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Quantitative Theorems for Regular Systems of Equations

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INTRODUCTION

An important topic in Ramsey theory deals with solution sets of (systems of) homogeneous linear equations. Pioneered by the early work of Schur [19] and van der Waerden [21], the subject received a major thrust with the fundamental results of Rado [17, 9] and, more recently, Deuber [4]. Essentially, these results guarantee for certain systems L, the existence of a function $N_L: \mathbb{Z}^+ \to \mathbb{Z}^+$, so that for any integer $r > 0$ and any partition of $[N_L(r)] := \{1, 2, ..., N_L(r)\} = C_1 \cup \cdots \cup C_r$ into r classes, some class C_i must contain a solution set for L. These systems are said to be *partition regular*. Often, the classes are called *colors*, the partition an *r-coloring*, and the corresponding solution sets *monochromatic*.

In this paper, we investigate how the number of monochromatic solution sets of L grows for r-colorings of $[N]$ as $N \to \infty$. It will turn out (Theorem 1) that for every partition regular system $L = L(x_1, ..., x_n)$, if $v_L(N)$ denotes the number of n-tuples $(x'_1, ..., x'_n)$ which satisfy L, where $1 \leqslant x'_i \leqslant N$ for all i, then there exists for each r, an absolute constant $c_r(L)$ so that for any r-coloring of $[N]$ there are always at least $c_r(L)\, v_L(N)$ monochromatic solution sets to L. In other words, in any r-coloring of

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$[N]$, the number of *monochromatic* solution sets is a positive fraction of the *total* number of solution sets.

We also prove analogous results (Theorem 2) for systems of equations which always have solutions in any set $X \subseteq \mathbb{Z}^+$ with positive upper density. Such systems will be said to be *density regular*; an example of such a system is

$$x_1 - x_2 = x_2 - x_3 = \cdots = x_{k-1} - x_k \tag{$*$}$$

The solution sets to (*) (with distinct x_i) are just the k-term arithmetic progressions. The fact that (*) is density regular is exactly Szemerédi's celebrated theorem [20]. Of course, in general, if L is density regular then it is partition regular.

We will conclude the paper by discussing a number of related results and open problems.

Three Equations

Before presenting our main results, we first discuss three homogeneous linear equations which will be useful in illustrating the concepts we will need later:

$$x + y = z, \tag{1}$$

$$x + y = 2z, \tag{2}$$

$$x + y = 3z. \tag{3}$$

Although superficially similar, these equations exhibit the three different types of behavior we will focus on in this paper.

To begin with, Eq. (3) is not partition regular. To see this, consider the following 4-coloring χ of $\mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, write $n = 5^{a_n}(5k_n + b_n)$, where $a_n \geqslant 0$ and $b_n = 1, 2, 3$, or 4. Define $\chi(n) = b_n$. It is easy to check that (3) has no monochromatic solution under the coloring χ.

Next, we consider (2). Any solution (x, y, z) to (2) with $x \neq y$ forms a 3-term arithmetic progression. The classic theorem of van der Waerden [21] shows that for all k and r, there exists a number $W(k, r)$ so that for any r-coloring of $[W(k, r)]$ there is a monochromatic k-term arithmetic progression. Let $W := W(3, r)$ and assume that $[N]$ is r-colored. Consider the set of W-term arithmetic progressions $AP(a, d) = \{a + dx \colon 0 \leqslant x < W\}$, where $1 \leqslant a < N/2$ and $N/4W < d < N/2W$. Clearly, each such $AP(a, d)$ is contained in $[N]$ and, by the choice of W, must contain some monochromatic 3-term arithmetic progression $P_{a,d}$. However, each such $P_{a,d}$ can occur in at most $\binom{W}{2}$ different arithmetic progressions $AP(a', d')$, since the first term of $P_{a,d}$ might be the ith term of $AP(a', d')$ and the last term of $P_{a,d}$ might be the jth term of $AP(a', d')$, and there are at most $\binom{W}{2}$

possible choices for $i<j$. Thus, since there are essentially $N^2/8W$ $AP(a, d)$'s then $[N]$ must contain at least $N^2/4W^3$ monochromatic 3-term arithmetic progressions. Therefore, if $v_{(2)}(N)$ denotes the minimum possible number of monochromatic solutions to Eq. (2) in any r-coloring of $[N]$, then we have shown:

FACT 1.

$$v_{(2)}(N) > c_r N^2 \tag{4}$$

for an absolute positive constant c_r (depending only on r).

Observe that this is to within a constant factor the most we could hope for, since there are only $c'N^2$ 3-term arithmetic progressions altogether in $[N]$.

Finally, we treat Eq. (1), which is the most difficult of the three. One reason for this appears to be that while (2) is *density* regular, (1) is only *partition* regular and not density regular. It turns out that the analog to (4) also holds here. Namely, if $v_{(1)}(N)$ denotes the minimum possible number of monochromatic solutions to Eq. (1) in any r-coloring of $[N]$, then we have:

FACT 2.

$$v_{(1)}(N) > c'_r N^2 \tag{5}$$

for an absolute positive constant c'_r (depending on r).

