset\mathscr{F}$ holds. Define

$$\mathscr{A}:=\{F-\{a\}:F\in\mathscr{F},F\cap A=\{a\}\},$$
$$\mathscr{B}:=\{F-\{b\}:F\in\mathscr{F},F\cap A=\{b\}\}.$$

Then $\mathscr{A}$, $\mathscr{B}$ are cross-intersecting families on $X-A$. By Theorem A,

$$|\mathscr{A}|+|\mathscr{B}|\leqslant 1+\binom{n-2}{k-1}-\binom{n-k-1}{k-1}.$$

Consequently,

$$|\mathscr{F}|\leqslant 1+\binom{n-2}{k-1}-\binom{n-k-1}{k-1}+\binom{n-2}{k-2}=1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1},$$

as desired.

Next consider the case when $\mathscr{F}$ is shifted (see Appendix). Note that $\{2, 3, ..., k+1\} \in \mathscr{F}$. Now define

$$\mathscr{A} := \{F - \{1\} : F \cap \{1, 2\} = \{1\}, F \in \mathscr{F}\},$$

$$\mathscr{B} := \{F - \{2\} : F \cap \{1, 2\} = \{2\}, F \in \mathscr{F}\},$$

$$\mathscr{C} := \{F - \{1, 2\} : \{1, 2\} \subset F \in \mathscr{F}\},$$

$$\mathscr{D} := \{F - \{1, 2\} : \{1, 2\} \cap F = \varnothing\}.$$

Then by Theorem A and $\{3, 4, ..., k+1\} \in \mathscr{A} \cap \mathscr{B}$, $|\mathscr{A}| + |\mathscr{B}| \leqslant 1 + \binom{n-2}{k-1} - \binom{n-k-1}{k-1}$ holds.

On the other hand, $\mathscr{C}$, $\mathscr{D}$ are cross-intersecting and $\mathscr{D}$ is 2-intersecting. Thus, $\mathscr{D}^c := \{X - D : D \in \mathscr{D}\} \subset \binom{X - \{1,2\}}{n-k-2}$ is $(n-2) - (2k-2) = (n-2k)$-intersecting. By the Intersecting Kruskal–Katona theorem (cf. [Ka2]), $|\mathscr{S} := \sigma_{k-2}(\mathscr{D}^c)| \geqslant |\mathscr{D}^c| = |\mathscr{D}|$ (see Appendix) and by the cross-intersecting property $\mathscr{S} \cap \mathscr{C} = \varnothing$. Therefore,

$$|\mathscr{C}| + |\mathscr{D}| \leqslant |\mathscr{S}| + |\mathscr{C}| \leqslant \left|\binom{X - \{1, 2\}}{k-2}\right| = \binom{n-2}{k-2}.$$

Again, we obtain $|\mathscr{F}| = |\mathscr{A}| + |\mathscr{B}| + |\mathscr{C}| + |\mathscr{D}| \leqslant 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$.

Now to the general case. Apply repeatedly to $\mathscr{F}$ the shift operator (see Appendix) S_{ij}, $1 \leqslant i < j \leqslant n$. Either we obtain a shifted non-trivial intersecting family of the same size (and we are done by the second case) or at some point the family stops to be non-trivial. That is for some $\mathscr{G} \subset \binom{X}{k}$, $\mathscr{G}$ non-trivial intersecting, $|\mathscr{F}| = |\mathscr{G}|$ we have that $\bigcap_{H \in S_{ij}(\mathscr{G})} H \neq \varnothing$. In this case, clearly $\{i\} = \bigcap_{H \in S_{ij}(\mathscr{G})} H$ and consequently $\{i, j\} \cap G \neq \varnothing$ for all $G \in \mathscr{G}$. Thus we are done by the first special case. ∎

