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Iterated Combinatorial Density Theorems

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1. INTRODUCTION

There are many Ramsey-type results in combinatorics for which stronger density versions actually hold. An excellent example of this phenomenon is given by the classical theorem of van der Waerden on arithmetic progressions (see [W, GRS]):

THEOREM (van der Waerden). *For any partition of* $\mathbb{N} = \{1, 2, 3, ...,\} = C_1 \cup C_2 \cup \cdots \cup C_r$ *into finitely many classes, some class* C_i *must contain arbitrarily long arithmetic progressions.*

This theorem is an immediate consequence of the following density result of Szemerédi (see [Sz, GRS]).

THEOREM (Szemerédi). *If* $X \subseteq \mathbb{N}$ *has positive upper density, i.e.,*

$$\bar{d}(X) := \limsup_{n \to \infty} \frac{|X \cap \{1, 2, ..., n\}|}{n} > 0,$$

then X *must contain arbitrarily long arithmetic progressions.*

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(This implication is immediate since some C_i must have $\bar{d}(C_i) \geqslant 1/r$.)

However, there are other partition-type results for which the corresponding natural density theorem is not valid. For example, a theorem of Schur [S, GRS] asserts the following.

THEOREM (Schur). *For any partition of* $\mathbb{N} = C_1 \cup \cdots \cup C_r$, *some* C_i *contains* x, y, z *with* $x + y = z$.

The set $\{2x+1 : x \in \mathbb{N}\}$ shows that the straightforward density version of Schur's theorem does not hold, since this set has (upper) density $\frac{1}{2}$ and contains no solution to $x + y = z$.

It was recently pointed out by Bergelson that nevertheless, it is still possible to prove a result which can be viewed as a form of a density version of Schur's theorem. This is the following.

THEOREM [B]. *For any partition of* $\mathbb{N} = C_1 \cup \cdots \cup C_r$, *some* C_i *having* $\bar{d}(C_i) > 0$ *satisfies for any* $\varepsilon > 0$,

$$\bar{d}(x \in \mathbb{N} : \bar{d}(y \in \mathbb{N} : x, y, z \in C_i, x + y = z) \geqslant \bar{d}(C_i)^2 - \varepsilon) > 0. \qquad (1.1)$$

What (1.1) says is that some C_i must contain "many" x so that for each of these x there are "many" y in C_i so that $x + y$ is also in C_i. However, Bergelson's result does not guarantee that the upper densities of either set is bounded away from 0 as a function only of the number r of classes.

Our goal in this paper is to prove a variety of iterated density results of this type for a number of different combinatorial and algebraic structures. A simple example of such a result is the following.

THEOREM. *For each* $r \in \mathbb{N}$ *there exists* $\delta = \delta(r) > 0$ *so that for any partition of the set* $\binom{\mathbb{N}}{2}$ *of pairs of elements of* $\mathbb{N}$ *into* r *classes* C_i *we have for some* C_i,

$$\underline{d}(x \in \mathbb{N} : \bar{d}(y \in \mathbb{N} : \bar{d}(z \in \mathbb{N} : \{x, y\}, \{x, z\}, \{y, z\} \in C_i) > \delta) > \delta) > \delta, \qquad (1.2)$$

where $\underline{d}(X) := \liminf_{n \to \infty} |X \cap \{1, 2, ..., n\}|/n$.

In general, we will always prove the stronger theorem having the outside density a lower density (rather than an upper density).

We will frequently use the traditional "chromatic" terminology of Ramsey theory, namely, classes correspond to *colors*, a partition into r classes is an *r-coloring*, and homogeneous objects are *monochromatic*. Hence, by identifying pairs from $\mathbb{N}$ as *edges* of the complete graph on $\mathbb{N}$, the above result can be restated as:

THEOREM. *For each $r \in \mathbb{N}$ there exists $\delta = \delta(r) > 0$ so that for any r-coloring of the edges of $K_{\mathbb{N}}$,*

$$\underline{d}(x \in \mathbb{N} : \bar{d}(y \in \mathbb{N} : \bar{d}(z \in \mathbb{N} : \{x, y, z\} \text{ is monochromatic}) > \delta) > \delta) > \delta.$$

An important class of results in Ramsey theory deal with so-called induced, restricted Ramsey theorems (see [NR2]). In attempting to construct iterated density versions for these theorems, however, it is not immediately obvious what the underlying large "universal" object should be, e.g., what infinite graph should take the place of $K_{\mathbb{N}}$ above for the case of triangle-free graphs. We will argue that for the case of graphs, the appropriate objects in this case are the so-called countable universal K_r-free graphs. We use these for our iterated density results, and in a similar way, after proving the existence and uniqueness of these universal objects for vector spaces, we establish the corresponding iterated density theorem (Section 3). We have also included (in Section 7) new proofs of the finite unions and finite sums theorems which are considerably simpler than previous proofs.

2. AN ITERATED DENSITY RAMSEY THEOREM

For $k \in \mathbb{N}$, let $\binom{X}{k}$ denote the set of k-element subsets of the set X. The standard form of Ramsey's theorem is the following.

THEOREM (Ramsey [Ra]). *For any k and r in $\mathbb{N}$, and for any partition of $\binom{\mathbb{N}}{k} = C_1 \cup \cdots \cup C_r$, there is an infinite set $X \subseteq \mathbb{N}$ such that for some C_i, $\binom{X}{k} \subseteq C_i$.*

A naive attempt at a density version of Ramsey's theorem cannot succeed since, for example, when $k = 2$ the graph B with vertex set $\mathbb{N}$ and edge set $\{\{2i, 2j+1\} : i, j \in \mathbb{N}\}$ has essentially half of all possible edges (say, restricted to $[N] = \{1, 2, ..., N\}$) and yet contains no triangle. Our next result furnishes an iterated density form of Ramsey's theorem.

