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The Number of Submatrices of a Given Type in a Hadamard Matrix and Related Results

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Let a, b be fixed positive integers. It is proved that if every relatively large submatrix of a ± 1-matrix N contains about the same number of both entries, then all 2^{ab} a by b matrices occur asymptotically the same number of times as a submatrix of N. A similar statement is shown to hold for graphs with uniformly distributed edges. © 1988 Academic Press, Inc.

1. INTRODUCTION

All matrices we consider have entries $+1$ and -1 only. With this convention a Hadamard matrix (of order n) is simply an n by n matrix M satisfying the matrix equation $MM^T = nI_n$, where I_n is the identity matrix of order n.

DEFINITION 1.1. A t by n matrix M is called an H-matrix if it satisfies

$$MM^T = nI_t. \tag{1}$$

Note that (1) is equivalent to saying that the rows of M are pairwise orthogonal. Clearly, every t by n submatrix of an n by n Hadamard matrix is an H-matrix, but the converse is not true in general.

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One of the important properties of H-matrices, namely that each t by s submatrix of them has nearly the same number of plus and minus 1's, is formulated in the following lemma.

LEMMA 1.2. *Suppose that N is a t by s submatrix of a t by n H-matrix M. Denoting by $p(N)$ the number of plus one's in N we have*

$$\left|p(N)-\frac{st}{2}\right| \leqslant \frac{1}{2}\sqrt{nst}. \tag{2}$$

DEFINITION 1.3. Call a matrix M *n-uniform* if for every s, t all t by s submatrices N of M satisfy (2).

With this terminology Lemma 1.2 says that H-matrices are n-uniform. Note that every submatrix of an n-uniform matrix is n-uniform.

Let $A=(a_{\nu\tau})$ be an a by b matrix and $N=(n_{\nu\tau})$ a t by s matrix. If $1 \leqslant i_1 \leqslant \cdots \leqslant i_a \leqslant t$ and $1 \leqslant j_1 < \cdots < j_b \leqslant s$ satisfy $a_{\nu\tau}=n_{i_\nu j_\tau}$ for all $1 \leqslant \nu \leqslant a$ and $1 \leqslant \tau \leqslant b$ then we say that the submatrix of N spanned by these rows and columns is isomorphic to A. Let $h(A, N)$ denote the number of submatrices of N which are isomorphic to A.

Let $\mathscr{A}$ be the set of all 2^{ab} matrices with a rows and b columns. Clearly, for every t by s matrix N we have

$$\sum_{A \in \mathscr{A}} h(A, N)=\binom{t}{a}\binom{s}{b}.$$

Our main result says that if t and s are "large" and a and b are "small" then $h(A, n)=(1+o(1))\binom{t}{a}\binom{s}{b} 2^{-ab}$ holds for every $A \in \mathscr{A}$ and every n-uniform matrix N.

THEOREM 1.4. *Let $t=t(n)$ and $s=s(n)$ be positive integers, functions of the positive integer n, satisfying $t(n)\, s(n)/n \rightarrow \infty$ as $n \rightarrow \infty$. Let a and b be fixed non-negative integers and A an a by b matrix. Then for every n-uniform t by s matrix N one has*

$$h(A, N)=(1+o(1))\binom{t}{a}\binom{s}{b} 2^{-ab}.$$

Recall that a t by s matrix can be interpreted as the adjacency matrix of a bipartite graph with vertex set $T \cup S$, T corresponding to the rows, S to the columns and edges corresponding to the $+1$'s.

Call a bipartite graph G n-uniform if the corresponding adjacency matrix is n-uniform. By a *bipartite isomorphism* we mean an isomorphism which maps vertices in the first (second) class on vertices in the first (second) class, respectively. Let $\mathrm{Aut}_p(G)$ denote the group of bipartite automorphisms of G. The following result is a direct consequence of Theorem 1.5.

