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On a Problem of Chvátal and Erdös on Hypergraphs Containing No Generalized Simplex

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Let X be an n-element set and $\mathscr{F}$ a family of k-subsets of X. Let r be an integer, $k > r \geqslant 2$. Suppose that $\mathscr{F}$ does not contain $r+1$ members having empty intersection such that any r of them intersect non-trivially. Chvátal and Erdös conjectured that for $(r+1)\,k \leqslant rn$ we have $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$. In this paper we first prove that this conjecture holds asymptotically (Theory 1). In Theorems 4 and 5 we prove it for $r=2$, $k \geqslant 5$, $n > n_0(k)$; $k \geqslant 3r$, $n > n_0(k, r)$, respectively.

1. Introduction

Let X be a finite set of elements and let $\mathscr{F}$ be a family of k-element subsets of X. Let r be an integer, $r \geqslant 2$.

We say that $\mathscr{F}$ contains an r-dimensional generalized simplex (or simply r-simplex) if we can find $F_1,..., F_{r+1} \in \mathscr{F}$ such that

$$\bigcap_{i=1}^{r+1} F_i = \varnothing,$$

but for any $1 \leqslant j \leqslant r+1$,

$$\bigcap_{i \neq j} F_i \neq \varnothing.$$

It is easy to see that there is no r-simplex with $k < r$. For $k = r$ the F_i's are necessarily the vertex-sets of the faces of an r-dimensional simplex, i.e., the different r-subsets of an $(r+1)$ set.

A special case of Turan's problem is: (see [7]). What is the maximum number of edges $\mathscr{F}$ can have if it contains no k-simplex?

For $k = 2$ the answer follows from Turan's more general theorem (see Turan [7]); it is $[n/2][(n+1)/2]$.

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For $k \geqslant 3$ the problem is involved, but evidently this maximum is at least $[n/k]^k$, i.e., more than $c_k\binom{n}{k}$ for some $c_k > 0$. What happens if $k > r$?

Erdös [4] made the following

CONJECTURE 1. *Let $3 \leqslant k \leqslant \frac{2}{3}n$, and suppose $\mathscr{F}$ contains no 2-simplex. Then $|\mathscr{F}| = \binom{n-1}{k-1}$.*

Chvátal [2] made the more general

CONJECTURE 2. *Let $r < k \leqslant (r/(r+1))\, n$, and suppose $\mathscr{F}$ contains no r-simplex. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$.*

Chvátal [2] proved this conjecture for $k = r + 1$.

The validity of Conjecture 2 in the case $((r-1)/r)\, n < k$ follows from Lemma 1 in [5]. It was proved by Bermond and Frankl [1] for an infinity of special values, but always $n < k^2$.

The aim of this paper is to deal with the case $n > n_0(k)$. First we prove

THEOREM 1. *Let $r < k \leqslant (r/(r+1))\, n$, and suppose $\mathscr{F}$ contains no r-simplex. Then $|\mathscr{F}| \leqslant (1 + o(1))\binom{n-1}{k-1}$.*

Acccording to the result of Chvátal we may assume $k \geqslant r + 2$.

In the proof we make use of the following consequence of a theorem of Duke and Erdös [3].

THEOREM 2. *Let $r + 2 \leqslant k$ and suppose $\mathscr{F}$ contains no 3 numbers F_1, F_2, F_3 such that for some r-element subset D of F_1 we have*

$$F_1 \cap F_2 = F_1 \cap F_3 = F_2 \cap F_3 = D.$$

then for some constant C_k we have

$$|f| \leqslant c_k \binom{n-2}{k-2}.$$

In Section 3, we prove

THEOREM 3. *Suppose $\mathscr{F}$ contains no 2-simplex (i.e., triangle), $k \geqslant 5$, $n \geqslant n_0(k)$. Then one of the following holds:*

(i) $|\mathscr{F}| < \binom{n-1}{k-1}$.

(ii) *For some $y \in X$ we have $y \in F$ for every $F \in \mathscr{F}$.*

(iii) *For some $x \in X$ $d_{\mathscr{F}}(x) < \frac{1}{2}\binom{n-1}{k-2}$ holds ($d_{\mathscr{F}}(x)$ is the nuber of edges of $\mathscr{F}$ containing x).*

Next we deduce the conjecture of Erdös from Theorem 3.