Assume l, $r \in \mathbb{N}$ and $\binom{\mathbb{N}}{k} = C_1 \cup \cdots \cup C_r$ is a partition of $\binom{\mathbb{N}}{k}$ into r classes. For $a_1 < \cdots < a_{l-1}$ in $\mathbb{N}$, define

$$\Gamma(a_1, ..., a_{l-1}) := \left\{ x \in \mathbb{N} : \binom{\{a_1, ..., a_{l-1}, x\}}{k} \text{ is homogeneous} \right\}$$

and for $0 \leqslant i < l-1$,

$$\Gamma(a_1, ..., a_i) := \{x \in \mathbb{N} : \bar{d}(\Gamma(a_1, ..., a_i, x)) > \delta\}, \tag{2.1}$$

where $\delta = \delta(k, l, r) = 2^{-R}$ and $R := R(k, l, r)$, the Ramsey number (see

[GRS]) guaranteeing homogeneous k-sets of an l-set using partitions into r classes. For $i=0$, we denote the expression in (2.1) by Γ.

THEOREM 2.1. *For all $k \leqslant l$ and $e \in \mathbb{N}$,*

$$\underline{d}(\Gamma) > \delta. \tag{2.2}$$

Comment. It is perhaps useful at this point to consider (2.2) for specific small values of k and l, say $k=2$ and $l=3$. In this case, since $R(2,3,r) < (r+1)!$, for example, then (2.2) asserts that for $\delta := 2^{-(r+1)!}$ and for any partition of the edges of the complete graph with vertex set $\mathbb{N}$ into r classes $C_1 \cup \cdots \cup C_r$, we must have

$$\underline{d}\{x : \bar{d}\{y : \bar{d}\{z : \text{all edges of triangle } \{x, y, z\} \text{ are in the same } C_i\} > \delta\} > \delta\} > \delta.$$

Proof of Theorem 2.1. Assume $\underline{d}(\Gamma) \leqslant \delta$. Thus,

$$\bar{d}(\mathbb{N} \setminus \Gamma) \geqslant 1 - \delta.$$

Note that

$$x \notin \Gamma(a_1, ..., a_i) \Rightarrow \bar{d}(\Gamma(a_1, ..., a_i, x)) \leqslant \delta.$$

Define S_i, $i = 1, 2, ...$, as follows:

$$S_1 = \{s_1\}, \qquad \text{where} \quad s_1 \in \mathbb{N} \setminus \Gamma \text{ is arbitrary.}$$

Suppose $S_j = \{s_1, s_2, ..., s_j\}$ has been defined. Form $S_{j+1} = S_j \cup \{s_{j+1}\}$ by choosing s_{j+1} (if possible) so that:

(i) $s_{j+1} \in \mathbb{N} - \bigcup_{u=0}^{l-1} \bigcup_{1 \leqslant i_1 < \cdots < i_u \leqslant j} \Gamma(s_i, ..., s_{i_u}) := B_j$,

(ii) For no $Y \in \binom{S_{j+1}}{l}$ is $\binom{Y}{k}$ homogeneous.

Note that since

$$\bar{d}(B_j) \geqslant 1 - \left\{ \binom{j}{0} + \binom{j}{1} + \cdots + \binom{j}{l-1} \right\} \delta \geqslant 1 - 2^j \delta$$

then we never get stuck because of (i). However, by Ramsey's theorem, we must eventually halt because of condition (ii), say, with the formation of S_i, for some $t < R$. By the definition of S_t, for each $c \in B_t$, there is a set $X(c) \in \binom{S_t}{l-1}$ such that $\binom{X(c) \cup \{c\}}{k}$ is homogeneous. Thus, there exists a set $X_0 = \{s_{j_1} < \cdots < s_{j_{l-1}}\} \in \binom{S_t}{l-1}$ such that

$$\bar{d}\{c \in B_t : X(c) = X_0\} \geqslant \bar{d}(B_t) \binom{t}{l-1}^{-1} \geqslant (1 - 2^t \delta) \binom{t}{l-1}^{-1} > 2^{-R} = \delta \tag{2.3}$$

by the choice of δ. However, by construction,

$$\begin{aligned} s_{j_{l-1}} &\notin \Gamma(s_{j_1}, ..., s_{j_{l-2}}) \\ &\Rightarrow \bar{d}(\Gamma(s_{j_1}, ..., s_{j_{l-1}})) \leqslant \delta \\ &\Rightarrow \bar{d}\left\{u \colon \binom{X_0 \cup \{u\}}{k} \text{ is homogeneous}\right\} \leqslant \delta \end{aligned}$$

which contradicts (2.3). This proves (2.2). ∎

3. Vector Spaces

Let q be a fixed prime power and let $\mathbb{V}$ be the standard infinite-dimensional vector space over $GF(q)$, that is, the points of $\mathbb{V}$ are all infinite sequences $(x_1, ..., x_n, ...,)$ with $x_i \in GF(q)$, having only a finite number of nonzero entries, and addition and multiplication by $\gamma \in GF(q)$ done coordinatewise. Throughout this section we consider q as fixed and do not refer to it specifically.

For vector spaces we use notation similar to that for sets. In particular, if U is a vector space then $\left[{U \atop k}\right]$ is the collection of all k-dimensional subspaces W of U. We use the notation $W < U$ to indicate that W is a subspace of U. Let $\left[{a \atop k}\right]_q = \prod_{0 \leqslant i < k} (q^a - q^i)/(q^k - q^i)$ be the Gaussian coefficient, i.e., the number of k-subspaces in an a-dimensional space. Note the trivial inequality $\left[{a \atop k}\right]_q < q^{ak}$.

Let $e_i \in \mathbb{V}$ be the sequence with 1 in the ith coordinate and 0 elsewhere. Set $V_n = \langle e_1, ..., e_n \rangle$. That is, V_n is the subspace of vectors having zeros in all but possibly the first n coordinates.

If $\mathscr{F} \subseteq \left[{\mathbb{V} \atop k}\right]$ then one defines the upper density $\bar{d}(\mathscr{F})$ by

$$\bar{d}(\mathscr{F}) = \limsup_{n \to \infty} \left| \mathscr{F} \cap \begin{bmatrix} V_n \\ k \end{bmatrix} \right| \Big/ \left| \begin{bmatrix} V_n \\ k \end{bmatrix} \right|.$$

The lower density $\underline{d}(\mathscr{F})$ is defined similarly, except that lim sup is replaced by lim inf. We will primarily use this definition for $k = 0$, i.e., when $\mathscr{F}$ is a set of points.

Let us recall the basic Ramsey theorem for vector spaces.