COROLLARY 1.6. *Let G be an n-uniform bipartite graph with t and s vertices in the two classes. Suppose that $s = s(n)$, $t = t(n)$ satisfy $n = o(st)$. Let H be an arbitrary bipartite graph with a vertices in the first and b in the second class. Then the number $h(H, G)$ of induced subgraphs of G which are bipartite isomorphic to H satisfies*

$$h(H, G) = (1 + o(1)) \binom{t}{a}\binom{s}{b} 2^{-ab}\, a!\, b!\, |\mathrm{Aut}_p H|^{-1}.$$

The following definition extends the notion of n-uniformity to arbitrary graphs. For A, B disjoint subsets of the vertex set of a graph we denote by $e(A, B)$ the number of edges between A and B.

DEFINITION 1.7. A graph is called n-uniform if for any two disjoint subsets A and B of its vertex set we have

$$\left| e(A, B) - \frac{|A|\,|B|}{2} \right| \leqslant \frac{1}{2}\sqrt{n|A|\,|B|}.$$

THEOREM 1.8. *Let δ be an arbitrary positive real and let l, n be positive integers satisfying $l \leqslant (\frac{1}{2} - \delta) \log_2 n$. Let H be a graph on l vertices and G an n-uniform graph on n vertices. Then the number $h(H, G)$ of induced subgraphs of G isomorphic to H satisfies*

$$h(H, G) = \binom{n}{l} 2^{-\binom{l}{2}} \frac{l!}{|\mathrm{Aut}\, H|} (1 + o(1)).$$

2. PROOF OF LEMMA 1.2

Let $v_i(u_i)$ be the ith row of $M(N)$, respectively, $1 \leqslant i \leqslant t$. Also, let d_j denote the number of $+1$ entries in the jth column of N, $1 \leqslant j \leqslant s$. Obviously $p(N) = d_1 + \cdots + d_s$ holds. Let d be the average of d_j, i.e., $d = p(N)/s$.

Counting the squarelength of $v_1 + v_2 + \cdots + v_t$, we obtain

$$tn = \|v_1 + \cdots + v_t\|^2 \geqslant \|u_1 + \cdots + u_t\|^2$$
$$= \sum^{s} (2d_j - t)^2 \geqslant s(2d - t)^2.$$

Comparing the two extreme sides, we infer

$$\left|d-\frac{t}{2}\right| \leqslant \sqrt{\frac{nt}{4s}},$$

or, using $sd=p(N)$

$$\left|p(N)-\frac{st}{2}\right| \leqslant \sqrt{nts}/2. \quad \blacksquare$$

A weaker form of this lemma appears already in [3], where it is attributed to Lindsey. It was discovered by Alon [1] as well.

3. Proof of Theorem 1.5.

First we prove a lemma.

Let $\mathbf{v}_i$ be the ith row of N. Let $t' \leqslant t$ and let $\mathbf{u}_1, ..., \mathbf{u}_{t'}$ be arbitrary t' distinct rows of N.

Lemma 3.1. *For all i, $1 \leqslant i \leqslant t'$ one has*

$$\left|\sum_{1 \leqslant j \leqslant t', j \neq i} (\mathbf{u}_i, \mathbf{u}_j)\right| \leqslant \sqrt{ns(t'-1)/2}.$$

Proof. Let $N^+(N^-)$ be the matrix formed by $t'-1$ rows (all but the ith one) and those columns where $\mathbf{u}_i$ has $+1$ (-1), respectively. Let $s^+(s^-)$ denote the number of columns of $N^+(N^-)$, respectively. Then we have, using n-uniformity and the concavity of $\sqrt{x}$

$$\begin{aligned}\left|\frac{1}{2}\sum_{j\neq i}(\mathbf{u}_i, \mathbf{u}_j)\right| &= \left|p(N^+)-\frac{s^+(t'-1)}{2}+\frac{s^-(t'-1)}{2}-p(N^-)\right| \\ &\leqslant \frac{1}{2}\left(\sqrt{ns^+(t'-1)}+\sqrt{ns^-(t'-1)}\right) \\ &\leqslant \sqrt{ns(t'-1)/2}. \quad \blacksquare\end{aligned}$$