THEOREM 4. *Suppose $\mathscr{F}$ contains no 2-simplex, $k \geqslant 5$, $n \geqslant n_0'(k)$. Then either every member of $\mathscr{F}$ contains a fixed element y of X or we have* $|\mathscr{F}| < \binom{n-1}{k-1}$.

In the last section we give a sketch of proof for

THEOREM 5. *Suppose $\mathscr{F}$ contains no r-simplex, $k > 3r$, $n \geqslant n_0(k, r)$. Then either every member of $\mathscr{F}$ contains a fixed element y of X or we have* $|\mathscr{F}| < \binom{n-1}{k-1}$.

The proof of Theorem 5 heavily depends on a refinement of a result in [6].

2. THE PROOF OF THEOREM 1

First we divide $\mathscr{F}$ into two subfamilies. Let us define

$$\mathscr{F}_0 = \{F \in \mathscr{F} \mid \forall G \subset F, |G| = k-1, \exists F' \in \mathscr{F}, F' \neq F, G \subset F'\},$$

$$\mathscr{F}_1 = \mathscr{F} - \mathscr{F}_0$$

$$= \{F \in \mathscr{F} \mid \exists G = G(F), |G| = k-1, \forall F' \in \mathscr{F}, F' \neq F, G \not\subset F'\}.$$

Let us set

$$\mathscr{G} = \{G(F) \mid F \in \mathscr{F}_1\}.$$

Then by the definition of $\mathscr{F}_1$ we have

$$|\mathscr{F}_1| = |\mathscr{G}| \leqslant \binom{n-1}{k-1} + \binom{n-1}{k-2}. \tag{1}$$

Suppose we can find F, F', $F'' \in \mathscr{F}_0$ and $D \subset F$, $|D| = r$ such that

$$F \cap F' = F \cap F' = F' \cap F'' = D. \tag{2}$$

Let $D = \{d, d_2, ..., d_r\}$ and for $i = 1, ..., r$ let F_1 be a set in $\mathscr{F}$ different to F which contains $F - \{d_i\}$. The existence of F_i is assumed by the definition of $\mathscr{F}_0$.

As $F_1 - F$ consists of one element, x_1, lying outside of D, and $(F' - D) \cap (F'' - D) = \varnothing$, we have either $x_1 \notin F'$ or $x_1 \in F''$.

Hence it is possible to choose $F_{r+1} = F'$ or F'' such that $x_1 \notin F_{r+1}$. We assert F_1, 1,..., 1, F_{r+1} is an r-simplex.

As $F_1 \cap F_{r+1} = D - \{d_1\}$, and $d_i \notin F_i$ for $1 \leqslant i \leqslant r$, we deduce $\bigcap_{i=1}^{r+1} F_i = \varnothing$.

By the definition of the F_i's we have

$$d_j \in \bigcap_{i \neq j} F_i \qquad \text{for } 1 \leqslant j \leqslant r,$$

and

$$(F - D) \subset \bigcap_{i \neq r+1} F_i,$$

which proves that $F_1, \ldots, F_{r+1}$ indeed form an r-simplex, a contradiction, establishing that there are no F, F', $F'' \in F_0$, and $D \subset F$, $|D| = r$ which satisfy (2).

An application of Theorem 2 yields

$$|\mathscr{F}_0| \leqslant c_k \binom{n-2}{k-2}. \tag{3}$$

Combining (1) and (3) we obtain

$$|\mathscr{F}| = |\mathscr{F}_1| + |\mathscr{F}_0| \leqslant \binom{n-1}{k-1} + (1 + c_k) \binom{n-2}{k-2} = (1 + o(1)) \binom{n-1}{k-1}.$$

Q.E.D.

3. The Proof of Theorem 3

Without loss of generality we may assume

$$|\mathscr{F}| \geqslant \binom{n-1}{k-1}, \tag{4}$$

and that $\bigcap_{F \in \mathscr{F}} F = \varnothing$.