Proof. To begin, it is known (cf. [9]) that for each r, there is a number $S = S(r)$ so that in any r-coloring of the set $2^{[S]}$ of subsets of $[S]$, one can always find two nonempty disjoint subsets $I, J \subseteq [S]$ such that I, J and $I \cup J$ all have the same color.

Next, for $1 \leqslant i \leqslant S$, choose a_i with $1 \leqslant a_i \leqslant N/S$ so that

$$a_i \equiv 2^{i-1} (\text{mod } 2^S). \tag{6}$$

Note that the 2^S sums $\sum_{i \in I} a_i$, $I \subseteq [S]$, are all distinct modulo 2^S, and therefore, distinct. Thus, since each integer $\sum_{i \in I} a_i$, $I \subseteq [S]$, is in $[S]$ and so, has been assigned one of the r colors, we can assign the same color to the corresponding *subset* $I \subseteq [S]$. By the definition of $S = S(r)$, we can find in this r-coloring of $2^{[S]}$, disjoint nonempty subsets I_0, $J_0 \subseteq [S]$ so that I_0, J_0 and $I_0 \cup J_0$ all have the same color.

Now, since there are essentially $N/S \cdot 2^S$ ways of choosing a_i, then there are altogether

$$\prod_{i=1}^{S} (N/S \cdot 2^S) = \frac{N^S}{S^S \cdot 2^{S^2}}$$

ways of choosing all the a_i's, $1 \leqslant i \leqslant S$. On the other hand, consider some solution to (1), say $a+b=c$. We claim that there are at most $c_1 N^{S-2}$ choices for $\bar{a} = (a_1, ..., a_S)$ satisfying the required conditions. In view of (6), if

$$a = \sum_{i \in I_0(\bar{a})} a_i, \qquad b = \sum_{j \in J_0(\bar{a})} a_j$$

then the sets $I_0(\bar{a})$ and $J_0(\bar{a})$ are uniquely determined. This gives two equations for $a_1, ..., a_S$. Thus, we lose two degrees of freedom in choosing $\bar{a}$, so that we only have $c_1 N^{S-2}$ choices instead of $c_2 N^S$. This implies that there must therefore be at least $c'_r N^2$ different monochromatic solutions to (1), and (5) is proved. ∎

As in Fact 1, (5) is to within a constant factor best possible.

In the next two sections, we will prove the corresponding extensions of (5) and (4) for (partition and density, respectively) regular systems of homogeneous linear equations over $\mathbb{Z}$.

Partition Regular Systems.

We begin by recalling several relevant facts concerning partition regular systems (see also [4, 10]).

For an l by k matrix $A = (a_{ij})$ of integers, denote by $L = L(A)$ the system of homogeneous linear equations

$$\sum_{j=1}^{k} a_{ij} x_j = 0, \qquad 1 \leqslant i \leqslant l. \tag{7}$$

We can abbreviate this by writing

$$A\bar{x} = \bar{0}, \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = (x_1, ..., x_k)^t.$$

We say that L is *partition regular* if for any r-coloring of $\mathbb{Z}^+$, there is always a solution to (7) with all x_i having the same color. The matrix A is said to satisfy the *columns condition* if it is possible to re-order the column vectors $\bar{a}_1, \bar{a}_2, ..., \bar{a}_k$ so that for some choice of indices $1 \leqslant k_1 < k_2 < \cdots < k_t = k$, if we set

$$A_i := \sum_{j=k_{i-1}+1}^{k_i} \bar{a}_j$$

then

(i) $A_1 = 0$,

(ii) For $1 < i \leqslant t$, A_i can be expressed as a *rational* linear combination of $\bar{a}_j$, $1 \leqslant j \leqslant k_{i-1}$.

A classical result of Rado asserts the following.

THEOREM [17, 10]. *The system $A\bar{x} = \bar{0}$ is partition regular if and only if A satisfies the columns condition.*

Let us call a set $X \subseteq \mathbb{Z}^+$ *large* if for any partition regular system $A\bar{x} = 0$ and any finite coloring of X, there is always a monochromatic solution to $A\bar{x} = 0$. It was shown by Deuber [4] (settling a conjecture of Rado) that large sets have the following partition property: If X is large and $X = X_1 \cup \cdots \cup X_r$ then for some i, X_i is large. We next introduce some notation due to Deuber [4].

DEFINITION.

$N_{m,p,c} := \{(\lambda_1, ..., \lambda_m)$: for some $i < m$, $\lambda_j = 0$ for $j < i$, $\lambda_i = c > 0$ and $|\lambda_k| \leqslant p$ for $k > i\}$.