Let us define $d=d(n)=\lceil (s(n)\, t(n)/n)^{1/2} \rceil$. Then $d(n)$ tends to infinity as $n \to \infty$. Define $m=m(n)=\lfloor t/d \rfloor$. Let $N_1, ..., N_d$ be m by s submatrices of N consisting of the first m, second m, etc., rows.

Because of the choice of d and m we have $\binom{t}{a}=(1+o(1))\binom{d}{a} m^a$, i.e., almost all a by b submatrices have all their rows among the rows of $N_1, ..., N_d$ and still at most one row from each N_i. Therefore, it will be sufficient to show that for all choices of $1 \leqslant i_1 < \cdots < i_a \leqslant d$ and each a by b matrix A there are $(1+o(1))\, m^a \binom{s}{b} 2^{-ab}$ submatrices of N, isomorphic to A and which have their jth row in N_{i_j}, $1 \leqslant j \leqslant a$.

Let $h(A)$ denote the number of such submatrices. We are going to prove by induction on a that $h(A) = (1 + o(1))\, m^a \binom{s}{b}\, 2^{-ab}$ holds.

The case $a = 0$ is trivial. Suppose that $r \geqslant 0$, the statement is proved that $a \leqslant r$ and let $a = r + 1$.

To simplify notation we will suppose that $i_j = j$, $1 \leqslant j \leqslant a$. Note that $ms/n > (t/d - 1)\, s/n > (ts/n)/\lceil \sqrt{ts/n} \rceil - 1$, i.e., $ms/n \to \infty$ as $n \to \infty$.

For $1 \leqslant \nu$, $\mu \leqslant m$ let $s(\nu, \mu)$ denote the number of common positions of the νth and μth rows of N_a. If $\mathbf{v}_\nu$, $\mathbf{v}_\mu$ are the corresponding row vectors, then $2s(\nu, \mu) - s = (\mathbf{v}_\nu, \mathbf{v}_\mu)$ holds.

LEMMA 3.2. *$s(\nu, \mu) = (\frac{1}{2} + o(1))\, s$ holds for all but $o(m^2)$ choices of $1 \leqslant \nu \neq \mu \leqslant m$.*

Proof. Let γ be an arbitrary positive number and let t_ν be the number of rows in N_a which agree with the νth row in at least $\frac{1}{2}(s + \gamma s)$ places. Applying Lemma 3.1 to the corresponding row vectors gives

$$t_\nu \gamma s \leqslant \sqrt{n s t_\nu / 2} \qquad \text{or} \qquad t_\nu \leqslant \frac{n}{2 s \gamma^2} = m \frac{n}{ms} \frac{2^{-1}}{\gamma^2} = o(m).$$

Similar computation applies to the number of rows coinciding with the ν'th row in at most $\frac{1}{2}(s - \gamma s)$ places.

Since ν was arbitrary, the statement of the lemma follows. ∎

For every choice $\mathbf{j} = (j_1, \ldots, j_b)$, $1 \leqslant j_1 < \cdots < j_b \leqslant s$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_b)$, $\varepsilon_\mu = +1$ or -1, let $f(\mathbf{j}, \boldsymbol{\varepsilon})$ denote the number of rows i of N_a, $1 \leqslant i \leqslant m$, so that the j_μth entry in this row is ε_μ for $1 \leqslant \mu \leqslant b$.

LEMMA 3.3. *The average $\bar{f}$ of the $\binom{s}{b} 2^b$ quantities $f(\mathbf{j}, \boldsymbol{\varepsilon})$ is $m \cdot 2^{-b}$, moreover*

$$\sum_{\mathbf{j}, \boldsymbol{\varepsilon}} (f(\mathbf{j}, \boldsymbol{\varepsilon}) - \bar{f})^2 = o\left(m^2 \binom{s}{b} 2^{-b}\right)$$

holds.