Let us divide $\mathscr{F}$ now into three subfamilies.

$$\mathscr{F}_0 = \{F \in \mathscr{F} \mid \forall G \subset F, |G| = k-1, \exists F' \in \mathscr{F}, F' \neq \mathscr{F}, G \subset F'\},$$

$$\mathscr{F}_2 = \{F \in \mathscr{F} \mid \exists G_1, G_2 \subset F, |G_1| = |G_2| = k-1, G_1 \neq G_2,$$
$$\forall F_1, F_2 \in \mathscr{F}, F_1 \neq F \neq F_2, G_1 \not\subset F_1, G_2 \not\subset F_2\},$$

$$\mathscr{F}_1 = \mathscr{F} - (\mathscr{F}_0 \cup \mathscr{F}_2),$$

i.e., for the members of $\mathscr{F}_1$ there is exactly one $(k-1)$-element subset which is contained in no other member of $\mathscr{F}$. By the definitions we have

$$|\mathscr{F}_1| + 2\,|\mathscr{F}_2| \leqslant \binom{n}{k-1} = \binom{n-1}{k-1} + \binom{n-1}{k-2}. \tag{5}$$

In view of (3) we have

$$|\mathscr{F}_0| \leqslant c_k \binom{n-2}{k-2}. \tag{6}$$

Is $|\mathscr{F}| = |\mathscr{F}_0| + |\mathscr{F}_1| + |\mathscr{F}_2|$ from (4), (5) and (6) we derive

$$|\mathscr{F}_2| = (1 + c_k) \binom{n-2}{k-2}. \tag{7}$$

Now using (6) and (7) we get from (4)

$$|\mathscr{F}_1| \geqslant \binom{n-1}{k-1} - (1 + 2c_k) \binom{n-2}{k-2} = \binom{n}{k-1} - (2 + 2c_k) \binom{n-2}{k-2}. \tag{8}$$

For an $F \in \mathscr{F}_1$ let $G(F)$ denote the $(k-1)$ subset which is contained in no other member of $\mathscr{F}$; we call $G(F)$ the kernel of F. Let us set $\{x(F)\} = F - G(F)$; we call $x(F)$ the complement of $G(F)$.

Let us define $\mathscr{G} = \{G(F) \mid F \in \mathscr{F}_1\}$.

Obviously $|\mathscr{G}| = |\mathscr{F}_1|$. For a 2-element subset E of X let us set $d(E) = |\{G \in \mathscr{G} \mid E \subset G\}|$.

Our next aim is to prove that all but $c'_k n$ 2-element subsets of X have degree $d(E) \geqslant \frac{3}{4}\binom{n-2}{k-3}$. For this purpose let us set

$$\mathscr{E}_1 = \left\{E \subset X \mid |E| = 2, d(E) \geqslant \frac{3}{4} \binom{n-2}{k-3}\right\},$$

$$\mathscr{E}_2 = \left\{E \subset X \mid |E| = 2, d(E) < \frac{3}{4} \binom{n-2}{k-3}\right\}.$$

Let us count the number of pairs (E, G), $E \subset G \in \mathscr{G}$, $|E| = 2$ in two different ways. We obtain

$$\begin{aligned} \binom{k-1}{2} |\mathscr{G}| &\leqslant |\mathscr{E}_1| \binom{n-2}{k-3} + |\mathscr{E}_2| \frac{3}{4} \binom{n-2}{k-3} \\ &= \binom{n}{2} \binom{n-2}{k-3} - \frac{1}{4} |e_2| \binom{n-2}{k-3}. \end{aligned} \tag{9}$$

Using (8) we obtain from (9)

$$\binom{n}{k-1} - (2 + 2c_k) \binom{n-1}{k-2} \leqslant \binom{n}{k-1} - \frac{|\mathscr{E}_2| \binom{n-2}{k-3}}{4 \binom{k-1}{2}}$$

or

$$|\mathscr{E}_2| \leqslant 4 \binom{k-1}{2}(2+2c_k)\frac{n-1}{k-2} < 4(k-1)(2+2c_k)\,n = c'_r n. \quad (10)$$

Our next observation is that if F, F', $F'' \in \mathscr{F}_1$ satisfy (2), with of course $r = 2$, then $D \subset G(F)$ is impossible. The proof is word for word the same as the proof of the impossibility of (2).