Vector Space Ramsey Theorem (Graham, Leeb, and Rothschild [GLR]). *Let $l \geqslant k > 0$, $r \geqslant 2$ be integers. Then there exists $d = d(l, k, r)$ such that for all partitions $\left[{U \atop k}\right] = C_1 \cup \cdots \cup C_r$ of the k-subspaces of a d-dimensional space U, one can find $W \in \left[{U \atop l}\right]$ with $\left[{W \atop k}\right] \subset C_i$ for some i.*

Let us now consider an arbitrary but fixed partition $\left[{\mathbb{V} \atop k}\right] = C_1 \cup \cdots \cup C_r$

of all k-spaces of $\mathbb{V}$. As usual, a subspace U is called monochromatic if $\left[{U \atop k}\right] \subset C_i$ holds for some i.

Fix $l > k$ and for every $W \in \left[{\mathbb{V} \atop l-1}\right]$ define

$$\Gamma(W) = \{w \in \mathbb{V} : \langle W, w \rangle \text{ is monochromatic}\}.$$

Let $\delta < 0$ be a small positive constant—we shall fix the value of δ later. Define by backward recursion on $l-2 \geqslant i > 0$ and $U \in \left[{\mathbb{V} \atop i}\right]$,

$$\Gamma(U) = \{u \in \mathbb{V} : \bar{d}(\Gamma(\langle U, u \rangle)) > \delta\}.$$

Finally, $\Gamma = \Gamma(\mathbf{0}) = \{u \in \mathbb{V} : \bar{d}(\Gamma(\langle u \rangle)) > \delta\}$, where $\mathbf{0} = \{(0, 0, ..., 0, ...,)\}$ is the zero-space.

THEOREM 3.1. *For* $\delta = q^{-ld}/4$ *we have*

$$\underline{d}(\Gamma) \geqslant \delta.$$

Proof. The proof is similar to the proof in the preceding section, although technically more complex. We argue indirectly and suppose that $\underline{d}(\Gamma) < \delta$, i.e., $\bar{d}(\mathbb{V} - \Gamma) > 1 - \delta$ holds.

Our aim is to find as large a subspace W as possible with the following properties:

(i) W has no monochromatic l-subspaces;

(ii) for all $U < W$ with $1 \leqslant \dim U < l$ one has $\bar{d}(\Gamma(U)) \leqslant \delta$.

By (i) and the vector space Ramsey theorem we know that $\dim W < d(l, k, r)$ holds. On the other hand, $W = \langle 0 \rangle$, the zero-space, satisfies (i) and (ii). The contradiction establishing the validity of the theorem follows from the next lemma.

LEMMA 3.2. *Let* $W < \mathbb{V}$ *be a subspace satisfying* (i), (ii) *and* $\dim W < d = d(l, k, r)$. *Then there exists* U *with* $\dim U = \dim W + 1$ *satisfying* (i) *and* (ii).

Proof of the lemma. Choose n so large that $W < V_n$ and $|\Gamma(W_0) \cap V_n| / |V_n| \leqslant 2\delta$ holds for all $W_0 < W$ (note that $\Gamma(W_0) = \emptyset$ holds automatically if W_0 is not monochromatic, and, in particular, for $\dim W_0 \geqslant l$). Among such n select n so that $|\Gamma \cap V_n| / |V_n| \leqslant 2\delta$ also holds.

Let $U_1, ..., U_m$ be all the subspaces of V_n satisfying $\dim U_i = \dim W + 1$ and $W < U_i$. Note that $V_n - W = (U_1 - W) \cup \cdots \cup (U_m - W)$ is a partition and $|U_i - W| = q^{\dim W + 1} - q^{\dim W}$ and consequently, $m > q^{n-d}$ holds. If some of the U_i satisfy (i) and (ii) then we have nothing to prove. Suppose that this is not the case. Then for $1 \leqslant i \leqslant m$, we can choose a subspace $T_i < U_i$ such that either $\dim T_i = l$ and T_i is monochromatic, or $1 \leqslant \dim W_i < l$ and $\bar{d}(\Gamma(T_i)) > \delta$.

Set $W_i = T_i \cap W$. Then $\dim W_i = \dim T_i - 1$ holds because $T_i \nless W$. Choose $u_i \in (T_i - W_i)$. Then by definition, $u_i \in \Gamma(W_i)$ holds. That is, $\{u_1, ..., u_m\} \subset \bigcup_{W_i < W} \Gamma(W_i)$, yielding

$$q^{-d} \leqslant \frac{m}{q^n} \leqslant \sum_{\substack{W_i < W \\ \dim W_i < l}} |\Gamma(W_i) \cap V_n| / |V_n| \leqslant 2\delta \sum_{0 \leqslant i < l} \begin{bmatrix} d \\ i \end{bmatrix}_q < 4\delta q^{d(l-1)}.$$

Comparing the extreme sides gives $\delta > q^{-dl}/4$, a contradiction. ∎

4. Universal Collections of Vector Spaces

For $\mathscr{F} \subset [{V \atop k}]$ and $\mathscr{G} \subset [{W \atop k}]$, an embedding of $\mathscr{G}$ into $\mathscr{F}$ is a linear injection $\phi: W \to V$ such that $\phi(\mathscr{G}) = \mathscr{F}|_{\phi(W)}$.

Let $\mathscr{G}|_U$ denote $\{G \in \mathscr{G} : G < U\}$.

Definition 4.1. Call a family $\mathscr{F} \subset [{V \atop k}]$ *universal* if for every collection $\mathscr{G} \subset [{W \atop k}]$, every $W_0 < W$ with $\dim W_0 = \dim W - 1$, and every embedding ϕ of $\mathscr{G}_{|W_0}$ into $\mathscr{F}$, there is an embedding of $\mathscr{G}$ into $\mathscr{F}$ which coincides with the previous embedding on W_0.

Theorem 4.2. *For every $k \geqslant 1$ there is up to isomorphism exactly one universal family $\mathscr{F} \subset [{V \atop k}]$.*

Proof of Existence. Let $T_1, T_2, ..., T_m, ...,$ be a linear ordering of all the elements of $[{V \atop k}]$ satisfying the property that $T_i < V_n$, $T_j \nless V_n$ implies $i < j$.