Proof. It is clear from the definition that $\sum_{\boldsymbol{\varepsilon}} f(\mathbf{j}, \boldsymbol{\varepsilon}) = m$ holds for every fixed $\mathbf{j}$, proving $\bar{f} = m \cdot 2^{-b}$.

Using Lemma 3.2 we deduce

$$\begin{aligned} \sum_{\mathbf{j}, \boldsymbol{\varepsilon}} f(\mathbf{j}, \boldsymbol{\varepsilon})(f(\mathbf{j}, \boldsymbol{\varepsilon}) - 1) &= \sum_{1 \leqslant \mu \neq \nu \leqslant m} \binom{s(\mu, \nu)}{b} \\ &= m(m-1) \binom{s/2}{b} (1 + o(1)) \\ &= m^2 \binom{s}{b} 2^{-b} (1 + o(1)). \end{aligned}$$

Writing for short f instead of $f(\mathbf{j}, \varepsilon)$ we infer

$$\sum (f-\bar{f})^2 = \sum f(f-1) + \sum f - 2\bar{f} \sum f + \bar{f}^2 \binom{s}{b} 2^b$$
$$= \sum f(f-1) + \bar{f} \binom{s}{b} 2^b - \bar{f}^2 \binom{s}{b} 2^b$$
$$= o\left(m^2 \binom{s}{b} 2^{-b} \right). \qquad \blacksquare$$

Now we are in position to prove the theorem. Let $a = r+1$ and let A be an arbitrary $r+1$ by b matrix. Let A_0 be the submatrix of A formed by the first r rows and let $\boldsymbol{\delta}$ be the last row of A. For fixed $\mathbf{j}$ let $h(A, \mathbf{j})$ denote the number of vectors $\mathbf{i} = (i_1, ..., i_{r+1})$ so that the submatrix spanned by the i_vth row of N_v, $v = 1, ..., r+1$, and the columns corresponding to $\mathbf{j}$ is isomorphic to A, $h(A_0, \mathbf{j})$ is defined analogously.

Obviously, $h(A) = \sum_{\mathbf{j}} h(A, \mathbf{j})$ and $h(A_0) = \sum_{\mathbf{j}} h(A_0, \mathbf{j})$ hold.

Using the convexity of the function $y = x^2$, we have

$$(h(A) - \bar{f}h(A_0))^2 = \left(\sum_{\mathbf{j}} h(A_0, \mathbf{j})(f(\mathbf{j}, \boldsymbol{\delta}) - \bar{f}) \right)^2$$
$$\leqslant h(A_0) \sum_{\mathbf{j}} h(A_0, \mathbf{j})(f(\mathbf{j}, \boldsymbol{\delta}) - \bar{f})^2$$
$$\leqslant h(A_0)\, m^r \sum_{\mathbf{j}, \boldsymbol{\varepsilon}} (f(\mathbf{j}, \boldsymbol{\varepsilon}) - \bar{f})^2$$
$$= o\left(h(A_0)\, m^{r+2} \binom{s}{b} 2^{-b} \right)$$
$$= o\left(m^{2r+2} \binom{s}{b}^2 2^{-2b} \right).$$

Taking square roots and using $\bar{f} = m2^{-b}$, we obtain

$$h(A) - mh(A_0)\, 2^{-b} = o\left(m^{r+1} \binom{s}{b} 2^{-b} \right).$$

Using the induction hypothesis, i.e., $h(A_0) = (1 + o(1))\, m^r \binom{s}{b} 2^{-rb}$, $h(A) = (1 + o(1))\, m^{r+1} \binom{s}{b}\, 2^{-(r+1)n}$ follows. $\blacksquare$

4. A Lemma for l-Partite Graphs

Recall that a graph is called l-partite if its vertex set is partitioned into l classes and all edges join distinct classes. By the notation $G = (\bigcup_{i=1}^{l} A_i, E)$ we mean an l-partite graph with edge set E and with vertex set partitioned into $A_1 \cup \cdots \cup A_l$.