For $E \subset X$, $|E| = 2$ and $x \in (X - E)$ let us set

$$\mathscr{G}(x, E) = \{G \in \mathscr{G} \mid E \subset G, x \text{ is the complement of } G\},$$
$$g(x, E) = |\mathscr{G}(x, E)|.$$

Let us consider now a fixed $E \in \mathscr{E}_1$. Let $x_1, \ldots, x_m$ be those elements of $X - E$ for which $g(x, E) > 0$. We may suppose

$$g(x_1, E) \geqslant g(x_2, E) \geqslant \cdots \geqslant g(x_m, E). \quad (11)$$

We want to prove

$$\sum_{i=3}^{m} g(x_i, E) \leqslant (4k-1)(k-2)\binom{n-3}{k-4}. \quad (12)$$

We may assume $m \geqslant 3$. Let $G_0 \in \mathscr{G}(x_m, E)$. Let us set $H = (G_0 - E) \cup \{x_m\}$. Let i be the greatest integer such that there exists $F_1 \in \mathscr{F}_1$ satisfying $F_1 \cap H = \varnothing$, $G(F_1) \in \mathscr{G}(x_i, E)$.

As the number of sets $G \in \mathscr{G}$, $E \subset G$, $G \cap H \neq \varnothing$ is at most $(k-2)\binom{n-3}{k-4}$ we deduce

$$\sum_{\substack{i<j<m \\ x_j \notin H}} g(x_j, E) \leqslant (k-2)\binom{n-3}{k-4}. \quad (13)$$

If there is no such i we set $i = 0$, and (13) remains valid.

If $i \geqslant 3$ we set $H' = F_1 - E$.

Now by the observation after (10) it is impossible to find an i', $1 \leqslant i' < i$ such that there is an $F_2 \in \mathscr{F}$, satisfying $G(F_2) \in G(x_{i'}, E)$, $F_2 \cap (H \cup H') = \varnothing$. Indeed $G_0 \cup \{x_m\} = F_0$, F_1, F_2 satisfy $F_2 \cap F_1 = F_2 \cap F_0 = F_1 \cap F_0 \subset G_0 = G(F_0)$. Hence we deduce as we deduced (13):

$$\sum_{\substack{1 \leqslant j < i \\ x_j \notin (H \cup H')}} g(x_j, E) \leqslant 2(k-2)\binom{n-3}{k-4}. \quad (14)$$

Equations (13), (14) yield, in view of (11),

$$\sum_{j=3}^{m} g(x_j, E) \leqslant 3(k-2)\binom{n-3}{k-4} + 2(k-1)\, g(x_3, E). \tag{15}$$

To prove (12) it suffices now to prove

$$g(x_3, E) \leqslant 2(k-2)\binom{n-3}{k-4}. \tag{16}$$

Suppose that (16) does not hold. Then we can find $G_3 \in \mathscr{G}(x_3, E)$ such that $\{x_1, x_2\} \cap G_3 = \varnothing$.

Now in view of (11) we can find $G_2 \in \mathscr{G}(x_2, E)$ such that $G_2 \cap ((G_3 - E) \cup \{x_1, x_3\}) = \varnothing$, and there is a $G_1 \in \mathscr{G}(x_1, E)$ such that $G_1 \cap ((G_2 \cup G_3 \cup \{x_2, x_3\}) - E)) = \varnothing$.

Setting $F_3 = G_3 \cup \{x_3\}$, $F_2 = G_2 \cup \{x_2\}$, $F_1 = G_1 \cup \{x_1\}$ we have $F_3 \cap F_2 = F_3 \cap F_1 = F_2 \cap F_1 = E \subset G(F_1)$, a contradiction, proving (16), and consequently (12).

Hence we have, as $E \in \mathscr{E}_1$

$$g(x_1, E) + g(x_2, E) \geqslant \frac{3}{4}\binom{n-2}{k-3} - (4k-1)(k-2)\binom{n-3}{k-4}. \tag{17}$$

Let us define

$$A(E) = \left\{ E \cup \left\{ x_i \mid g(x_i, E) > 2(k-2)\binom{n-3}{k-4} \right\} \right\}.$$

Now in view of (16), (17) and $n > n_0(k)$ we have $|A(E)| = 3$ or 4. Let us set

$$\mathscr{A} = \{A(E) \mid E \in E_1\}.$$

LEMMA. *If $k \geqslant 5$ then for any $A_1, A_2, A_3 \in \mathscr{A}$ we have*

$$A_1 \cap A_2 \cap A_3 \neq \varnothing.$$

Proof of the Lemma. Suppose we have found $E_1, E_2, E_3 \in \mathscr{E}_1$ such that

$$A(E_1) \cap A(E_2) \cap A(E_3) = \varnothing.$$

Let us define for $i = 1, 2, 3$,

$$\mathscr{G}_i = \{G - E_i \mid G \in \mathscr{G}(x, E_i) \text{ for some } x \in (A(E_i) - E_i),$$
$$(G - E_i) \cap (A(E_1) \cup A(E_2) \cup A(E_3)) = \varnothing\}.$$