Define a probability space $\mathbf{P}$ on all (uncountably many) collections $\mathscr{F} \subset [{V \atop k}]$ by setting

$$p(T_i \in \mathscr{F}) = \tfrac{1}{2} \qquad \text{for all} \quad i \geqslant 1$$

and letting these events be completely independent (for a detailed explanation of this procedure for sets see [C]).

Claim. $p(\mathscr{F} \in \mathbf{P}$ is universal$) = 1$.

Proof of Claim. Since there are only countably many choices for W, W_0, $\mathscr{G}$, and ϕ from Definition 4.1, it is sufficient to show that for each particular choice the extendibility holds with probability 1. Set $\dim W = d$.

Set $\phi(W_0) = U_0$ and let $U_1, U_2, ..., U_m, ...,$ be infinitely many distinct spaces in $[{V \atop \dim W_0 + 1}]$ containing U_0. Let $\phi_1, \phi_2, ...,$ be arbitrary extensions of the map $\phi: W_0 \to U_0$ to a 1–1 linear map $\phi_i: W \to U_i$, $i = 1, 2, ...$. Thus,

$$p(\phi_i \text{ is an embedding of } \mathscr{G} \text{ into } \mathscr{F}) = 2^{-[{d \atop k}]_q + [{d-1 \atop k}]_q}.$$

Moreover, these events are completely independent for $i=1, 2, \dots$. Thus, with probability 1, infinitely many of them are embeddings of $\mathscr{G}$ into $\mathscr{F}$ extending the embedding ϕ. ∎

Proof of Uniqueness. Let $\mathscr{F}$ and $\mathscr{H}$ be two universal families of k-spaces of $\mathbb{V}$. Let $f_1, f_2, \dots$, and $h_1, h_2, \dots$, be two bases for $\mathbb{V}$. We shall define an isomorphism $\phi: \mathbb{V} \to \mathbb{V}$ from $\mathscr{F}$ to $\mathscr{H}$ inductively. First we fix $\phi(0, \dots,) = (0, \dots,)$.

Suppose that ϕ is defined on a subspace W_0 and choose i minimal so that $f_i \notin W_0$. Set $W = \langle W_0, f_i \rangle$ and use the universality of $\mathscr{H}$ to extend ϕ to $\phi': W \to \mathbb{V}$.

Then choose j minimal so that $h_j \notin \phi(W)$.

Let ψ be the inverse of ϕ, i.e., $\psi(\phi(W))$ is the identity on W. Apply the universality of $\mathscr{F}$ to extend the embedding ψ of $\mathscr{H}_{|\phi(W)}$ to an embedding of $\mathscr{H}_{|\langle \phi(W), h_j \rangle}$ and let ϕ'' be the inverse of this embedding.

Iterating this procedure indefinitely, we obtain the desired isomorphism. ∎

Call a family $\mathscr{G} \subset \left[{W \atop k}\right]$ *K_t-free* if $\left[{U \atop k}\right] \not\subset \mathscr{G}$ holds for all $U \in \left[{W \atop t}\right]$.

DEFINITION 4.3. Let $t > k \geqslant 1$ be positive integers. Call a K_t-free family $\mathscr{F} \subset \left[{\mathbb{V} \atop k}\right]$ *universal K_t-free* if the conditions of Definition 4.1 hold for all K_t-free families $\mathscr{G}$.

THEOREM 4.4. *For all $t > k \geqslant 1$ there exists a unique universal K_t-free family $\mathscr{F} \subset \left[{\mathbb{V} \atop k}\right]$.*

The proof is rather similar to that of Theorem 4.2.

Proof of Existence. Let $T_1, T_2, \dots$, be a linear ordering of $\left[{\mathbb{V} \atop k}\right]$ as before. We will define a probability space $\mathbf{P}(t)$. However, this time we have to be more careful. For every finite $(0, 1)$-sequence $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ define the family $\mathscr{F}(\boldsymbol{\varepsilon}) = \{T_i : \varepsilon_i = 1\}$ and the event

$$E(\boldsymbol{\varepsilon}) = \{T_{n+1} \in \mathscr{F} \text{ holds for } \mathscr{F} \in \mathbf{P}(t) \text{ assuming that } \mathscr{F}(\boldsymbol{\varepsilon}) \subset \mathscr{F}\}.$$

Next we define $p(E(\boldsymbol{\varepsilon}))$ inductively:

$$p(E(\boldsymbol{\varepsilon})) = 1 \qquad \text{for the empty sequence and}$$

$$p(E((\varepsilon_1, \dots, \varepsilon_n))) = \begin{cases} \frac{1}{2} & \text{if } \mathscr{F}((\varepsilon_1, \dots, \varepsilon_n, 1)) \text{ is } K_t\text{-free} \\ 0 & \text{if } \mathscr{F}((\varepsilon_1, \dots, \varepsilon_n, 1)) \text{ is not } K_t\text{-free}; \end{cases}$$

$$p(E((\varepsilon_1, \dots, \varepsilon_n, 0))) = 1 - p(E((\varepsilon_1, \dots, \varepsilon_n, 1))), \qquad n \geqslant 0.$$

In a less formal way, we build up a random family $\mathscr{F}$ step by step. In step n we first check whether the addition of T_n maintains K_t-freedom. If not, we do not add T_n; if so, then we add it with (independent) probability $\frac{1}{2}$. As in the case of Theorem 4.1, we only have to show that a fixed quadruple $(W, W_0, \phi, \mathscr{G})$ can be extended with probability 1. Set dim $W=d$.

Choose b to satisfy $\phi(W_0)<V_b$ and let $R_1, R_2, \ldots$, be d-dimensional spaces satisfying $\phi(W_0)<R_i<\mathbb{V}$ and $R_i \nless V_b$. Set $Q_i=\langle V_b, R_i\rangle$. Suppose that $Q_1, Q_2, \ldots$, are all distinct. This implies that the k-subspaces of $\phi(W_0)$ precede the remaining k-subspaces of R_i in the ordering $T_1, T_2, \ldots$.