Definition 4.1. Let n be a positive integer and $G = (\bigcup_{i=1}^{l} A_i, E)$ an l-partite graph. Then G is called n-uniform if for every $1 \leqslant i < j \leqslant l$ and all $B_i \subseteq A_i$, $B_j \subseteq A_j$ we have

$$\left| e(B_i, B_j) - \frac{|B_i|\,|B_j|}{2} \right| \leqslant \frac{\sqrt{n|B_i|\,|B_j|}}{2}.$$

Definition 4.2. Let $H = (W, F, <)$ be a graph with edge set F, with l vertices and with vertex set $W = \{w_1, ..., w_l\}$ ordered by $w_1 < \cdots < w_l$. Let $G = (\bigcup_{i=1}^{l} A_i, E)$ be an l-partite graph. We say that H is *partite isomorphic* to an induced subgraph H' of G if $H' = (W', E')$ $W' = \{w'_1, ..., w'_l\}$, $w'_i \in A_i$, $i = 1, ..., l$, and the mapping $w_i \mapsto w'_i$ is an isomorphism.

Let $p(H, G)$ denote the number of induced subgraphs of G which are partite isomorphic to H.

Lemma 4.3. *Let $\varepsilon > 0$ and let n, l, m be positive integers satisfying*

$$nl^4 2^{2(l-1)} \leqslant m^{2-3\varepsilon}. \tag{4.1}$$

Let $G = (\bigcup_{i=1}^{l} A_i, E)$ be an n-uniform l-partite graph with

$$|A_1| = \cdots = |A_l| = m.$$

Let $H = (W, F, <)$ be a graph with l vertices. Then we have

$$\left| p(H, G) - \frac{m^l}{2^{\binom{l}{2}}} \right| \leqslant l \left(\frac{2^l}{m}\right)^{\varepsilon} \frac{m^l}{2^{\binom{l}{2}}}.$$

Remark. Note that (4.1) implies the much weaker inequality $(mnl)^{1/3} < m^{1-\varepsilon}/l$, which we shall often use.

Proof. We prove the statement by induction on l. For $l = 2$ the fact that G is n-uniform and (4.1) yield

$$\left| e(A_0, A_1) - \frac{m^2}{2} \right| \leqslant \frac{\sqrt{n}}{2} m \leqslant \frac{m^{2-3\varepsilon/2}}{2}.$$

As $p(H, G) = e(A_0, A_1)$ for H being an edge (for H being a pair of isolated vertices the argument is clearly the same) the statement is proved.

Supposing the statement is true for l, we prove it for $l+1 \geqslant 3$.

Let $G=(\bigcup_{i=0}^{l} A_i, E)$ be an $(l+1)$-partite, n-uniform graph which satisfies (4.1) with l replaced by $l+1$. Then, by n-uniformity, the number $s_x(i)$, $(\bar{s}_x(i))$ of vertices of A_0 that are joined to less than $\frac{1}{2}m - \frac{1}{2}m^{1/2}x$ (more than $\frac{1}{2}m + \frac{1}{2}m^{1/2}x$) vertices of A_i satisfies

$$\frac{xm^{1/2}s_x(i)}{2} \leqslant \frac{\sqrt{nms_x(i)}}{2}$$

and thus

$$s_x(i) \leqslant \frac{n}{x^2}, \left(\bar{s}_x(i) \leqslant \frac{n}{x^2}\right),$$

respectively. Hence, the number s_x of points for which there exists an i with the first (or second) property is bounded by