A set $\mathscr{S} \subseteq \mathbb{Z}^+$ is called an $(m, p, c)-$*set* if

$$\mathscr{S} = \left\{ \sum_{i=1}^{m} \lambda_i y_i : (\lambda_1, \lambda_2, ..., \lambda_m) \in N_{m,p,c} \right\}$$

for some choice of $y_1, y_2, ..., y_m > 0$.

As shown by Deuber, sets of solutions for partition regular systems $A\bar{x} = 0$ correspond to subsets of (m, p, c)-sets in the following way.

Remark 1. Let A be an l by k matrix satisfying the columns condition, and let $A_1, A_2, ..., A_t$ be the column vector sums coming from the definition of the columns condition. We can assume without loss of generality that A has rank l. Then there exist $k - l$ linearly independent solutions to $A\bar{x} = \bar{0}$ which (by the columns condition) have the following form[1]:

	k_1	$k_2 - k_1$	$k_t - k_{t-1}$
$\bar{w}_1$	$= (1, 1, ..., 1,$	$0, 0, ..., 0$,..., ...	$0, 0, ..., 0)'$
$\bar{w}_2$	$= (\alpha_{21}, ..., \alpha_{2k_1},$	$1, 1, ..., 1$, ..., ...	$0, 0, ..., 0)'$
$\vdots$			
$\bar{w}_t$	$= (\alpha_{t1}$	..............., ...	$\alpha_{tk_{t-1}}, 1, 1, ..., 1)'$
$\bar{w}_{t+1}$	$= (\alpha_{t+1,1}$	..............., ...	$\alpha_{t+1,k})'$
$\vdots$			
$\bar{w}_{k-l}$	$= (\alpha_{k-l,1}$	..............., ...	$\alpha_{k-l,k})'$

[1] $\bar{x}'$ denotes the transpose of $\bar{x}$; we will occasionally omit this if it is clear from context.

where all the α_{ij} are rational. Multiplying all the entries by a sufficiently large integer c, we obtain linearly independent vectors of the following form:

$$
\begin{aligned}
\bar{v}_1 &= (c, c, ..., \quad c, \quad 0, ..., \quad 0, ..., \quad 0, ..., 0)' \\
\bar{v}_2 &= (\beta_{21}, ..., \quad \beta_{2,k_1}, c, ..., \quad c, 0, ..., ... 0)' \\
&\vdots \\
\bar{v}_{t+1} &= (\beta_{t+1,1}, ..., ..., ..., ... \beta_{t+1,k})' \\
&\vdots \\
\bar{v}_{k-l} &= (\beta_{k-l,1}, ..., ..., ..., ... \beta_{k-l,k})',
\end{aligned}
\tag{8}
$$

where all entries are integers. Set $p = |\max \beta_{ij}|$. Since every solution to $A\bar{x} = \bar{0}$ can be expressed as a linear combination of the vectors $\bar{v}_1, \bar{v}_2, ..., \bar{v}_{k-l}$, say, $\bar{x} = \sum_{i=1}^{k-l} y_i \bar{v}_i$, then in fact each solution of $A\bar{x} = \bar{0}$ is always a subset of some $(k-l, p, c)$-set, and conversely, as claimed.

We are now ready to give the following quantitative version of Rado's theorem.

THEOREM 1. *Let A be an l by k matrix of rank l which satisfies the columns condition. Then for any r there exists $c_r(A) > 0$ such that in any r-coloring of $[N]$ there are at least $c_r(A)\, N^{k-l}$ monochromatic solutions to the partition regular system $A\bar{x} = \bar{0}$.*

If we let $v_L(N, r)$ denote the minimum possible number of monochromatic solution sets to a system L whenever $[N]$ is r-colored (so that $v_L(N) = v_L(N, 1)$), then we have as an immediate consequence:

COROLLARY 1. *If L is partition regular then for any r there exists $c_r(L) > 0$ so that*

$$v_L(N, r) \geqslant c_r(L)\, v_L(N).$$

Proof of Theorem 1. The proof will use the following version of Deuber's theorem.

THEOREM [5]. *For every choice of m, p, c, and r there exist M, P, and C such that for any r-partition of*

$$J = \left\{ \sum_{i=1}^{M} \lambda_i Y_i : (\lambda_1, ..., \lambda_M) \in N_{M,P,C} \right\},$$

say

$$J = J_1 \cup J_2 \cup \cdots \cup J_r,$$

there exist pairwise disjoint sets $B_1, B_2, \ldots, B_m \subseteq [M]$, *and*

$$y_i = \sum_{j \in B_i} \xi_j Y_j, \qquad 1 \leqslant |\xi_j| \leqslant P,\ 1 \leqslant i \leqslant m,$$

such that all linear combinations

$$\sum_{i=1}^{m} \lambda_i y_i, \qquad (\lambda_1, \lambda_2, \ldots, \lambda_m) \in N_{m,p,c},$$

belong to a single class J_k *for some* k.