Let $\phi_i\colon W\to R_i$ be an arbitrary 1–1 linear extension of ϕ. Then for $\mathscr{F}\in \mathbf{P}(t)$ we have

$$p(\mathscr{F}_{|Q_i}=\mathscr{F}_{|V_b}\cup\phi_i(\mathscr{G}))\geqslant 2^{-\left[{b+1 \atop k}\right]+\left[{b \atop k}\right]}$$

because $\mathscr{F}_{|V_b}\cup\phi_i(\mathscr{G})$ is K_t-free and once $\mathscr{F}_{|V_b}$ is fixed, for any of the $\left[{b+1 \atop k}\right]-\left[{b \atop k}\right]$ k-subspaces $T<Q_i$, $T\nless V_b$ with probability at least $\frac{1}{2}$, we make the right choice.

The resulting probability is a small but positive number, so with probability 1 the desired event occurs infinitely many times. ▌

The proof of the uniqueness is exactly the same as in the case of Theorem 4.1 and will be omitted. ▌

5. Iterated Density Results for Induced and Restricted Substructures

Let $1\leqslant k\leqslant l$ be integers. For $\mathscr{F}\subset\left[{\mathbb{V} \atop k}\right]$, let $\left[{\mathscr{F} \atop l}\right]\subset\left[{\mathbb{V} \atop l}\right]$ be the collection of all complete l-spaces W in $\mathscr{F}$, i.e., those W satisfying $\left[{W \atop k}\right]\subset\mathscr{F}$. The induced restricted Ramsey theorem for spaces can be stated as follows.

Theorem 5.1 (Frankl, Graham, and Rödl [FGR2]). *Let $1\leqslant k\leqslant l<t$, $r\geqslant 2$ be integers and let $\mathscr{F}\subset\left[{\mathbb{V} \atop k}\right]$ be a K_t-free family of k-spaces. Then there exists a K_t-free collection $\mathscr{G}\subset\left[{\mathbb{V} \atop k}\right]$ such that for all r-colorings of $\left[{\mathbb{V} \atop l}\right]$ one can find some $U<\mathbb{V}$ with $\mathscr{G}_{|U}\cong\mathscr{F}$ and such that all l-spaces in $\left[{\mathscr{G}_{|U} \atop l}\right]$ have the same color.*

We shall use this theorem to establish the desired iterated density result for colorings of the universal families of spaces.

Let us start with the harder one. Let $\mathscr{F}$ be as in Theorem 5.1 and consider an arbitrary r-coloring of $\left[{\mathscr{H} \atop l}\right]$, where $\mathscr{H}$ is a random element of $\mathbf{P}(t)$. Let T be a subspace satisfying $\mathscr{F}\subset\left[{T \atop k}\right]$, define $d=\dim T$ and let $e_1, \ldots, e_d$ be a basis for T.

For $R\in\left[{\mathbb{V} \atop d-1}\right]$ define

$$\Gamma(R)=\{f\in\mathbb{V}:\mathscr{H}_{|\langle R,f\rangle}\text{ is a monochromatic induced copy of }\mathscr{F}\}.$$

As in Section 3, we define by backward induction for $Q \in [{\mathbb{V} \atop i}]$,

$$\Gamma(Q) = \{f \in \mathbb{V} - Q : \bar{d}(\Gamma(\langle Q, f \rangle)) > 0\} \qquad \text{for} \quad d-1 > i \geqslant 0.$$

Finally, we set $\Gamma = \Gamma(0)$.

THEOREM 5.2. *$\underline{d}(\Gamma) > 0$ holds for almost all $\mathscr{H} \in \mathbf{P}(t)$.*

Proof. From Theorem 4.3 we already know that almost all $\mathscr{H} \in \mathbf{P}(t)$ are K_t-free universal. However, to prove this theorem we need slightly more (which follows from the proof of that result). Namely, let $\mathscr{A} \subset [{W \atop k}]$ be K_t-free, $W_0 < W$, $W = \langle f_1, ..., f_d \rangle$, and $W_0 = \langle f_1, ..., f_{d-1} \rangle$. In addition, let $e_1, ..., e_{d-1}$ be elements of $\mathbb{V}$ such that $\phi : f_i \to e_i$, $1 \leqslant i \leqslant d-1$, defines an induced embedding of $\mathscr{A}_{|W_0}$ into $\mathscr{H}$. Then with probability 1 the lower density of $f_d \in \mathbb{V}$ such that $\phi : f_i \to e_i$, $1 \leqslant i \leqslant d$, embeds $\mathscr{A}$ as an induced family is positive.

Using this fact, the proof of Theorem 5.2 is similar to that of Theorem 3.1.

We suppose to the contrary that $\underline{d}(\Gamma) = 0$. Let $\mathscr{G}$ be as in Theorem 5.1 and let $W = \langle w_1, ..., w_b \rangle$ be such that $\dim W = b$ and $\mathscr{G} \subset [{W \atop k}]$.

We want to find (by induction) linearly independent elements $u_1, ..., u_i \in \mathbb{V}$ such that the following hold:

(i) $\phi : w_j \to u_j$ defines an induced embedding of $\mathscr{G}_{|\langle w_1, ..., w_i \rangle}$ into $\mathscr{H}_i = \mathscr{H}_{|\langle u_1, ..., u_i \rangle}$;

(ii) $\mathscr{H}_i$ contains no monochromatic induced copy of $\mathscr{F}$;

(iii) for all subspaces Q of dimension less than d of $\langle u_1, ..., u_i \rangle$ one has $\bar{d}(\Gamma(Q)) = 0$.

If we can show that this is possible up to $i = b$, then we get the desired contradiction. Namely, (ii) contradicts Theorem 5.1 for $i = b$.

However, as we said in the beginning of the proof, with probability 1 the set of possible continuations u_{i+1} has positive lower density. Omitting all elements which would cause a violation of (ii) or (iii) means omitting one set of lower density zero and a finite number of sets of upper density zero. Thus, the set of permissible choices of u_{i+1} will have positive upper density—the desired contradiction. ∎

Remarks. Obviously, the same proof works for the countably universal K_t-free graphs as well (the existence of those graphs was proved by Henson [H]; cf. [C] for many interesting results on universal and universal K_3-free graphs).

We believe that in this case, unlike Theorem 3.1, it is not possible to get positive lower bounds for $\underline{d}(\Gamma)$. However, with basically the same proof one can get lower bounds of the form $\delta(\varepsilon)$ if we only require the statement to hold for $\mathscr{H} \in \mathbf{P}(t)$ with probability at least $1-\varepsilon$.

