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Note

On Set Intersections

LÁSZLÓ BABAI

Department of Algebra, Eötvös University, H-1088 Budapest, Muzeum krt. 6–8, Hungary

AND

PÉTER FRANKL

Mathematical Institute of the Hungarian Academy of Science, H-1053 Budapest, Realtanoda u. 13–15, Hungary

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Let L be a finite set of nonnegative integers. Let k and n be natural numbers satisfying $n \geqslant k > \max L$. We call a family $\mathscr{F}$ of k-subsets of an n-set X an (n, k, L)-system if $|F \cap F'| \in L$ for any $F, F' \in \mathscr{F}$, $F \neq F'$. We are interested in the maximum cardinality an (n, k, L)-system can have. We denote it by $f(n, k, L)$.

Ryser [4] proved that $f(n, k, \{l\}) \leqslant n$. This result has been generalized by Ray–Chaudhuri and Wilson [3] to

$$f(n, k, L) \leqslant \binom{n}{|L|}.$$

Deza, Erdös and Frankl [1] obtained that for $n > n_0(k)$,

$$f(n, k, L) \leqslant \prod_{l \in L} \frac{n-l}{k-l}.$$

Deza, Erdös and Singhi [2] proved that

$$f(n, k, \{0, l\}) \leqslant n \qquad \text{whenever} \qquad l \nmid k.$$

In the present note we are discussing the possible generalizations of this last result.

THEOREM 1. *Suppose that the greatest common divisor of the members of L does not divide k. Then $f(n, k, L) \leqslant n$.*

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Proof. Let p^f be a prime power which divides each $l \in L$ but does not divide k.

Let $\mathscr{F} = \{F_1, ..., F_m\}$ be an (n, k, L)-system consisting of subsets of $X = \{1, ..., n\}$. Let $a_j = (\alpha_{1j}, ..., \alpha_{nj})$ be the characteristic vector of F_j, i.e., $\alpha_{ij} = 1$ if $i \in F_j$ and $\alpha_{ij} = 0$ otherwise. We assert that the vectors $a_1, ..., a_m$ are linearly independent over the rationals. This implies the inequality $m \leqslant n$, hence the theorem.

Assume, to the contrary, that $\sum_j \gamma_j a_j = 0$, where we may suppose the γ_j's to be integers with g.c.d. $(\gamma_1, ..., \gamma_m) = 1$. For $1 \leqslant q \leqslant m$, consider the inner product $0 = (a_q, \sum_j \gamma_j a_j) = \sum_j \gamma_j (a_q, a_j) = \sum_j \gamma_j \mid F_q \cap F_j \mid \equiv \gamma_q \mid F_q \mid$ (mod p^f). As $\mid F_q \mid = k \not\equiv 0$ (mod p^f), we infer $p \mid \gamma_q$. This holds for $q = 1, ..., m$, a contradiction.

THEOREM 2. *Let $L = \{l_0, ..., l_{s-1}\}$ with $l_0 = 0$. Suppose we can choose $l_{i_1}, ..., l_{i_t}$ not necessarily different members of $L - \{0\}$ such that $\sum_{q=1}^{t} l_{i_q} = k$. Then for $n \geqslant 2k^2$ we have $f(n, k, L) \geqslant n^2/4k^2$.*

Proof. Let p be the greatest prime not exceeding n/k. Then of course $p \geqslant n/2k$. Let us choose k pairwise disjoint p-subsets X_r of the n-set X; $X_r = \{x_r^1, ..., x_r^p\}$ $(r = 1, ..., k)$. For $1 \leqslant i, j \leqslant p$ set

$$F_{i,j} = \{x_r^{h(r,i,j)}: r = 1, ..., k\},$$

where

$$h(r, i, j) \equiv i + (q - 1) j \qquad (\text{mod } p)$$

for

$$\sum_{v=1}^{q} l_{i_v} < r \leqslant \sum_{v=1}^{q+1} l_{i_v} \qquad (0 \leqslant q < t).$$

Let $\mathscr{F} = \{F_{i,j}: 1 \leqslant i, j \leqslant p\}$.

We have $t \leqslant p$ since $t \leqslant k \leqslant n/2k \leqslant p$. Using this, one readily verifies that $\mathscr{F}$ is an (n, k, L)-system. We conclude that

$$f(n, k, L) \geqslant p^2 \geqslant n^2/4k^2.$$

COROLLARY. *Let n, k be positive integers and $L = \{l_0, l_1, ..., l_{s-1}\}$, where $0 = l_0 < l_1 < \cdots < l_{s-1}$, and $s \geqslant 3$. Assume that $n \geqslant 2k^2$ and $k \geqslant (s - 2) l_{s-1} l_{s-2}$. Then $f(n, k, L) \geqslant n^2/4k^2$ or $f(n, k, L) \leqslant n$ according to whether* g.c.d.$(l_1, ..., l_{s-1})$ *divides k or not.*

Proof. By Theorem 1 we may suppose that g.c.d.$(l_1, ..., l_{s-1})$ divides k. This implies that $\sum_{i=1}^{s-1} \gamma_i l_i = k$ for some integers $\gamma_1, ..., \gamma_s$. Let us choose the γ_i's such that the sum of the negative γ_i's is maximal. We assert that none of the γ_i's is negative.

For, assume $\gamma_j < 0$. Then

$$\sum_{i \neq j} \gamma_i l_i > k \geqslant (s-2)\, l_{s-1} l_{s-2}\,.$$

This implies that $\gamma_q l_q > l_{s-1} l_{s-2}$ for some $q \neq j$, hence $(\gamma_q - l_j)\, l_q > l_{s-1} l_{s-2} - l_j l_q \geqslant 0$. Now, setting $\delta_j = \gamma_j + l_q$, $\delta_q = \gamma_q - l_j$ and $\delta_i = \gamma_i$ for $i \neq j, q$, we arrive at a contradiction since $\sum_{i=1}^{s-1} \delta_i l_i = k$ but the sum of the negative γ_i's is strictly less than the sum of the negative δ_i's.

This proves that k can be written as a nonnegative integer linear combination of the l_i's and $f(n, k, L) \geqslant n^2/4k^2$ follows by Theorem 2.

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