Now in view of (17) and the definition of the $A(E_i)$'s for $n > n_0(k)$ for $i = 1, 2, 3$ we have

$$|\mathscr{G}_i| > \frac{2}{3}\binom{n}{k-3}. \tag{18}$$

Let $X = \{x_1, x_2, ..., x_n\}$, $B_1 = \{x_1, x_2, ..., x_{k-3}\}$, $B_2 = \{x_{k-3}, x_{k-2}, ..., x_{2k-7}\}$, $B_3 = \{x_{2k-7}, x_{2k-6}, ..., x_{3k-12}, x_1\}$ (i.e., $|B_1| = |B_2| = |B_3| = k-3$, and they form a triangle, that is,

$$B_1 \cap B_2 \cap B_3 = \varnothing, \qquad B_1 \cap B_2 \neq \varnothing \neq B_1 \cap B_3, B_2 \cap B_3 \neq \varnothing).$$

For a permutation Π of X we set

$$\Pi(B_i) = \{\pi(x) \mid x \in B_i\}, \qquad i = 1, 2, 3.$$

For the number N_i of permutations Π satisfying $\Pi(B_i) \in \mathscr{G}_i$ we obtain, using (18),

$$N_i = (k-3)!\,(n-k+3)!\,|\mathscr{G}_i| > \tfrac{2}{3}n! \qquad (i = 1, 2, 3). \tag{19}$$

Now (19) implies the existence of a permutation Π_0 such that $\Pi_0(B_i) \in \mathscr{G}_i$ holds $i = 1$, 2, and 3.

By the definition of the $\mathscr{G}_i$ there are sets $C_i \subseteq A(E_i)$ such that $\Pi_0(B_i) \cup C_i = F_i \in \mathscr{F}_1$ $(i = 1, 2, 3)$. However, F_1, F_2, F_3 form a 2-dimensional simplex, a contradiction, proving the lemma.

LEMMA. *If $k \geqslant 5$ then there exists a $y \in X$ such that $y \in A$ holds for every $A \in \mathscr{A}$.*

Proof of the Lemma. If we can find $A_1, A_2 \in \mathscr{A}$ such that $|A_1 \cap A_2| = 1$, then in view of the preceding lemma $A_1 \cap A_2 \subset A$ for any $A \in \mathscr{A}$, and we are done. Hence we may assume that for every $A_1, A_2 \in \mathscr{A}$ we have

$$|A_1 \cap A_2| \geqslant 2. \tag{20}$$

We know that $\mathscr{A}$ consists of 3- and 4-element sets, and by the definition of $\mathscr{A}$ we have in view of (10) for $n > n_0(k)$

$$|\mathscr{A}| \geqslant \frac{|E_1|}{6} \geqslant \frac{n^2}{20}. \tag{21}$$

Suppose that for $A_1, A_2 \in \mathscr{A}$ we have $A_1 \cap A_2 = \{y_1, y_2\}$. Let us set

$$A_1 - \{y_1, y_2\} = D_1, \qquad A_2 - \{y_1, y_2\} = D_2, \qquad \{y_1, y_2\} = D_3.$$

In view of (2) for $A \in \mathscr{A}$, $D_3 \not\subset A$ we have

$$A \cap D_i \neq \emptyset \qquad \text{for } i = 1, 2, 3.$$

Hence the number of such A's is less than $|D_1| \cdot |D_2| \cdot |D_3| \cdot n \leqslant 8n$. Consequently by (21) the set $D = \{A - D_3 \mid D_3 \subset A \in \mathscr{A}\}$ has cardinality at least $n^2/30$. As it consists of 1- and 2-sets and $n > n_0(k)$, it contains four pairwise disjoint members D_4, D_5, D_6, D_7. But then for every $A \cap \mathscr{A}$, $D_3 \subset A$ implies, in view of (20), $A \cap D_i \neq \emptyset$, $3 \leqslant i \in 7$, which is impossible since $|A| \leqslant 4$. Thus, in this case $D_3 \subset A$ for every $A \in \mathscr{A}$.