In the case of universal (not K_t-free) spaces or graphs, one can get explicit lower bounds for $\underline{d}(\Gamma)$ in terms of r, k, and l.

In case of graphs one has to use the induced Ramsey theorem (cf. [D, EHP, R]). For hypergraphs or for colorings of complete subgraphs (and for restricted versions as well) one needs the stronger Ramsey theorems of Nešetřil and Rödl (see [NR]).

6. (m, p, c)-Sets

In 1973, Deuber [D1] introduced certain combinatorial structures, called (m, p, c)-sets, in connection with his fundamental work on solution sets of systems of homogeneous linear equations. In particular, Deuber showed that (m, p, c)-sets enjoy a Ramsey theorem (see the theorem below). In this section, we prove an iterated density theorem for (m, p, c)-sets.

We begin with a few definitions. For m, p, $c \in \mathbb{N}$, define:

$$D(m, p, c) := \{(\lambda_1, ..., \lambda_m) : \lambda_i \text{ are integers, and for some } i < m,$$
$$|\lambda_k| \leqslant p, j < i, \lambda_i = c, \lambda_j = 0, j > i\}.$$

For $x_1, ..., x_m \in \mathbb{N}$,

$$\langle x_1, ..., x_m \rangle := \left\{ \sum_{i=1}^{m} \lambda_i x_i : (\lambda_1, ..., \lambda_m) \in D(m, p, c) \right\}.$$

A subset $S \subseteq \mathbb{N}$ which can be written as $S = \langle x_1, ..., x_n \rangle$ for some $x_i \in \mathbb{N}$ is called an (m, p, c)-set (where the values of p and c are understood from the context).

Theorem (Deuber [D1]). *For all m, p, c, $r \in \mathbb{N}$ there exist M, P, $C \in \mathbb{N}$ such that in any r-colored (M, P, C)-set there must always exist a monochromatic (m, p, c)-set.*

Fix m, p, c, r, $m \geqslant 2$, $r \geqslant 2$ and fix an r-coloring of $\mathbb{N}$. Let $\delta = \delta(m, p, c, r) > 0$ be chosen appropriately (to be specified later). Define

$$\Gamma(x_1, ..., x_{m-1}) := \{x \in \mathbb{N} : \langle x_1, ..., x_{m-1}, x \rangle \text{ is monochromatic}\},$$

and for $0 \leqslant i < m-1$,

$$\Gamma(x_1, ..., x_i) := \{x \in \mathbb{N} : \bar{d}(\Gamma(x_1, ..., x_i, x)) \geqslant \delta\}.$$

In particular, for $i=0$,

$$\Gamma=\Gamma(\varnothing)=\{x\in\mathbb{N}:\bar{d}(\Gamma(x))\geqslant\delta\}.$$

THEOREM 6.1.

$$\underline{d}(\Gamma)\geqslant\delta. \tag{6.1}$$

Proof. Suppose (6.1) does not hold, i.e., $\underline{d}(\Gamma)<\delta$. Thus, if $\bar{\Gamma}:=\mathbb{N}\backslash\Gamma$ then

$$\bar{d}(\bar{\Gamma})>1-\delta.$$

We now construct a sequence of (k, P, C)-sets S_k, $k=1,2,\ldots$, as follows. To begin with, select $s_1\in\bar{\Gamma}$ arbitrarily and set $S_1=\langle s_1\rangle'$, i.e., S_1 is a $(1,P,C)$-set (in general, (k,P,C)-sets will be denoted by primed notation $\langle x_1,\ldots,x_k\rangle'$). Now suppose $S_j=\langle s_1,\ldots,s_j\rangle'$ has been defined. Select s_{j+1}, if possible, so that with $S_{j+1}:=\langle s_1,\ldots,s_j,s_{j+1}\rangle'$ we have

(i) $s_{j+1}>3MPc\,s_j$;

(ii) if $\langle v_1,\ldots,v_k\rangle\subseteq S_{j+1}$ for some $k\leqslant m$ then

$$v_k\notin\Gamma(v_1,\ldots,v_{k-1});$$

(iii) S_{j+1} contains no monochromatic (m,p,c)-set.

By Deuber's theorem, this process must stop with the formation of S_t for some $t<M$. We claim that the reason why S_{t+1} could not be formed is because (iii) could not be satisfied. (It is obvious that it is not (i) which stops S_{t+1}.) To see it is not because of (ii) we argue as follows.

First observe that (by (i)) any $x\in S_t=\langle s_1,\ldots,s_t\rangle'$ has a unique representation as

$$x=\lambda_1s_1+\cdots+\lambda_{w-1}s_{w-1}+Cs_w,\qquad|\lambda_i|\leqslant P.$$

In this case, we say that $\operatorname{rank}(x):=w$.

CLAIM. *If $\langle u_1,\ldots,u_k\rangle\subseteq S_t$ for some $k\leqslant m$ then all the quantities cu_i have distinct ranks.*

Proof of Claim. Suppose the contrary and let $i<j$ be so that $\operatorname{rank}(cu_i)=\operatorname{rank}(cu_j)=w$. Thus

$$cu_i=\lambda_1s_1+\cdots+\lambda_{w-1}s_{w-1}+Cs_w\in S_t,$$

$$cu_j=\lambda_1's_1+\cdots+\lambda_{w-1}'s_{w-1}+Cs_w\in S_t,$$

and also

$$u_i + cu_j = \frac{1}{c}(\lambda_1 s_1 + \cdots + \lambda_{w-1} s_{w-1}) + \lambda_1' s_1 + \cdots + \lambda_{w-1}' s_{w-1}$$

$$+ C(1+1/c)\, s_w \in \langle u_1, ..., u_k \rangle \subseteq S_t.$$

However, by the choice of the s_t,

$$MPs_{w-1} + Cs_w < u_i + cu_j < -MPs_w + Cs_{w+1},$$

which implies $u_i + cu_j \notin \langle s_1, ..., s_t \rangle'$, a contradiction, and the claim is proved.

