Discrete Mathematics 83 (1990) 201–204
North-Holland

BINARY CODES AND QUASI-SYMMETRIC DESIGNS

A.R. CALDERBANK and P. FRANKL

AT&T Bell Laboratories, Math. and Stat. Research Center, 600 Mountain Ave., Murray Hill, New Jersey 07974, USA

Received 3 October 1987
Revised 1 February 1988

We obtain a new necessary condition for the existence of a $2-(v, k, \lambda)$ design where the block intersection sizes $s_1, s_2, \ldots, s_n$ satisfy $s_1 \equiv s_2 \equiv \cdots \equiv s_n \equiv s(\text{mod } 2)$. This condition eliminates quasi-symmetric $2-(20, 10, 18)$ and $2-(60, 30, 58)$ designs. Quasi-symmetric $2-(20, 8, 14)$ designs are eliminated by an ad hoc coding theoretic argument.

A $2-(v, k, \lambda)$ design $\mathfrak{B}$ is said to be quasi-symmetric if there are two block intersection sizes s_1 and s_2. The parameters of the complementary design $\mathfrak{B}^*$ are related to the parameters of $\mathfrak{B}$ as follows:

$$\begin{array}{llllll} \mathfrak{B}: & v & k & \lambda_1 & \lambda_2 & s_i \\ \mathfrak{B}^*: & v & v-k & \lambda_0-\lambda_1 & \lambda_0-2\lambda_1+\lambda_2 & v-2k+s_i \end{array} \tag{1}$$

Here λ_i denotes the number of blocks through a given i points (and $\lambda = \lambda_2$). Calderbank [1] used Gleason's theorem on self-dual codes to obtain new necessary conditions for the existence of $2-(v, k, \lambda)$ designs where the block intersection sizes $s_1, s_2, \ldots, s_n$ satisfy $s_1 \equiv s_2 \equiv \cdots \equiv s_n (\text{mod } 2)$. We use (1) to restate these conditions as follows:

Theorem 1. *Let $\mathfrak{B}$ be a $2-(v, k, \lambda)$ design where the block intersection sizes $s_1, s_2, \ldots, s_n$ satisfy $s_1 \equiv s_2 \equiv \cdots \equiv s_n \equiv s(\text{mod } 2)$. If $\lambda_1 \not\equiv \lambda_2 (\text{mod } 4)$, then after possibly taking complements either*

(i) $k \equiv 0(\text{mod } 4)$, $v \equiv 1(\text{mod } 8)$, $\lambda_1 \equiv 0(\text{mod } 8)$, *or*

(ii) $k \equiv 0(\text{mod } 4)$, $v \equiv -1 (\text{mod } 8)$, $2\lambda_2 + \lambda_1 \equiv 0(\text{mod } 8)$.

Note that Calderbank [1, Lemma 1] proved that $k \equiv s(\text{mod } 2)$ and $\lambda_1 \equiv \lambda_2 (\text{mod } 2)$ for designs $\mathfrak{B}$ satisfying the hypotheses of Theorem 1. The next lemma is just proved by simple counting but nevertheless it is very useful.

Lemma 2. *If $\mathfrak{B}$ be a $2-(v, k, \lambda)$ design where the block intersection sizes $s_1, s_2, \ldots, s_n$ satisfy $s_1 \equiv s_2 \equiv \cdots \equiv s_n \equiv s(\text{mod } 2)$. Let b_i, $i = 1, 2, \ldots, \lambda_0$ be the blocks of $\mathfrak{B}$ and let $z \in \mathbb{F}_2^v$ (z is the characteristic vector of an arbitrary subset).*

(A) *If $(z, b_i) \equiv 0(\text{mod } 2)$ for all i, and if the weight of z satisfies* ωt(z) $\not\equiv 0 \pmod 4$, *then either*

(i) $\omega t(z) \equiv 1 \pmod 4$, *and* $\lambda_1 \equiv 0 \pmod 8$,

(ii) $\omega t(z) \equiv -1 \pmod 4$, *and* $2\lambda_2 + \lambda_1 \equiv 0 \pmod 8$, *or*

(iii) $\omega(t) \equiv 2 \pmod 4$, *and* $\lambda_1 \equiv \lambda_2 \pmod 4$.

(B) *If* $(z, b_i) \equiv 1 \pmod 2$ *for all i, then either*

(iv) $\omega t(z) \equiv 0 \pmod 4$, *and* $\lambda_0 \equiv 0 \pmod 8$

(v) $\omega t(z) \equiv 1 \pmod 4$, *and* $\lambda_0 \equiv \lambda_1 \pmod 8$

(vi) $\omega t(z) \equiv 2 \pmod 4$, $\lambda_0 \equiv 0 \pmod 4$, *and* $\lambda_0 + 2(\lambda_2 + \lambda_1) \equiv 0 \pmod 8$, *or*

(vii) $\omega t(z) \equiv -1 \pmod 4$, *and* $\lambda_0 + \lambda_1 - 2\lambda_2 \equiv 0 \pmod 8$.