The only remaining case is when for every $A_1, A_2 \in \mathscr{A}$ we have

$$|A_1 \cap A_2| \geqslant 3. \tag{22}$$

But (22) and $|A_1| \leqslant 4$ imply $|\mathscr{A}| \leqslant 4n$, a contradiction, proving the lemma.

Let y be the (or one of the) element(s) contained in every $A \in \mathscr{A}$. Let us set

$$\mathscr{E} = \{E \in \mathscr{E}_1 \mid y \notin E\}.$$

Obviously we have $|\mathscr{E}| \geqslant |\mathscr{E}_1| - (n-1)$. Let us set $B = \{b \in (X - y) \mid d_E(b) = n/2\}$, where $d_E(b)$ denotes the degree of b in the graph $\mathscr{E}$.

Using that, in view of (10) we have

$$|\mathscr{E}| \geqslant |\mathscr{E}_1| - (n-1) = \binom{n}{2} - |\mathscr{E}_2| - (n-1) \geqslant \binom{n}{2} - (c'_k + 1)\, n;$$

we deduce

$$2\,|\mathscr{E}| = \sum_{x \in X} d_E(x) = |B|\,\frac{n}{2} + (n - |B|)\, n = n^2 - |B|\,\frac{n}{2}.$$

Or equivalently,

$$|B| \leqslant \frac{(c'_k + 1)\, n + n}{n/2} = 2c'_k + 4. \tag{23}$$

Proposition. *If $F \in \mathscr{F}$, $y \notin F$ then*

$$|F \cap B| \geqslant k - 1. \tag{24}$$

Suppose for some F (24) is not true. Let z_1, z_2 be two different elements of $F - B$.

As $z_1, z_2 \in B$ we can find $v_1, v_2 \in (X - F)$ such that $\{z_1, v_1\}, \{z_2, v_2\} \in \mathscr{E}$.

By the definition of $A(E)$, and $A(\{a_i, v_i\})$ for $i=1,2$ we can find G_1, G_2 such that

$$G_i \in \mathscr{G}(y, \{a_i, v_i\}), \qquad G_i \cap F = \{z_i\} \qquad (i=1,2).$$

Now setting $F_1 = \{y\} \cup G_1$, $F_2 = \{y\} \cup G_2$ these two sets and F form a triangle, a contradiction, proving the proposition.

Let d denote the number of sets $F \in \mathscr{F}$ satisfying $y \notin F$. We have

COROLLARY.

$$d < n - 2^{2c'_k+4}. \tag{25}$$

In view of (23) and (24) we have

$$d \leqslant (n - |B|) \binom{|B|}{k-1} + \binom{|B|}{k} < n\, 2^{2c'_k+4},$$

proving (25).

Now we are in position to prove that in $\mathscr{F}$ there is a vertex of degree not exceeding $\frac{1}{2}\binom{n-2}{k=2}$.

For this purpose let $F \in \mathscr{F}$, $y \notin F$. Such an edge exists by our assumptions. We claim that for at least $k-1$ vertices x of F

$$d_{\mathscr{F}}(x) < \frac{1}{2} \binom{n-2}{k-2} \tag{26}$$

holds. Suppose it is not true and let x_1, x_2 be two different vertices of F for which (26) is not true. Let us set $\mathscr{D}_i = \{F \in \mathscr{F} \mid \{x_i, y\} \subset F\}$. In view of (25) and $n > n_0(k)$ for $i=1,2$ we have

$$|\mathscr{D}_i| > (k-1) \binom{n-3}{k-3}. \tag{27}$$

Hence we can find $F_i \in \mathscr{D}_i$ such that $F_i \cap F = \{x_i\}$ for $i=1,2$; that is, F, F_1, F_2 form a triangle which proves Theorem 3.

4. The Proof of the Conjecture of Erdős

In view of Theorem 1 there is an $n_0^* = n_0^*(k)$ such that for $n > n_0^*(k)$ and an $\mathscr{F}$ without triangles we have

$$|\mathscr{F}| < 2 \binom{n-1}{k-2}.$$

Let us choose $n_0'(k) = 2\max\{n_0^*(k), n_0(k)\}$, where $n_0(k)$ is the bound in Theorem 3.