As a consequence, if $\langle u_1, ..., u_k \rangle \subseteq \langle s_1, ..., s_j \rangle'$ and $\operatorname{rank}(cu_1) < \cdots < \operatorname{rank}(cu_k) = j$ then $\langle u_1, ..., u_{k-1} \rangle \subseteq \langle s_1, ..., s_{j-1} \rangle'$. Thus, to satisfy (ii) in trying to form S_{t+1}, we must choose s_{t+1} so that if $\langle u_1, ..., u_k \rangle \subseteq \langle s_1, ..., s_{t+1} \rangle'$ with $\operatorname{rank}(cu_1) < \cdots < \operatorname{rank}(cu_k)$ then $u_k \notin \Gamma(u_1, ..., u_{k-1})$. This is immediate (by induction) if $\operatorname{rank}(cu_k) \leqslant t$, so assume $\operatorname{rank}(cu_k) = t+1$. Therefore, $\langle u_1, ..., u_{k-1} \rangle \subseteq \langle s_1, ..., s_t \rangle'$. By the definition of s_t,

$$u_{k-1} \notin \Gamma(u_1, ..., u_{k-2}) = \{x \in \mathbb{N} : \bar{d}(\Gamma(u_1, ..., u_{k-2}, x)) \geqslant \delta\}.$$

Hence, $\bar{d}(\Gamma(u_1, ..., u_{k-2}, u_{k-1})) < \delta$. Now $u_k = \lambda_1 s_1 + \cdots + \lambda_t s_t + Cs_{t+1}$ for some choice of $\lambda_1, ..., \lambda_t$ with $|\lambda_i| \leqslant P$. Thus, for δ suitably small, we can certainly find a (large) s_{t+1} so that $u_k \notin \Gamma(u_1, ..., u_{k-1})$, i.e., so that (ii) holds.

Therefore, S_t has the property that for a set X of upper density at least $\frac{1}{2}$ (for δ sufficiently small), for each $x \in X$, $\langle s_1, ..., s_t, x \rangle'$ contains some *monochromatic* (m, p, c)-set $U(x) = \langle u_1(x), ..., u_m(x) \rangle$, where we can assume $\operatorname{rank}(cu_1(x)) < \cdots < \operatorname{rank}(cu_m(x))$. Since S_t contains no monochromatic (m, p, c)-set then $U(x) \not\subseteq S_t$ so that $\operatorname{rank}(cu_m(x)) = t+1$. Hence, $U'(x) := \langle u_1(x), ..., u_{m-1}(x) \rangle \subseteq S_t$. Therefore, if K denotes the number of $(m-1, p, c)$-sets contained in S_t, then for some $(m-1, p, c)$-set $U^* = \langle u_1^*, ..., u_{m-1}^* \rangle$ with $\operatorname{rank}(cu_1^*) < \cdots < \operatorname{rank}(cu_{m-1}^*)$, we have for $X^* = \{x \in X : u_i(x) = u_i^*, 1 \leqslant i \leqslant m-1\}$,

$$\bar{d}(X^*) \geqslant \frac{1}{K} \bar{d}(X) \geqslant \frac{1}{2K}.$$

In particular, since $x \in X^* \Rightarrow \langle u_1^*, ..., u_{m-1}^*, u_m(x) \rangle$ is monochromatic and $\operatorname{rank}(cu_m(x)) = t+1$ in $\langle s_1, ..., s_t, x \rangle$ then $\bar{d}\{y \in \mathbb{N} : \langle u_1^*, ..., u_{m-1}^*, y \rangle$ is monochromatic$\} \geqslant 1/2CK > \delta$ for δ sufficiently small.

However,

$$\begin{aligned}
&\langle u_1^*, ..., u_{m-1}^* \rangle \subseteq S_t \\
&\Rightarrow u_{m-1}^* \notin \Gamma(u_1^*, ..., u_{m-2}^*) = \{ y \in \mathbb{N} : \bar{d}(\Gamma(u_1^*, ..., u_{m-2}^*, y) \geqslant \delta \} \\
&\Rightarrow \bar{d}(\Gamma(u_1^*, ..., u_{m-2}^*, u_{m-1}^*)) < \delta \\
&\bar{d}\{ y \in \mathbb{N} : \langle u_1^*, ..., u_{m-1}^*, y \rangle \text{ is monochromatic} \} < \delta
\end{aligned}$$

which is a contradiction. Thus, we must have $\underline{d}(\Gamma) \geqslant \delta$ and the theorem is proved. ∎

We point out that our theorem for (m, p, c)-sets has as an immediate consequence iterated density results for solution sets to partition regular systems of homogeneous linear equations, the simplest perhaps being the single equation $x + y = z$. The statement in this case is:

THEOREM. *For all $r \in \mathbb{N}$ there exists $\delta = \delta(r) > 0$ so that for any r-coloring of $\mathbb{N}$,*

$$\underline{d}(x \in \mathbb{N} : \bar{d}(y \in \mathbb{N} : x, y \text{ and } x + y \text{ are monochromatic}) \geqslant \delta) \geqslant \delta.$$

In fact, a more careful analysis shows that in this case we can take $\delta(r) = 2^{-(r+1)!}$. This, of course, is an iterated density version of the previously mentioned theorem of Schur [S], which asserts that in any finite coloring of $\mathbb{N}$, monochromatic solutions to $x + y = z$ always exist.

7. FINITE UNIONS AND FINITE SUMS

In this section we will first give new proofs for the "finite unions" and "finite sums" theorems: Let us first recall these two important Ramsey-type theorems (see [GRS]). As usual, for $x \in \mathbb{N}$, $[x]$ denotes the set $\{1, 2, ..., x\}$:

$U(k, r)$: For all k, $r \in \mathbb{N}$ there exists $u = u(k, r)$ such that for all r-colorings of the power set $2^{[u]}$, one can find k pairwise disjoint nonempty subsets $A_1, ..., A_k$ so that all possible nonempty unions $\bigcup_{i \in I} A_i$, $\varnothing \neq I \subseteq [k]$, have the same color.

$S(k, r)$: For all k, $r \in \mathbb{N}$ there exists $s = s(k, r)$ such that for all r-colorings of $[s]$, one can find k distinct integers $a_1, ..., a_k$ in $[s]$ so that all possible nonempty sums $\sum_{i \in I} a_i$, $\varnothing \neq I \subseteq [k]$, have the same color.