$$\begin{aligned} s_x &\leqslant \sum_{i=1}^{l} s_x(i) \leqslant \frac{ln}{x^2}, \\ \bar{s}_x &\leqslant \sum_{i=1}^{l} \bar{s}_x(i) \leqslant \frac{ln}{x^2}, \end{aligned} \tag{4.2}$$

respectively. Setting $x_0 = (ln)^{1/3}/m^{1/6}$, this yields that there are at least $m - 2(mnl)^{1/3}$ vertices v of A_0 which, for every $i = 1, 2, ..., l$, are joined to at least $\frac{1}{2}m - \frac{1}{2}(mnl)^{1/3}$ but not more than $\frac{1}{2}m + \frac{1}{2}(mnl)^{1/3}$ vertices of A_i. Let v be any of these vertices and let I be the set of indices i such that

$$\{w_0, w_i\} \in F = E(H).$$

Let H_0 be the induced subgraph of H on the vertex set $\{w_1, ..., w_l\}$. For $i = 1, 2, ..., l$ let B_i^v be the set of points of A_i which are (which are not) joined to v, depending on whether $i \in I$ or $(i \notin I)$, respectively.

Using (4.1) we infer

$$|B_i^v| \geqslant \frac{1}{2}m - \frac{1}{2}(mnl)^{1/3} > \frac{m}{2}\left(1 - \frac{1}{(l+1)\, m^{\varepsilon}}\right).$$

Using (4.1) again we obtain

$$\begin{aligned} l^4 2^{2(l-1)} n &\leqslant \frac{m^{2-3\varepsilon}}{4} \frac{l^4}{(1+l)^4} \\ &\leqslant \frac{1}{m^{3\varepsilon}} \left(\frac{m - (mnl)^{1/3}}{2}\right)^2 \\ &\leqslant \left(\frac{m - (mnl)^{1/3}}{2}\right)^{2-3\varepsilon}. \end{aligned}$$

Thus the induction assumption can be applied to the l-partite graph G_0 with vertex set $\bigcup_{i=1}^{l} D_i^v$, where $D_i^v \subseteq B_i^v$, $|D_i^v| = \lceil m/2(1-1/m^\varepsilon)\rceil$.

Using the inequality

$$\frac{l}{m-(mnl)^{1/3}} \geqslant \frac{l}{m}\frac{1}{1-(1/2(l+1))} = \frac{2l(l+1)}{(2l+1)\,m}$$

we infer

$$\begin{aligned} p(H,G) &\geqslant (m-2(mnl)^{1/3})\,p(H_0,G_0)\\ &\geqslant m\left(1-\frac{1}{lm^\varepsilon}\right)\frac{1}{2^l}\frac{(m(1-(1/2lm^\varepsilon)))^l}{2^{\binom{l}{2}}}\left(1-l\left(\frac{2^{l+1}}{m-(mnl)^{1/3}}\right)^\varepsilon\right)\\ &\geqslant \frac{m^{l+1}}{2^{\binom{l+1}{2}}}\left(1-\frac{1}{m^\varepsilon}\right)\left(1-\frac{2l(l+1)}{2l+1}\left(\frac{2^{l+1}}{m}\right)^\varepsilon\right)\\ &> \frac{m^{l+1}}{2^{\binom{l+1}{2}}}\left(1-(l+1)\left(\frac{2^{l+1}}{m}\right)^\varepsilon\right). \end{aligned}$$

Now we derive the upper bound. Replace the bipartite graph between A_i and A_j by its complement whenever (w_i, w_j) is not an edge of H. Then $p(H,G)$ is simply the number of K_{l+1} in the new graph. Thus we may assume that $H = K_{l+1}$ holds.