Proof II. Since the theorem clearly holds for $n = 2k$, we assume that $n \geqslant 2k+1$. We may assume that $n \in F \in \mathscr{F}$ holds for some F. Let us define $Y := X - \{n\}$, $m := |Y|$, $a := k$,

$$\mathscr{A} := \{F : n \notin F \in \mathscr{F}\} \subset \binom{Y}{a},$$

$$\mathscr{B} := \{F - \{n\} : n \in F \in \mathscr{F}\} \subset \binom{Y}{a-1}.$$

Then $\mathscr{A}$ and $\mathscr{B}$ are non-empty cross-intersecting families. Since $\mathscr{A}$ is intersecting itself, $|\mathscr{A}| \leqslant \binom{m-1}{a-1}$ holds. First suppose that $|\mathscr{A}| = \binom{m-1}{a-1}$. If $m = 2a$, then $|\mathscr{A}| = \frac{1}{2}\binom{m}{a}$. Hence for all $B \in \mathscr{B}$ and for all $y \in Y - B$, $B \cup \{y\} \in \mathscr{A}$

holds. Therefore, $\mathscr{B}$ is also intersecting, and so we may that $m \in B$ holds for all $B \in \mathscr{B}$. Since $\mathscr{F}$ is non-trivial, there exists $A \in \mathscr{A}$ such that $m \notin A$. So,

$$|\mathscr{B}| \leqslant \left| \binom{Y-\{m\}}{a-2} - \binom{Y-A}{a-1} \right| = \binom{m-1}{a-2} - \binom{m-a}{a-1}.$$

This implies that $|\mathscr{F}| = |\mathscr{A}| + |\mathscr{B}| \leqslant \binom{m}{a-1} - \binom{m-a}{a-1} = \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$. If $m > 2a$, then we may assume that $m \in A$ holds for all $A \in \mathscr{A}$, that is, $\mathscr{A} = \{A \in \binom{Y}{a} : m \in A\}$. Since $\mathscr{F}$ is non-trivial, there exists $B \in \mathscr{B}$ such that $m \notin B$. But, for all $A_0 \in \binom{Y-(B \cup \{m\})}{a-1}$, $A := A_0 \cup \{m\} \in \mathscr{A}$ must hold, a contradiction.

Next suppose that $|\mathscr{A}| < \binom{m-1}{a-1}$. Then by Theorem 3, we have

$$|\mathscr{F}| = |\mathscr{A}| + |\mathscr{B}| \leqslant \binom{m}{a-1} - \binom{m-a}{a-1} + 1 = \binom{n-1}{k-1} - \binom{n-k-1}{a-1} + 1,$$

as desired. ∎

Appendix

Let n, k be integers and let X be an n-element set. We define the colex order $<$ on $\binom{X}{k}$ by setting $A < B$ if $\max\{i : i \in A - B\} < \max\{i : i \in B - A\}$. The shift operator S_{ij}, $1 \leqslant i < j \leqslant n$, on $\binom{X}{k}$ is defined as follows: Let $\mathscr{F} \subset \binom{X}{k}$. For $F \in \mathscr{F}$, define

$$s_{ij}(F) := \begin{cases} (F - (j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F, \text{ and } (F - \{j\}) \cup \{i\} \notin \mathscr{F} \\ F & \text{otherwise,} \end{cases}$$

and $S_{ij}(\mathscr{F}) := \{s_{ij}(F) : F \in \mathscr{F}\}$. It is easily checked that (i) $|S_{ij}(\mathscr{F})| = |\mathscr{F}|$ and (ii) $S_{ij}(\mathscr{F})$ is intersecting if $\mathscr{F}$ is intersecting. A family $\mathscr{F} \subset \binom{X}{k}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ holds for all $1 \leqslant i < j \leqslant n$. For a family $\mathscr{F} \subset \binom{X}{k}$ and an integer $l \leqslant k$, we define the lth shadow of $\mathscr{F}$ by $\sigma_l(\mathscr{F}) := \{G \in \binom{X}{l} : G \subset \exists F \in \mathscr{F}\}$.

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