The implication $U(k, r) \Rightarrow S(k, r)$ is immediate. Indeed, an r-coloring of $[2^u]$ induces an r-coloring of $2^{[u]}$ in the usual way, namely, to $X \in 2^{[u]}$ assign the same color that $\sum_{x \in X} 2^{x-1} \in [2^u]$ has. The definition of u

implies the existence of pairwise disjoint nonempty $A_1, A_2, ..., A_k$ with all unions $\bigcup_{i \in I} A_i$, $\varnothing \neq I \subseteq [u]$, having the same color. This clearly implies that the integers $a_1, a_2, ..., a_k$ with $a_j := \sum_{i \in A_j} 2^{i-1}$ have monochromatic sums.

We will now prove the implication

$$S(k, r) \Rightarrow U(2k, r). \tag{7.1}$$

Since $S(1, r)$ is obvious, we can thereby obtain the chain of implications

$$\begin{aligned} S(1, r) &\Rightarrow U(2, r) \Rightarrow S(2, r) \Rightarrow U(4, r) \Rightarrow \cdots \\ &\Rightarrow S(2^d, r) \Rightarrow U(2^{d+1}, r) \Rightarrow S(2^{d+1}, r) \Rightarrow \cdots, \end{aligned}$$

which implies both $U(k, r)$ and $S(k, r)$ for all k and r.

Proof of (7.1). Let $s = s(k, r)$ be the integer needed for $S(k, r)$. Let n be an integer (whose existence is guaranteed by Ramsey's theorem) with the property that for any r-coloring of $2^{[n]}$, there is a subset $X \subseteq 2^{[n]}$ of size $3s$ so that the *color* of any subset $Y \subseteq X$ depends only on the *size* $|Y|$ of Y.

Now consider an arbitrary r-coloring of $2^{[n]}$. In particular, this is an r-coloring of all subsets of $[n]$ of the form $S = [a_1, b_1) \cup [a_2, b_2) \cup \cdots \cup [a_i, b_i)$, where $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_i < b_i \leqslant n$ and $i \leqslant s$. To each such subset associate the $2i$-set $S^* = \{a_1, b_1, ..., a_i, b_i\} \subseteq [n]$ and assign it the same color that S has. By the choice of n there is an associated $3s$-element set $X^* = \{a_1, b_1, c_1, ..., a_s, b_s, c_s\} \subseteq [n]$ (in increasing order) with the property that the (original) color of any subset of $2^{[n]}$ consisting of i pairwise disjoint half-open intervals with distinct endpoints from X^* depends only on i, for $i \leqslant s$. This therefore defines an r-coloring of $[s]$. By the choice of s, $S(k, r)$ implies the existence of positive integers $x_1, ..., x_k$ so that all nonzero sums $\sum_{i \in I} x_i$, $I \subseteq [k]$, have the same color c.

Consider the $2k$ pairwise disjoint subsets

$$\begin{aligned} X_i &:= \bigcup_{x_1 + \cdots + x_{i-1} < j \leqslant x_1 + \cdots + x_i} [a_j, b_j) \\ Y_i &:= \bigcup_{x_1 + \cdots + x_{i-1} < j \leqslant x_1 + \cdots + x_i} [b_j, c_j), \qquad 1 \leqslant i \leqslant k. \end{aligned}$$

Since $[a_j, b_j) \cup [b_j, c_j) = [a_j, c_j)$ then it follows that any nonempty union of the X_i and Y_j has color c. This shows that $S(k, r) \Rightarrow U(2k, r)$ and the proof is complete. ∎

Finite unions. For a family $\mathscr{F}$ of finite subsets of $\mathbb{N}$, we can define

$$\bar{d}(\mathscr{F}) = \limsup_{n \to \infty} \frac{|\mathscr{F} \cap 2^{[n]}|}{2^n}$$

with $\underline{d}(\mathscr{F})$ defined analogously. For $A_1, ..., A_k \in \mathscr{F}$ define $\mathscr{B}^*(A_1, ..., A_k) := \{\bigcup_{i \in I} A_i : \emptyset \neq I \subseteq [k]\}$, the Boolean algebra generated by $A_1, ..., A_k$ (with the empty set deleted). There is an iterated density version of the finite unions theorem which can be stated as follows. First fix an r-coloring of all finite subsets of $\mathbb{N}$. For finite sets $A_1, ..., A_{m-1}$, define:

$$\Gamma(A_1, ..., A_{m-1}) := \{X : \mathscr{B}^*(A_1, ..., A_{m-1}, X) \text{ is monochromatic}\},$$
$$\Gamma(A_1, ..., A_i) := \{X : \bar{d}(\Gamma(A_1, ..., A_i, X)) \geqslant \delta\}, \qquad 0 \leqslant i < m-1,$$

where

$$\Gamma = \Gamma(\emptyset) = \{X : \bar{d}(\Gamma(X)) \geqslant \delta\}$$

and δ is a suitably small positive number, which depends on m and the particular coloring as well.

THEOREM. *For any r-coloring of the finite subsets of* $\mathbb{N}$, $\underline{d}(\Gamma) \geqslant \delta$.

The proof follows much the same lines as the earlier proofs. That is, we assume $\underline{d}(\Gamma) < \delta$ and sequentially construct a maximal family $F_t = \{S_1, S_2, ..., S_t\}$ of disjoint sets so that $S_j \notin \Gamma(S_{i_1}, ..., S_{i_u})$, $0 \leqslant i_1 < \cdots < i_u < j$, $1 \leqslant j \leqslant t$, and for all $X \notin \Gamma(S_{i_1}, ..., S_{i_u})$, $0 \leqslant i_1 < \cdots < i_u \leqslant t$, $\mathscr{B}^*(S_1, ..., S_t, X)$ contains no monochromatic $\mathscr{B}^*(V_1(X), ..., V_k(X))$. The finite unions theorem guarantees that $t < u(k, r)$. The argument is now completed by focussing on that portion of F_t which is common to a positive fraction of the $B^*(V_1(X), ..., V_k(X))$ (details are left to the reader).

An iterated density version of the finite sums theorem follows from Theorem 6.1. An iterated density version of the infinite version of the finite sums theorem (Hindman's theorem [Hi]) was proved recently by Bergelson and Hindman [BH] using, in part, an ergodic theoretic approach.

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