Proof. Part A is proved in [1, Lemma 4]. to prove part B let N_{2i-1}, $i = 1, 2, \ldots$ be the number of blocks meeting z in $2i - 1$ points. Since $\mathfrak{B}$ is a 2-design we have

$$\sum_i N_{2i-1} = \lambda_0, \tag{2}$$

$$\sum_i (2i-1) N_{2i-1} = \omega t(z) \lambda_1, \tag{3}$$

$$\sum_i \binom{2i-1}{2} N_{2i-1} = \binom{\omega t(z)}{2} \lambda_2,$$

and so

$$\sum_i [3 - 3(2i-1) + (2i-1)(2i-2)] N_{2i-1} = \sum_i 4(i-2)(i-1) N_{2i-1}$$
$$= 3\lambda_0 + \omega t(z)[(\omega t(z) - 1)\lambda_2 - 3\lambda_1].$$

Then (iv), (v), (vi), and (vii) follow from the fact that both sides of the above equation are congruent to zero modulo 8. □

Now let M be the incidence matrix of $\mathfrak{B}$ of size $(\lambda_0 \times v)$ and let R be the binary code spanned by the rows of M. Let $j = (1, 1, \ldots, 1)$.

Theorem 3. *Let* β *be a* $2-(v, k, \lambda)$ *design where the block intersection sizes* $s_1, s_2, \ldots, s_n$ *satisfy* $s_1 \equiv s_2 \equiv \cdots \equiv s_n \pmod 2$.

(i) *If* $\lambda_1 \equiv \lambda_2 \pmod 4$ *then either* $j \in R$ *or* $\lambda_1 \equiv 0 \pmod 8$.

(ii) *If* $k \equiv 2 \pmod 4$ *then there exists z such that* $(z, b_i) \equiv 1 \pmod 2$ *for all blocks* b_i.

Proof. (i) If $j \notin R$ then R is properly contained in $\langle R, j \rangle$ and there exists $z \in R^\perp \setminus \langle R, j \rangle^\perp$. Then $\omega t(z)$ is odd and part A of Lemma 2 implies $\lambda_1 \equiv 0 \pmod 8$ or $2\lambda_2 + \lambda_1 \equiv 0 \pmod 8$. Since $\lambda_1 \equiv \lambda_2 \pmod 4$ we have $\lambda_1 \equiv 0 \pmod 8$.

(ii) The code R is self-orthogonal and there is a doubly even kernel K with codimension 1. Hence $R^\perp$ is properly contained in $K^\perp$. If $z \in K^\perp \setminus R^\perp$ then $b_1 + b_i \in K$ for all i, and so $(z, b_1 + b_i) \equiv 0 \pmod 2$ for all i. Thus (z, b_i) is constant modulo 2, and so $(z, b_i) = 1 \pmod 2$ for all i. □

Theorem 3 eliminates two exceptional quasi-symmetric designs from the list compiled by Neumaier (see Neumaier [4], Calderbank [1, 2]).

Example 1. Here $v = 20$, $k = 10$, $\lambda_2 = 18$, $\lambda_1 = 38$, $\lambda_0 = 76$, $s_1 = 4$, $s_2 = 6$. Part (ii) of Theorem 3 implies there exists z such that $(z, b_i) \equiv 1 (\mathrm{mod}\ 2)$ for all blocks b_i. However, none of the congruence conditions listed in (iv), (v), (vi), or (vii) of Lemma 2 are satisfied.

Example 2. Here $v = 60$, $k = 30$, $\lambda_2 = 58$, $\lambda_1 = 118$, $\lambda_0 = 236$, $s_1 = 14$, $s_2 = 18$. Again non-existence follows directly from part (ii) of Theorem 3 and part (B) of Lemma 2.

The block graph $G(\mathfrak{B})$ of a quasi-symmetric design $\mathfrak{B}$ is obtained by joining two blocks b_i, b_j if $|b_i \cap b_j| = s_2$, where $s_1 < s_2$. Goethals and Seidel [3] proved that the block graph of a quasi-symmetric design is strongly regular. Denote the number of vertices by V, the valency by K, and the number of vertices joined to two given vertices by λ or μ according as the two given vertices are adjacent or non-adjacent. Example 1 leads to a block graph with parameters $V = 76$, $K = 35$, $\lambda = 18$ and $\mu = 14$. Since $V = 2(2K - \lambda - \mu)$ it follows that $\delta G(\mathfrak{B})$ is a regular 2-graph where δ is the coboundary operator (see Seidel [5], Taylor [7] or Seidel and Taylor [6] for details). It is not known whether there are any regular 2-graphs on 76 or on 96 vertices. The non-existence of Example 1 eliminates one method of constructing a regular 2-graph on 76 points. If Φ is a regular 2-graph on 96 vertices then $\Delta_\omega^* \Phi$ is a strongly regular graph with $V = 95$, $K = 54$, $\lambda = 33$ and $\mu = 27$. Here Δ_ω^* is the restriction of the contracting homotopy with respect to the point ω (see [5, 6, 7] for details). These are the parameters of the block graph $G(\mathfrak{B})$ of a quasi-symmetric $2-(20, 8, 14)$ design $\mathfrak{B}$. We now eliminate this design by an ad hoc coding theoretic argument.