First, we have at most m points $v \in A_0$ which are joined to at most $(m+m^{1-\varepsilon})/2$ points of A_i for every $i = 1, 2, \ldots, l$ and thus, according to the induction assumption, these vertices are contained altogether in at most

$$\begin{aligned} &m\,\frac{((m+m^{1-\varepsilon})/2)^l}{2^{\binom{l}{2}}}\left(1+l\left(\frac{2^{l+1}}{m+m^{1-\varepsilon}}\right)^\varepsilon\right)\\ &\qquad\leqslant \frac{m^{l+1}}{2^{\binom{l+1}{2}}}\left(1+\frac{1}{m^\varepsilon}\right)^l\left(1+l\left(\frac{2^{l+1}}{m}\right)^\varepsilon\right) \end{aligned} \tag{4.3}$$

copies of K_{l+1}.

Let V_x be the set of vertices v of A_0 for which there exists an i, $1 \leqslant i \leqslant l$, such that v is joined to more than $\frac{1}{2}(m + xm^{1/2})$ vertices of A_i. Using the induction assumption we see that the number N_x of K_{l+1} with one vertex in $V_x - V_{x+1}$ is at most

$$\begin{aligned} &|V_x - V_{x+1}|\left(\frac{m+(x+1)\,m^{1/2}}{2}\right)^l 2^{-\binom{l}{2}}\left(1+l\left(\frac{2^{l+1}}{m+xm^{1/2}}\right)^\varepsilon\right)\\ &\qquad< |V_x - V_{x+1}|\,\frac{m^l}{2^{\binom{l+1}{2}}}\left(1+\frac{x+1}{m^{1/2}}\right)^l\left(1+l\left(\frac{2^{l+1}}{m}\right)^\varepsilon\right). \end{aligned}$$

Recalling $|V_x| = \bar{s}_x$ and using Abel's summation, we obtain

$$\sum_{x=m^{1/2-\varepsilon}}^{m^{1/2}} N_x \leqslant \frac{m^l}{2^{\binom{l+1}{2}}}\left(1 + l\left(\frac{2^{l+1}}{m}\right)^{\varepsilon}\right)$$

$$\times\left(\bar{s}_{m^{1/2-\varepsilon}}(1 + m^{-\varepsilon} + m^{-1/2})^l\right.$$

$$\left. + \sum_{x=m^{1/2-\varepsilon}+1}^{m^{1/2}} \bar{s}_x\left(\left(1 + \frac{x+1}{m^{1/2}}\right)^l - \left(1 + \frac{x}{m^{1/2}}\right)^l\right)\right).$$

Recalling (4.2), i.e., $\bar{s}_x \leqslant ln/x^2$ we obtain

$$\bar{s}_{m^{1/2-\varepsilon}}(1 + m^{-\varepsilon} + m^{-1/2})^l$$

$$< \frac{nl}{m^{1-2\varepsilon}}(1 + m^{-\varepsilon} + m^{-1/2})^l$$

$$< m^{1-\varepsilon}\left(\frac{1 + m^{-\varepsilon} + m^{-1/2}}{4}\right)^l < m^{1-\varepsilon}.$$

Similarly, for the last summation using $\sum_{x=a+1}^{\infty} 1/x^2 < 1/a$ and $u^l - v^l < (u-v)\, lu^l$ we obtain

$$\sum_{x=m^{1/2-\varepsilon}+1}^{m^{1/2}} \bar{s}_x\left(\left(1 + \frac{x+1}{m^{1/2}}\right)^l - \left(1 + \frac{x}{m^{1/2}}\right)^l\right)$$

$$< \sum_x \frac{ln}{x^2}\frac{l}{m^{1/2}}2^l < \frac{l^2 n 2^l}{m^{1-\varepsilon}} < m^{1-2\varepsilon}.$$

Thus we proved

$$\sum_{x=m^{1/2-\varepsilon}}^{m^{1/2}} N_x \leqslant \frac{m^{l+1}}{2^{\binom{l+1}{2}}}\left(1 + l\left(\frac{2^{l+1}}{m}\right)^{\varepsilon}\right)\frac{2}{m^{\varepsilon}}.$$