Now, given our l by k matrix A of rank l satisfying the columns condition, we know by Remark 1 that the entries of the set of solution vectors of $A\bar{x} = \bar{0}$ all belong to some $(k-l, p, c)$-set. Set $m = k - l$ and let M, P, and C be the integers from Deuber's theorem. Choose $N > M$ to be very large. Consider all the M-tuples $(Y_1, Y_2, \ldots, Y_M)$ of integers Y_i satisfying

$$0 < Y_i \leqslant \frac{N}{MC} \qquad \text{and} \qquad Y_i \equiv (2P+1)^i \bmod (2P+1)^M \tag{9}$$

for $1 \leqslant i \leqslant M$. There are at least $c_1 N^M$ such M-tuples for some constant $c_1 > 0$ not depending on N. For such an M-tuple $(Y_1, Y_2, \ldots, Y_M)$, consider the (M, P, C)-set

$$J(Y_1, \ldots, Y_M) = \left\{ \sum_{i=1}^{M} \lambda_i Y_i : (\lambda_1, \ldots, \lambda_M) \in N_{M,P,C} \right\}.$$

Let $[N] = C_1 \cup \cdots \cup C_r$ be an r-coloring of $[N]$. By Deuber's theorem we can find disjoint subsets $B_1, \ldots, B_{k-l} \subseteq [M]$ and $y_i = \sum_{j \in B_i} \xi_j Y_j$, $1 \leqslant |\xi_j| \leqslant P$, so that all the linear combinations $\sum_{i=1}^{m} \lambda_i y_i$, $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in N_{m,p,c}$, have the same color. In particular, $\bar{x} = \sum_{i=1}^{k-l} y_i \bar{v}_i$ (from (8)) is a monochromatic solution to the system $A\bar{x} = \bar{0}$. This therefore gives, with multiplicity, at least $c_1 N^M$ monochromatic solutions (one for each choice of $(Y_1, \ldots, Y_M)$). Our proof will be complete if we can show that each of these solutions can occur at most $N^{M-(k-l)}$ times.

To see this, suppose $(x_1, \ldots, x_k)$ is some solution obtained above, i.e., for some choice of $(y_1, \ldots, y_{k-l})$, the x_i are *fixed* linear combinations of the y_i. Then, we must show that the same monochromatic (m, p, c)-set is obtained at most $N^{M-(k-l)}$ times. However, given y_i, its residue modulo $(2P+1)^M$ uniquely determines the λ_j, $1 \leqslant j \leqslant M$, from (9). Thus, the possible $Y_1, \ldots, Y_M$ must satisfy $k - l$ linear equations, which involve pairwise disjoint sets of unknowns among them. This gives the required bound and the proof is complete. ∎

Density Regular Systems

Suppose $X \subseteq \mathbb{Z}^+$ is a set having positive upper density, i.e., so that

$$\limsup_{N \to \infty} \frac{|X \cap [N]|}{N} > 0.$$

The system

$$A\bar{x} = \bar{0} \tag{10}$$

is said to be *density regular* if for any set X of positive upper density there is a vector $\bar{x}$ satisfying (10) and having all entries belonging to X.

If it happens that (10) has the vector $\bar{x} = \bar{1} = (1, 1, ..., 1)$ as a solution then, of course, for any $k \in \mathbb{Z}^+$, $\bar{x} = k \cdot \bar{1} = (k, k, ..., k)$ is also a solution. In this case, (10) is trivially density regular. However, the solution $k \cdot \bar{1}$ is normally not considered to be very interesting. For example, for the density regular system

$$x_1 - 2x_2 + x_3 = 0,$$

the solutions (x_1, x_2, x_3) are just the 3-term arithmetic progressions, provided the x_i are distinct.

With these considerations in mind, let us call the system (10) *irredundant*, if (10) does not imply that $x_i = x_j$ for $i \neq j$. Also, let us call a solution $\bar{x} = (x_1, ..., x_k)$ to (10) *proper* if all the x_i are distinct.

FACT 3. *If $A\bar{x} = \bar{0}$ is irredundant then it has a proper solution.*

Proof. For each choice of $i < j$, let $\bar{x}^{(ij)} = (x_1^{(ij)}, x_2^{(ij)}, ..., x_k^{(ij)})$ be a solution to (10) with $x_i^{(ij)} \neq x_j^{(ij)}$, which exists by hypothesis. Thus, for any integer N, $\bar{x}^* = (x_1^*, x_2^*, ..., x_k^*)$ with

$$x_t^* = \sum_{i<j} N^{ki+j} x_t^{(ij)}$$

is also a solution to (10) by linearity. However, if $N > \max_{i,j,t}(x_t^{(ij)})$ then all x_t^* are distinct. ∎

FACT 4. *An irredundant system $A\bar{x} = \bar{0}$ has a proper solution in every set X of positive upper density if and only if $A \cdot \bar{1} = 0$.*