Example 3. Here $v = 20$, $k = 8$, $\lambda_2 = 14$, $\lambda_1 = 38$, $\lambda_0 = 95$, $s_1 = 2$, $s_2 = 4$. We consider a binary code R_1 satisfying $R \leqslant R_1 \leqslant R^\perp$. Note that by Theorem 3, we have $j \in R$. Let z_4 be a codeword of weight 4 in R_1 and let $N_i(z_4)$ be the number of blocks meeting z_4 in i points. Simple counting arguments (see [1]) give

$$N_0(z_4) = 21, \qquad N_2(z_4) = 72, \qquad N_4(z_4) = 2. \tag{5}$$

Now let z_8 be a codeword of weight 8 in R_1 that is not a block ($N_8(z_8) = 0$) and let $N_i(z_8)$ be the number of blocks meeting z_8 in i points. We obtain

$$N_0(z_8) + N_6(z_8) = 3. \tag{6}$$

Let A_i denote the number of codewords of weight i in R_1. We prove that $A_8 - 95 = 31A_4$ by exhibiting a correspondence between codewords of weight 4 and codewords of weight 8 that are not blocks. Given a codeword x of weight 4 there are blocks $c_1, \ldots, c_{21}$ disjoint from x and blocks $d_1, \ldots, d_{72}$ that meet x in two points. Then $x + j + c_l$, $l = 1, 2, \ldots, 21$ and $x + d_l$, $l = 1, 2, \ldots, 72$ are codewords in R_1 of weight 8 that are not blocks. Thus a codeword of weight 4 determines 93 codewords of weight 8 that are not blocks. Now (6) implies that every codeword of weight 8 that is not a block occurs 3 times in this way (if $|y \cap z_8| = 6$ then $\text{wt}(y + z_8) = 4$, and if $|y \cap z_8| = 0$ then $\text{wt}(y + z_8 + j) = 4$). Set $R = R_1$. The weight enumerator $W(z)$ of the doubly even code R is given by

$$W(z) = 1 + A_4 z^4 + (31A_4 + 95)z^8 + (31A_4 + 95)z^{12} + A_4 z^{16} + z^{20}. \tag{7}$$

Gleason's theorem implies $\dim(R) \leq 9$ so there are two solutions: $A_4 = 1$, $\dim R = 8$, and $A_4 = 5$, $\dim R = 9$. The first solution violates (5) (since $A_4 \neq 0$ implies $A_4 \geq 2$) so there are 5 codewords of weight 4; w_1, w_2, w_3, w_4, w_5 say. Next we claim that $w_1, \ldots, w_5$ are disjoint. For otherwise we may suppose $w_3 = w_1 + w_2$, that $|w_4 \cap w_5| = 0$, and $|w_i \cap w_j| = 0$ for $i = 1, 2, 3$ and $j = 4, 5$. Hence every block through w_1, w_2, or w_3 is also a block through w_4 or w_5. But this is impossible since there are exactly two blocks through each codeword of weight 4.

The codewords w_1, w_2, w_3, w_4, w_5 are disjoint and there are 5 blocks of the form $w_i + w_j$. We may suppose that these blocks are $w_1 + w_5$ and $w_i + w_{i+1}$, $i = 1, 2, 3, 4$. Then $w_2 + w_5$ is a codeword of weight 8 that is not a block. Now (5) implies that any block disjoint from $w_2 + w_5$ is of the form $w_i + w_j$, and that no block meets $w_2 + w_5$ in 6 points. The only block disjoint from $w_2 + w_5$ is $w_3 + w_4$ and we have a contradiction to (6). This eliminates the design.

References

[1] A.R. Calderbank, The application of invariant theory to the existence of quasi-symmetric designs, J. Combin. Theory Ser. A 44 (1987) 94–109.

[2] A.R. Calderbank, Geometric invariants for quasi-symmetric designs, J. Combin. Theory Ser. A, 47 (1988) 101–110.

[3] J.-M. Goethals and J.J. Seidel, Strongly regular graphs derived from combinatorial designs, Canad. J. Math. 22 (1970) 597–614.

[4] A. Neumaier, Regular sets and quasi-symmetric 2-designs, in D. Jungnickel and K. Vedder, Eds., Combinatorial theory Lecture Notes in Math. 969, pp. 258–275 (Springer-Verlag, New York/Berlin, 1982).

[5] J.J. Seidel, A survey of two-graphs, Proc. Intern. Coll. Theorie Combinatorie (Roma 1973) (Accad. Naz. Lincei) 481–511.

[6] J.J. Seidel and D.E. Taylor, Two-graphs, a second survey, in L. Lovasz and V. Sós, Eds, Algebraic Methods in Graph theory (North-Holland, 1981) 689–711.

[7] D.E. Taylor, Regular 2-graphs, Proc. London Math. Soc. 35 (1977) 257–274.