Combining this with (4.3) the desired upper bound follows. ∎

5. Proof of Theorem 1.8

For computational convenience we suppose that l divides n and set $m = n/l$. For every partition π of the vertex set into l equal parts $A_1, ..., A_l$ let G be the corresponding l-partite subgraph of G. Choosing $\varepsilon = \delta/4$, $n > n_0(l, \delta)$, G_π satisfies the conditions of Lemma 4.3. Thus we infer for any fixed ordering of the vertices of H

$$p(H, G_\pi) = m^l 2^{-\binom{l}{2}}(1 + o(1)).$$

Running through all $l!$ orderings gives an extra factor of $l!/|\text{Aut } H|$. Since there are $l!\, n!/(m!)^l$ ordered partitions and every fixed copy of H occurs (as an l-partite subgraph) in $l!(n-l)!/(m-1)!^l$ partitions, we infer

$$h(H, G) = (1+o(1))\, m^l 2^{-\binom{l}{2}} \frac{l!\, n!}{m!^l} \frac{l!}{\text{Aut } H} \Big/ \frac{(m-1)!^l}{l!(n-l)!}$$

$$= (1+o(1)) \binom{n}{l} 2^{-\binom{l}{2}} \frac{l!}{|\text{Aut } H|}. \quad \blacksquare$$

6. Concluding Remarks

The condition $n = o(st)$ is clearly necessary in Theorem 1.5. Without this n-uniformity would not even imply that the number of plus and minus ones is asymptotically the same in the matrix. Besides Hadamard matrices (and their submatrices) random matrices provide a natural example for n-uniformity.

Speaking about random matrices one is led to consider the non-symmetric case, i.e., when the probability of each entry to be plus one is p, $p \neq \frac{1}{2}$. One can define that a matrix N is (p, n)-uniform if for every t by s submatrix M of it $|p(M) - pst| \leqslant \sqrt{p(1-p)\, nst}$ holds.

Theorem 1.5 can be extended to this case also: 2^{-ab} should be replaced by $p^{p(A)}(1-p)^{ab-p(A)}$.

An interesting example of n-uniform graphs and tournaments is provided by the Paley graphs and tournaments.

Let q be an odd prime power and let the vertex set of G be $GF(q)$ with i and j joined by an edge (arc from i to j) if $i-j$ is a quadratic residue and $q \equiv 1 \pmod 4$, $q \equiv -1 \pmod 4$, respectively. The fact that Paley tournaments contain all small tournaments as subtournaments was proved by Graham and Spencer [4]. This was extended—using the same method—to Paley graphs by Bollobas and Thomason [2]. Actually the proof can be adopted to yield the statement corresponding to Theorem 1.8 for the Paley tournaments and Paley graphs.

It is worth noting that the Paley graphs have stronger pseudo-random properties than those given in Theorem 1.8. For example, if G is a Paley graph, then for every fixed induced subgraph F of G and any not too large graph H, the number of induced subgraphs of G isomorphic to H that contain F is asymptotically the same as that number for a random graph of the same size as G. This follows easily from the results of [4]. Graphs arising from general Hadamard matrices do not have this property.

Finally, let us mention that in [5] it is proved that graphs satisfying much weaker uniformity conditions contain all small graphs as induced subgraphs.

REFERENCES

1. N. ALON, Geometric expanders, eigenvalues and sorting in rounds, *Combinatorica* **6** (1986).
2. B. BOLLOBAS AND A. THOMASON, Graphs which contain all small graphs, *European J. Combin.* **2** (1981), 13–15.
3. P. ERDÖS AND J. SPENCER, "Probabilistic Methods in Combinatorics," Academic Press, Budapest, 1974.
4. R. L. GRAHAM AND J. H. SPENCER, A constructive solution to a tournament problem, *Canad. Math. Bull.* **14** (1971), 45–48.
5. V. RÖDL, On universality of graphs with uniformly distributed edges, *Discrete Math.* **59** (1986), 125–134.