Proof. First, since X has positive upper density then by Szemerédi's theorem [20, 8], X contains arbitrarily long arithmetic progressions. Suppose $\bar{x}_0 = (b_1, b_2, ..., b_k)$ is a proper solution of $A\bar{x}_0 = \bar{0}$, i.e., all the b_k are distinct. Let $B := \max_k b_k$ and let $P = \{c + \lambda d \colon \lambda \in [B]\}$ be a B-term arithmetic progression in X. If $\bar{1}$ also satisfies $A \cdot \bar{1} = \bar{0}$ then so does the linear combination

$$\bar{x}^* = c \cdot \bar{1} + d\bar{x}_0 = (c + b_1 d, c + b_2 d, ..., c + b_n d),$$

which is proper, and furthermore, has all entries in $P \subseteq X$, as desired.

In the other direction, suppose $A\bar{x} = \bar{0}$ has a proper solution in every set of positive upper density. Let $N > \sum_{i,j} |a_{ij}|$, where a_{ij} ranges over all entries of A. Consider the set $Y = \{Ny + 1: y \in \mathbb{Z}^+\}$ with (upper) density $1/N$. Suppose $\bar{x} = (x_1, ..., x_n)$ satisfies $A\bar{x} = \bar{0}$, where each $x_k = Ny_k + 1 \in Y$. Thus,

$$0 = \sum_j a_{ij} x_j = \sum_j a_{ij}(Ny_j + 1) = N \sum_j a_{ij} y_j + \sum_j a_{ij}$$

for $1 \leqslant i \leqslant m$. By the choice of N, this implies that $\sum_j a_{ij} = 0$ for all i. This is exactly the statement that $A\bar{1} = \bar{0}$, as required. This completes the proof. ∎

THEOREM 2. *Let A be an l by k matrix of rank l so that $A\bar{x} = 0$ is irredundant and $A\bar{1} = \bar{0}$. Then for any $\varepsilon > 0$ there is a constant $c_\varepsilon = c_\varepsilon(A) > 0$ so that if $N > N_0(A, \varepsilon)$ and $X \subseteq [N]$ with $|X| > \varepsilon N$ then X must contain at least $c_\varepsilon N^{k-l}$ proper solutions $\bar{x}$ to $A\bar{x} = \bar{0}$.*

Proof. Let $\varepsilon > 0$ be arbitrary (but fixed) and let $X \subseteq [N]$ with $|X| > \varepsilon N$ be given, where it will be useful to think of N as being very large. Since A has rank l, the space of all (rational) solutions $\bar{x}$ to $A\bar{x} = \bar{0}$ has dimension $k - l$. Let $\bar{v}_0 = \bar{1}, \bar{v}_1, ..., \bar{v}_m$ be linearly independent integer solutions to $A\bar{x} = \bar{0}$, where $m := k - l - 1$ and for $\bar{v}_i = (v_{i1}, ..., v_{ik})^t$, we can assume without loss of generality, all $v_{ij} \geqslant 0$ (since if not, then we can repeatedly add $\bar{1}$ to $\bar{v}_i$ until this is true). Define $t := 1 + \max_{ij} v_{ij}$.

For $u \in \mathbb{Z}^+$ and each vector $\bar{y} = (y_1, ..., y_m)$ with $y_i \in \mathbb{Z}^+$, define the *m-box* $B_u(\bar{y})$ to be the set

$$\{(a_1 y_1, a_2 y_2, ..., a_m y_m): 0 \leqslant a_i < u, 1 \leqslant i \leqslant m\}.$$

Further, define the projection $\pi: B_u(\bar{y}) \to \mathbb{Z}$ by

$$\pi((a_1 y_1, ..., a_m y_m)) = \sum_{i=1}^{m} a_i y_i.$$

By the theorem of Furstenberg and Katznelson [7, 8], there is an integer T so that for any $\bar{Y} = (Y_1, ..., Y_m)$ with $Y_i \in \mathbb{Z}^+$, if $X^* \subseteq B_T(\bar{Y})$ with $|X^*| > (\varepsilon/2) |B_T(\bar{Y})| = (\varepsilon/2) T^m$ then there exists a "translated" m-box $\bar{A} + B_t(A_0 \bar{Y}) \subseteq X^*$, where $\bar{A} = (A_1 Y_1, ..., A_m Y_m)$ and $A_0, A_1, ..., A_m \in \mathbb{Z}^+$.

Now, consider the set of all integer vectors $\bar{Y} = (Y_1, Y_2, ..., Y_m)$ which satisfy the constraints:

(i) $0 \leqslant Y_i < \varepsilon^2 N/mT$, $1 \leqslant i \leqslant m$;

(ii) $Y_i \equiv T^{i-1} \pmod{T^m}$, $1 \leqslant i \leqslant m$.