Suppose Theorem 4 doesn't hold for some $n > n_0'(k)$, and some $\mathscr{F}(n)$. We apply Theorem 3, obviously in this case (iii) holds. Let us set

$$\mathscr{F}(n-1) = \{F \in \mathscr{F}(n), x \in F\}.$$

We have

$$|\mathscr{F}(n-1)| = |\mathscr{F}(n)| - d_{\mathscr{F}(n)}(x) > \binom{n-1}{k-1} - \frac{1}{2}\binom{n-2}{k-2}$$
$$= \binom{n-2}{k-1} + \frac{1}{2}\binom{n-2}{k-2}.$$

Now we consider the family $\mathscr{F}(n-1)$ on $n-1$ vertices and apply Theorem 3, omit a vertex of degree less than $\frac{1}{2}\binom{n-3}{k-2}$, obtain $\mathscr{F}(n-2)$, and so on until we obtain $\mathscr{F}([n/2])$.

Let us estimate the cardinality of $\mathscr{F}([n/2])$.

$$\left|\mathscr{F}\left(\left[\frac{n}{2}\right]\right)\right| > |\mathscr{F}(n)| - \sum_{i=1}^{[(n+1)/2]} \frac{1}{2}\binom{n-i-1}{k-2}$$
$$\geqslant \binom{[n/2]-1}{k-1} + \sum_{i=1}^{[(n+1)/2]} \frac{1}{2}\binom{n-i-1}{k-2}$$
$$\geqslant \binom{[n/2]-1}{k-1} + \frac{[n/2]}{2}\binom{[n/2]-2}{k-2} \geqslant 2\binom{[n/2]-1}{k-1},$$

a contradiction, since $\mathscr{F}([n/2]) \subset \mathscr{F}(n)$ contains no triangle, and $n \geqslant 2n_0^*(k)$. Thus Theorem 4 is proved.

5. The Conjecture of Chvátal

We partition $\mathscr{F}$ into $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{F}_2$ according to $F \in \mathscr{F}$ contains 0, 1 or at least two $(k-1)$ subsets which are not contained in any other $F' \in \mathscr{F}$. In Section 2 we proved there are no F, F', $F'' \in \mathscr{F}_0$, $D \subset F$, $|D| = r$ which satisfy (2), i.e., they form a Δ-system with kernel of cardinality r.

For $k > 3r$ applying the methods of [6] we can deduce that

$$|\mathscr{F}_0| \leqslant c_k \binom{n-r-1}{k-r-1}, \tag{28}$$

for some constant c_k.

Let us count the number of pairs (G, F), $G \subset F \in \mathscr{F}$, $|G| = k-1$.

$$|\mathscr{F}| \cdot k \leqslant |\mathscr{F}_1| + 2|\mathscr{F}_2| + \left(\binom{n}{k-1} - |\mathscr{F}_1| - 2|\mathscr{F}_2|\right)(n-k+1),$$

or equivalently

$$n|\mathscr{F}_1| + (2n-k)|\mathscr{F}_2| + k|\mathscr{F}_0| \leqslant (n-k+1)\binom{n}{k-1},$$

dividing by n and rearranging

$$|\mathscr{F}_0| + |\mathscr{F}_1| + |\mathscr{F}_2| \leqslant \binom{n-1}{k-1} + \frac{n-k}{n}(|\mathscr{F}_0| - |\mathscr{F}_2|). \tag{29}$$

As we may assume $|\mathscr{F}| \geqslant \binom{n-1}{k-1}$ we obtain from (29) and (28), $|\mathscr{F}_0| \leqslant c_k\binom{n-r-1}{k-r-1}$, and consequently

$$|\mathscr{F}_1| \geqslant \binom{n-1}{k-1} - 2c_k\binom{n-r-1}{k-r-1}. \tag{30}$$

Now we define G, as in the proof of Theorem 3, i.e., the family of the unique $(k-1)$ subsets of $\mathscr{F}_1$. Moreover let us define

$$\mathscr{H} = \{H \mid |H| = k-1, |\{F \in \mathscr{F}_1 \mid H \subset \mathscr{F}_1\}| \geqslant k+1\}.$$

Using (30) we derive (we count the pairs (F, H), $F \in \mathscr{F}_1$, $|H| = k-1$, $H \notin \mathscr{G}$)

$$|\mathscr{F}_1|(k-1) \leqslant |\mathscr{H}|(n-k-1) + \left(\binom{n-1}{k-2} + 2c_k\binom{n-r-1}{k-r-1} - |\mathscr{H}|\right)k. \tag{31}$$

From (31) we obtain for some constant c'_k.