Note that if $\bar{P} = (a_1 Y_1, ..., a_m Y_m) \in B_T(\bar{Y})$, $\bar{P}' = (a'_1 Y_1, ..., a'_m Y_m) \in B_T(\bar{Y})$ and $\pi(\bar{P}) = \pi(\bar{P}')$ then by (ii),

$$\sum_{i=1}^{m} a_i T^{i-1} \equiv \sum_{i=1}^{m} a'_i T^{i-1} \pmod{T^m},$$

which in turn implies $a_i = a'_i$ for all i, since $0 \leqslant a_i, a'_i < T$. Thus, π is 1-to-1 on $B_T(\bar{Y})$. Also, by (i)

$$0 \leqslant \pi(\bar{P}) < \varepsilon^2 N.$$

Let us call an integer $a \in [N]$ "good" if

$$B(a) := a + \pi(B_T(\bar{Y})) \subseteq [N]$$

and

$$|X \cap B(a)| > \frac{\varepsilon}{2} T^m.$$

It is easy to see that for a fixed constant $\delta = \delta(\varepsilon) > 0$, the set $A = \{a \in [N]$: a is good$\}$ satisfies

$$|A| > \delta N.$$

By the choice of T, for each $a \in A$, $X \cap B(a)$ contains the translated projection

$$Y_0 + \pi(B_t(A_0 \bar{Y}))$$

for some $Y_0, A_0 \in \mathbb{Z}^+$. Furthermore, by the choice of t, this in turn contains all components of the solution

$$\bar{x} = Y_0 \cdot \bar{1} + \sum_{i=1}^{m} A_0 Y_i \bar{v}_i$$

to $A\bar{x} = \bar{0}$. Since there are cN^{m+1} ways to choose the $Y_0, Y_1, ..., Y_m$ for a positive constant c (depending on ε and A) then the theorem will be proved if we can show that no solution $\bar{x}$ to $A\bar{x} = \bar{0}$ can arise this way in more than a *bounded* number of ways.

To see this, first note that since the $(k-l)$ by k matrix $V = (v_{ij})$ formed from the (linearly independent) solution vectors $\bar{v}_i$, $0 \leqslant i < k-l$, has rank

$k-l$ then we can assume without loss of generality (by relabeling, if necessary) that the $(k-l)$ by $(k-l)$ submatrix $V'=(v_{ij})_{0 \leqslant i,j<k-l}$ is nonsingular. Suppose $\bar{x}=(x_1, ..., x_k)$ has all its components x_i lying in some set $Y_0+\pi(B_T(\bar{Y})) \subseteq [N]$, where $\bar{Y}=(Y_1, ..., Y_m)$ satisfies (i) and (ii). For each of the $k!$ permutations σ on $[k]$, consider the vector $\bar{x}_\sigma=(x_{\sigma(1)}, ..., x_{\sigma(k)})$. If

$$\bar{x}_\sigma=\sum_{i=0}^{m} Y_i \bar{v}_i$$

then by the nonsingularity of V', the first $k-l$ coordinates of $\bar{x}_\sigma$ determine all the Y_i. Thus, each such $\bar{x}$ can arise from at most $k!$ choices for the Y_i.

Finally, we observe that almost all of these $c'N^{k-l}$ solutions $\bar{x}$ to $A\bar{x}=\bar{0}$ are proper solutions. This is because, by hypothesis, for $i \neq j$, the space of solutions $\bar{x}$ with $x_i=x_j$ corresponds to a nontrivial dependence between the coefficients Y_i, $1 \leqslant i \leqslant m$, resulting in at most $O(N^{k-l-1})$ such solutions.

This completes the proof of the theorem. ∎

Let $v_L^*(N; \varepsilon)$ denote the minimum possible number of proper solutions to a system $L=L(A)$ which can belong to a set $X \subseteq [N]$ having $|X|>\varepsilon N$. The following corollary is immediate.

COROLLARY 2. *If A is irredundant and $L=L(A)$ is density regular (i.e., $A \cdot \bar{1}=\bar{0}$) then for any $\varepsilon>0$ there exists $c_\varepsilon^*(L)>0$ such that*

$$v_L^*(N; \varepsilon) \geqslant c_\varepsilon^*(L)\, v_L(N),$$

where $v_L(N)$ denotes the total number of solutions L has in $[N]$.

Canonical Colorings

Suppose $[N]$ is colored with some arbitrary number of colors and we would like to know what types of colored k-term arithmetic progressions must always occur. Monochromatic arithmetic progressions are no longer guaranteed since we might, for example, always decide to give each $x \in [N]$ a distinct color. In this case, however, we would find a k-term arithmetic progression with all its terms having *distinct* colors. We call such a coloring a *one-to-one* coloring. It turns out that one of these two possibilities must always occur.

THEOREM (*Erdös and Graham* [6]; *see also* [14]). *For any $k \in \mathbb{Z}^+$, if N is sufficiently large and $[N]$ is arbitrarily colored then there must always exist a k-term arithmetic progression which is either monochromatic or colored one-to-one.*

We will call such colorings (for arithmetic progressions) *canonical.* The reader can find further information on canonical colorings for other structures in [12, 13, 3, 15, 16, 22].

In the spirit of the preceding results, one could ask for the number of canonically colored k-term arithmetic progressions which must occur in an arbitrary coloring of $[N]$. The answer is given by the following result.

THEOREM 3. *For any $k \in \mathbb{Z}^+$ there exists a constant $c_k > 0$ such that in any coloring of $[N]$ there are always at least $c_k N^2$ canonically colored k-term arithmetic progressions.*

Proof. Let $[N] = \bigcup_{i \in I} C_i$ be a coloring of $[N]$ and suppose $0 < \varepsilon < 1/k^3$ is fixed.

There are two possibilities:

(i) Suppose $|C_i| > \varepsilon N$ for some i. Then by Theorem 2 there are $c_\varepsilon N^2$ k-term arithmetic progressions which belong to C_i, and this case is finished.

(ii) Suppose $|C_i| \leqslant \varepsilon N$ for all $i \in I$. Since there are at most $\binom{|C_i|}{2}\binom{k}{2}$ k-term arithmetic progressions which hit C_i in at least two elements then the total number of k-term arithmetic progressions which are *not* colored one-to-one is at most

$$\sum_{i \in I} \binom{|C_i|}{2}\binom{k}{2}, \tag{11}$$

where, of course,

$$\sum_{i \in I} |C_i| = N.$$

Since

$$\sum_{i \in I} \binom{|C_i|}{2} < \frac{1}{2} \sum_{i \in I} |C_i|^2$$

then the expression in (11) is maximized by taking as many C_i as possible to be as large as possible (in this case, of size εN). Thus

$$\begin{aligned} \sum_{i \in I} \binom{|C_i|}{2}\binom{k}{2} &< \frac{1}{2}\binom{k}{2} \sum_{i \in I} |C_i|^2 \\ &\leqslant \frac{1}{2}\binom{k}{2} \frac{1}{\varepsilon} (\varepsilon N)^2 \\ &\leqslant \frac{\varepsilon k^2}{4} N^2. \end{aligned}$$

Since for N large enough, $[N]$ contains at least $N^2/2k$ distinct k-term progressions altogether, then there must be at least

$$\left(\frac{1}{2k} - \frac{\varepsilon k^2}{4}\right) N^2 \geqslant \frac{1}{4k} N^2$$

monochromatic k-term arithmetic progressions, and the proof is complete. ∎

The same techniques can be applied to density regular systems generally to give the following result.

THEOREM 4. *Suppose A is an irredundant l by k matrix of rank l and $A\bar{1}=\bar{0}$. Then for any $k \in \mathbb{Z}^+$ there is a constant $c_k(A)$ such that in any coloring of $[N]$ there are at least $c_k(A)\, N^{k-l}$ proper solutions $\bar{x}=(x_1, ..., x_k)$ to $A\bar{x}=\bar{0}$ in $[N]$ such that either the x_i all have the same color or they all have distinct colors.*

Next we prove (cf. Theorem 5, below) the corresponding extension of Theorem 1.

First we introduce some preliminaries. For m, p, c positive integers and $i \leqslant m$ set

$$N_{m,p,c}(i)=\{(\lambda_1, \lambda_2, ..., \lambda_m);\, \lambda_j=0 \text{ for } j<i,\ \lambda_i=c \text{ and } |\lambda_j| \leqslant p \text{ for } j>i\}.$$

Then

$$N_{m,p,c}=\bigcup_{i=1}^{m} N_{m,p,c}(i)$$

(cf. the definition of (m, p, c)-set). The next is a slight modification of the theorem proved by Lefmann [10, Satz 2.2].

THEOREM. *Let m, p, c be positive integers. Then there exist M, P, and C such that for any partition (into arbitrarily many classes) of*

$$J=\left\{\sum_{i=1}^{M} \lambda_i y_i;\, (\lambda_1, \lambda_2, ..., \lambda_M) \in N_{M,P,C}\right\},$$

say $J=J_1 \cup \cdots \cup J_s$, there exist pairwise disjoint sets $B_1, B_2, ..., B_m \subseteq [M]$ and

$$Y_i=\sum_{j \in B_i} x_j y_j, \qquad 1 \leqslant |x_j| \leqslant P,\ 1 \leqslant i \leqslant m,$$

such that one of the following possibilities holds:

(i) *all linear combinations*

$$\sum_{i=1}^{m} \lambda_i Y_i,\ (\lambda_1, \lambda_2, ..., \lambda_m) \in N_{m,p,c}$$

belong to a single class J_k for some $k \leqslant s$;

(ii) *all linear combinations*

$$\sum_{i=1}^{m} \lambda_i Y_i, \ (\lambda_1, ..., \lambda_m) \in N_{m,p,c},$$

belong to different partition classes, i.e., if $\sum_{i=1}^{m} \lambda_i Y_i \in J_k$ *and* $\sum_{i=1}^{m} \lambda_i' Y_i \in J_{k'}$, *then* $k = k'$ iff$(\lambda_1, ..., \lambda_m) = (\lambda_1', \lambda_2', ..., \lambda_m')$;

(iii) *for every* $j \leqslant m$, *all linear combinations* $\sum_{i=1}^{m} \lambda_i Y_i$, $(\lambda_1, ..., \lambda_m) \in N_{m,p,c}(j)$, *belong to a single partition class* J_{k_j}; *however,* $k_j \neq k_{j'}$ *for* $j \neq j'$.