$$|\mathscr{H}| \geqslant \binom{n-1}{k-2} - c'_k\binom{n-r-1}{k-r-1}.$$

Let us define for $B \subset X$, $|B| = r$

$$\mathscr{D}_G(B) = \{G \in \mathscr{G} \mid B \subset G\}, \qquad \mathscr{D}_H(B) = \{H \in \mathscr{H} \mid B \subset H\}.$$

Let us set further

$$\mathscr{B}=\left\{B\subset X\mid |B|=r,\ \middle|\ \mathscr{D}_{\mathscr{G}}(B)\leqslant\frac{r+1}{r+2}\binom{n-r}{k-r},\right.$$
$$\left.\mathscr{D}_{\mathscr{H}}(B)|\leqslant\frac{1}{2(r+2)}\binom{n-r}{k-r}\right\}.$$

Counting the number of pairs (B,E), $B\subset E$, $E\in(\mathscr{G}\cup\mathscr{H})$, $|B|=r$ we get

$$\begin{aligned}\binom{n}{k-1}\binom{k-1}{r}-c_k'\binom{n-r-1}{k-r-1}\binom{k-1}{r}&\\ \leqslant\binom{n}{r}\binom{n-r}{k-r-1}-|\mathscr{B}|\frac{1}{2(r+2)}\binom{n-r}{k-r}&.\end{aligned}\tag{32}$$

As $n>n_0(k,r)$, (32) yields $\mathscr{B}$ is empty, i.e., for every r-element subset B of X either

$$|\mathscr{D}_{\mathscr{G}}(B)>\frac{r+1}{r+2}\binom{n-r}{k-r}\qquad\text{or}\qquad|\mathscr{D}_{\mathscr{H}}(B)|>\frac{1}{2(r+2)}\binom{n-r}{k-r}.$$

Let us set

$$\mathscr{E}_1=\left\{B\subset X\mid |B|=r,\ |\mathscr{D}_{\mathscr{G}}(B)>\frac{r+1}{r+2}\binom{n-r}{k-r}\right\}.$$

From the proof of Theorem 1 it follows that there are no 3 sets F, F', $F''\in\mathscr{F}_1$ which form a Δ-system with kernel D, $|D|=r$, and $D\subset G(F)$. Hence we may proceed with $\mathscr{E}_1$ as in the proof of Theorem 1, and prove that there are two elements $x_1(E)$, $x_2(E)$ such that for almost every G, $E\subset G$ the complement of G is either $x_1(E)$ or $x_2(E)$, then we define for $E\in\mathscr{E}_1$ the set $A(E)$ satisfying

$$E\subset A(E),\qquad |A(E)-E|=1\quad\text{or}\quad 2.$$

Next we prove that the intersection of any $r+1$ member of $\mathscr{A}=\{A(E)\mid E\in\mathscr{E}_1\}$ is non-empty.

From this and $|A|>c(k,r)\binom{n}{r}$ we derive that there is a $y\in X$ such that $y\in A$ for every $A\in\mathscr{A}$. Then at last we are in a posititon to prove that every member of $\mathscr{F}$ contains y. Suppose the contrary and let $F_0\in\mathscr{F}$, $y\in F_0$. Let $\{x_1,x_2,\dots,x_r\}\subset F_0$, $x_{r+1}\in(X-F_0)$, $x_{r+1}\neq y$. Let us set further $E_i=\{x_j\mid 1\leqslant j\leqslant r+1, j\neq i\}$.

For $E_i\in\mathscr{E}_1$ we choose $G_i\in\mathscr{G}$ such that its complement is y and $G_i\cap F=E_i-\{x_{r+1}\}$ (its is possible by the definition of $A(E_i)$ and $y\in A(E_i)$). We put

$F_i = G_i \cup \{y\}$ in this case. For $E_i \notin \mathscr{E}_1$ we can find $H_i \in \mathscr{H}$ such that $H_i \cap F = E_i - \{x_{r+1}\}$ as $|\mathscr{D}_{\mathscr{H}}(E_i)| > (1/2(r+2))\binom{n-r}{k-r-1}$.

By the definition of $\mathscr{H}$ we can find $F_i \in \mathscr{F}_i$, $H_i \subset F_i$ such that

$$(F_i - H_i) \cap F = \varnothing.$$

Now the sets F_0, F_1, 1,..., 1, F_r form an r-dimensional simplex, a contradiction, which proves the theorem.

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