Now, let $A\bar{x} = \bar{0}$ be partition regular (i.e., A satisfies the columns condition). Let $\bar{v}_1, \bar{v}_2, ..., \bar{v}_{k-l}$ be the vectors described in (8). Then $\bar{x} = \sum_{i=1}^{k-l} y_i \bar{v}_i$ satisfies $A\bar{x} = \bar{0}$ for every $(k-l)$-tuple of positive integers $y_1, y_2, ..., y_{k-l}$. Suppose now that the set $[N]$ is partitioned into arbitrarily many classes, say, $[N] = N_1 \cup \cdots \cup N_s$. For an arbitrary $(k-l)$-tuple $(y_1, ..., y_{k-l})$ consider a k-tuple of integers $(x_1, ..., x_k)$ formed by entries of the vector $\bar{x} = \sum_{i=1}^{k-l} y_i \bar{v}_i$. We say that the partition $N_1 \cup \cdots \cup N_s$ restricted to $(x_1, ..., x_k)$ is *canonical* if one of the following possibilities holds:

(i) all $x_1, ..., x_k$ belong to a single class N_j for some $j \leqslant s$;

(ii) all $x_1, ..., x_k$ belong to different partition classes, i.e., if $x_i \in J_{j_i}$ and $x_{i'} \in J_{j_{i'}}$ then $i \neq i'$ implies $j_i \neq j_{i'}$;

(iii) let t be the integer from the definition of the columns condition and suppose $k_1, k_2, ..., k_t$ are the integers defined by (8). Then there exist distinct $j_1, j_2, ..., j_t$, $1 \leqslant j_i \leqslant s$, $1 \leqslant i \leqslant t$, such that

$$x_r \in J_{j_i},$$
$$k_{i-1} < r \leqslant k_i, \qquad i = 1, ..., t.$$

The same proof as that of Theorem 1 (with Deuber's theorem replaced by Lefman's theorem) now gives:

THEOREM 5. *Let A be an l by k matrix of rank l which satisfies the columns condition. Then there exists $c(A) > 0$ such that for any coloring of $[N]$ there are at least $c(A)\, N^{k-l}$ canonical solutions to the partition regular system $A\bar{x} = \bar{0}$.*

CONCLUDING REMARKS

Note that we did not investigate what can be said about various constant factors. These questions, in full generality, are certainly very hard.

However, for some equations one can get reasonable bounds. For example, in the case of Eq. (1), $x+y=z$, the constant c_r' in (5) satisfies

$$\frac{1}{R^3} \leqslant c_r' \leqslant (1+o(1))\, 15^{-r}, \tag{12}$$

where $o(1) \to 0$ as $r \to \infty$ and R is the smallest integer with the property that for any r-coloring of edges of the complete graph K_R by r colors there always exists a monochromatic triangle. The known bounds for R are

$$c_1(3.16\cdots)^r \leqslant R \leqslant \lfloor r!\, e \rfloor$$

(see [1, 2]), where $c_1 > 0$ is an appropriate constant.

The upper bound in (12) comes from the following coloring of N. For $x \in [N]$, write $x = \sum_{i \geqslant 0} a_i 15^i, 0 \leqslant a_i < 15$, to the base 15. Let $\mu = \mu(x)$ be the least i such that $a_i > 0$. If $\mu(x) \geqslant r/3$, assign to x the color 0. If $\mu(x) < r/3$ then assign to x the color

$$\begin{array}{lll} 3\mu(x)+1 & \text{if} & a_\mu \equiv \pm 1, \pm 5 \pmod{15} \\ 3\mu(x)+2 & \text{if} & a_\mu \equiv \pm 2, \pm 3, \pm 7 \pmod{15} \\ 3\mu(x)+3 & \text{if} & a_\mu \equiv \pm 4, \pm 6 \pmod{15}. \end{array}$$

In this way, we use $r+1$ colors, and the only color a monochromatic solution to $x+y=z$ can have is the color 0. Thus, x, y, and z are all congruent to 0 (mod $15^{\lfloor r/3 \rfloor}$) which implies the upper bound in (12). The same idea can be used with more complicated decompositions of Z/mZ into sum-free sets to give slight improvements of the upper bound in (12). However, even here the following principal problem remains. Is c_r' an exponential function of r?

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